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The "Notes on the preparation of papers" are printed in the last issue of every volume.

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a, α, $\propto$; B, β; r, γ; σ, 6; $+$, $\dagger$; i, $\imath$; a', a^1; the symbol a and the indefinite article a; also the handwritten Roman letters,
$c, C; e, l; I, J; k, K; o, O; p, P; s, S; u, U; v, V; w, W; x, X; z, Z.$

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preferred form	instead of	preferred form	instead of
$A^*, \tilde{b}, \gamma', \mathbf{v}$, etc.	$\bar{A}, \hat{b}, \hat{\gamma}, \vec{v}$, etc.	$\exp(-(x^2+y^2)/a^2)$	$e^{-((x^2+y^2)/a^2)}$
lim sup, proj lim	$\overline{\lim}$, $\underline{\lim}$		
$f: A \to B$	$A \xrightarrow{f} B$		
$\Sigma_{n=1}^{\infty}$	$\sum\limits_{n=1}^{\infty}$	$\dfrac{\cos(1/x)}{(a+b/x)^{1/2}}$	$\dfrac{\cos\frac{1}{x}}{\sqrt{a+\frac{b}{x}}}$

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A typical reference to an article in a journal would be like:

[1] Narasimhan M S and Ramanan S, Moduli of vector bundles on a compact Riemann surface, *Ann. Math.* **89** (1969) 14–51

A reference to a book would be on the following lines:

[2] Royden H, Invariant metrics on Teichmüller space, in: Contributions to analysis (eds) L Ahlfors *et al* (1974) (New York: Academic Press) pp. 393–399

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A combinatorial Lefschetz fixed point formula

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Abstract. We define a class of simplicial maps – those which are "*expanding directions preserving*" – from a barycentric subdivision to the original simplicial complex. These maps naturally induce a self map on the links of their fixed points. The local index at a fixed point of such a map turns out to be the Lefschetz number of the induced map on the link of the fixed point in relative homology. We also show that a weakly hyperbolic [4] simplicial map $\mathrm{sd}^n K \to K$ is expanding directions preserving.

Keywords. Lefschetz fixed point formula; fixed point indices; simplicial complexes.

1. Introduction

The Lefschetz fixed point theorem (LFPT) and the Lefschetz fixed point formula (LFPF) were developed, more or less, simultaneously but notwithstanding their intimate connection with one another they have developed in different directions by groups of investigators who form two disjoint classes.

Lefschetz proved his celebrated LFPT in the context of self-maps of a closed, triangulated oriented manifold M: he looked at the intersection number of the oriented cycles supported by the diagonal of M and the graph of f and proved the (algebraically counted) intersection number of these cycles was equal to the Lefschetz number $L(f)$ of f, defined by,

$$L(f) = \sum_{p\geq 0} \mathrm{Trace}\{f_* : H_p(M, \mathbb{Q}) \to H_p(M; \mathbb{Q})\}.$$

Thus the rational number, $L(f)$ which depended on the global nature of M and f was expressible in terms of the nature of the intersections of the graph of f with respect to the diagonal of M. In particular, if $L(f) \neq 0$ then the intersection set was non-empty and hence f had at least one fixed point. This is the original LFPT.

For a differentiable manifold M and a map $f : M \to M$ which is transverse to the diagonal at each fixed point p, its contribution to the intersection or equivalently to the Lefschetz number turns out to be,

$$i_p = \mathrm{sign}\ \det(I - \mathbf{D}f(p))$$

where $\mathbf{D}f(p)\colon T_pM \to T_p(M)$ is the derivative of f. This leads to the original LFPF:

$$L(f) = \sum_{\mathrm{Fix}(f)} \mathrm{sign}\ \det(I - \mathbf{D}f(p))$$

where $\mathrm{Fix}(f)$ is the fixed point set of f.

Soon afterwards Hopf discovered that the LFPT (that is, a fixed point exists whenever the Lefschetz number is nonzero) was, in fact, true for self-maps of an arbitrary compact polyhedron. In this case he could also attach an integer, $i(f;p)$, to each fixed-point, p, provided p belonged to a principal simplex (or equivalently p had a neighbourhood homeomorphic to an open ball in some Euclidean space). Now topologists, Lefschetz included, devoted themselves to extending the validity of the LFPT and proving the existence of a fixed-point index and the associated LFPF to more and more general classes of spaces. The index was axiomatically characterized by O'Neil [6] for a fairly large class of spaces. Dold [3] gave a beautiful formula for the fixed-point index of self maps of arbitrary compact ENR's which however was difficult to use for actual computations.

Parallelly in time, though diverging in direction, analysts and geometers looked at the LFPF in more specialized situations. In 1967 Atiyah and Bott [1] reinterpreted the terms on the right side of the original LFPF as a normalized Lefschetz number of the map induced by f on the sheaf of germs of differential forms in a neighbourhood of p and went on to extend this idea to a LFPF for elliptic complexes with a large number of spectacular applications. These developments led to a change in paradigm for the study of the fixed point index and LFPF. The focus was now on associating to the fixed point set (of some special class of maps) a cochain complex and an endomorphism thereof such that $i(f;p)$ would be the alternating sum of the traces of the maps induced on the cohomology of the complex i.e. $i(f,p)$ would be expressed as a local Lefschetz number. Such results have been obtained for holomorphic maps, ([7,9]) for isometries of spin manifolds [2] and most spectacularly, in the context of algebraic geometry and number theory by Deligne.

Curiously no topological result had been obtained in this context till 1993 when Goresky and McPherson [4], using the very sophisticated methods of intersection cohomology, obtained a LFPF for *weakly hyperbolic maps* of compact stratified spaces.

In this paper we study the LFPF in the classical topological setting using quite elementary methods. We consider the problem of defining the fixed point indices as local Lefschetz numbers of simplicial maps from iterated barycentric subdivisions to the original simplicial complex. The motivation for studying such maps comes from the fact that any continuous map on a compact connected polyhedron can be approximated by such a simplicial map.

In § 2 we consider simplicial maps $K \to K$ on a finite simplicial complex K. We show that for such simplicial maps, the fixed point set is a subcomplex of $\operatorname{sd} K$. In this case the fixed point index at a path component of the fixed point set turns out to be its Euler characteristic.

We consider simplicial maps on higher barycentric subdivisions of a finite simplicial complex in § 3. We show that the fixed point set of a simplicial map $\operatorname{sd}^n K \to K$, $n \geq 1$ is a finite set of points. Such simplicial maps induce a map on the stars of its fixed points. To obtain the fixed point index as a local Lefschetz number we impose certain conditions on the map which arise quite naturally from the problem. We call this special class of simplicial maps *expanding directions preserving (e.d.p).* We give a homological definition of the fixed point index at a fixed point of an e.d.p. map as the Lefschetz number of the induced map on the relative homology of the star modulo a subcomplex of the link of the fixed point and show that the LFPF holds in this case also.

The notion of weakly hyperbolic maps is independent of compact stratified spaces. In § 4 we show that if a simplicial map $\operatorname{sd}^n K \to K$, $n \geq 1$ is weakly hyperbolic then it i

e.d.p We also show by an example that a continuous map having an e.d.p simplicial approximation need not be weakly hyperbolic.

2. LFPF for a simplicial map $K \to K$

Throughout this paper K will be a finite simplicial complex whose geometric realisation $|K|$ is connected. The notations will be as in Spanier [8].

The Lefschetz fixed point formula expresses the Lefschetz number of a continuous map as a sum of its fixed point indices. In the category of compact connected polyhedra this result can be precisely stated as follows:

Let $f : X \to X$ be a continuous map on a compact polyhedron X. A subset C of $\mathrm{Fix}(f)$ is called an *isolated set of fixed points* of f if C is both compact and open in $\mathrm{Fix}(f)$ [5].

Let $\{C_1, \dots, C_k\}$ be a collection of isolated sets of fixed points of f such that $\mathrm{Fix}(f) = \cup_{j=1}^{k} C_j$. If $L(f)$ is the Lefschetz number of f, and for each j, $1 \le j \le k$, $i(f, C_j)$ is the fixed point index of f at C_j, then the Lefschetz fixed point formula states that,

$$L(f) = \sum_{j=1}^{k} i(f, C_j).$$

A collection of isolated sets of fixed points of a map whose union is the fixed point set of the map is clearly not unique. For instance we could define a map on the unit interval whose fixed point set is precisely $\{0, \frac{1}{n}\colon n = 1, 2, 3, \dots\}$ and for such a map and for arbitrary $m \ge 2$ a choice of a collection of isolated sets of fixed points could be $\{\{1\}, \dots \{\frac{1}{m-1}\}, \{0, \frac{1}{n} : n \ge m\}\}$.

We now show that for simplicial maps $K \to K$ the natural choice of a collection of isolated sets of fixed points whose union is the fixed point set of the map is the distinct path components of the fixed point set.

Let $f : K' \to K$ be a simplicial map. The subdivision of f is the simplicial map, $\mathrm{sd} f\colon \mathrm{sd}\, K' \to \mathrm{sd}\, K$, defined canonically through barycentric subdivisions.

Lemma 2.1. *Let $f : K \to K$ be a simplicial map. Then* $\mathrm{Fix}(f)$ *is a subcomplex of* $\mathrm{sd}\, K$.

Proof. It is quite easily seen that $\mathrm{Fix}(\mathrm{sd} f) = \mathrm{Fix}(f)$. In fact if x is a fixed point of f and σ is a simplex of $\mathrm{sd}\, K$ such that $x \in \langle\sigma\rangle$ then for all vertex $b(\tau)$ of σ, $\mathrm{sd} f(b(\tau)) = b(\tau)$. Hence $\mathrm{Fix}(\mathrm{sd} f)$ is a subcomplex of $\mathrm{sd}\, K$. ■

Since $\mathrm{Fix}(f)$ is a subcomplex of $\mathrm{sd}\, K$, the family of path components of $\mathrm{Fix}(f)$ is finite and we choose the family of isolated sets of fixed points of f whose union is $\mathrm{Fix}(f)$ to be the family of path components of $\mathrm{Fix}(f)$. Let $C_1, \dots, C_n$ be the distinct path components of $\mathrm{Fix}(f)$. Let K_j be the smallest subcomplex of K containing C_j for $1 \le j \le n$. Then if $\mathrm{N}(K_j, K)$ is the stellar neighbourhood of K_j, f maps $\mathrm{N}(K_j, K)$ into $\mathrm{N}(K_j, K)$. By the localisation and normalisation property of the fixed point index [3] it is clear that $i(f, C_j) = L(f|\mathrm{N}(K_j, K))$ and this is invariant of taking subdivisions i.e. $i(f, C_j) = i(\mathrm{sd} f, C_j)$. Hence by taking subdivisions if necessary, we can assume that each C_j is a subcomplex of K and if $j \ne l$, $\overline{\mathrm{N}}(C_j, K) \cap \overline{\mathrm{N}}(C_l, K) = \emptyset$.

Theorem 2.2. $L(f) = \sum_{j=1}^{n} i(f, C_j)$.

Proof. Let σ be a simplex of K such that $\mathrm{Fix} f \cap \langle\sigma\rangle \ne \emptyset$. Then $\sigma = f(\sigma)$. Since $\mathrm{Fix}(f)$ is a subcomplex, σ is a simplex of $\mathrm{Fix}(f)$. Hence a simplex of K contributes to the trace of f

if and only if it is a simplex of Fix(f). Therefore $L(f) = \sum_{j=1}^{n} i(f, C_j)$ by the Hopf trace theorem [8].

Remark 2.3. It is clear from the above proof that for all $1 \leq j \leq n$, $i(f, C_j)$ is the Euler characteristic of C_j.

3. LFPF for a simplicial map $\mathrm{sd}^n K \to K$, $n \geq 1$

The situation is quite different when we consider simplicial maps from higher barycentric subdivisions. Let $f : \mathrm{sd}^n K \to K$, $n \geq 1$ be a simplicial map.

Lemma 3.1. *Fix(f) is a finite set of points of $|K|$.*

Proof. Let σ be a simplex of $\mathrm{sd}^n K$ such that $\langle\sigma\rangle \subset \langle f(\sigma)\rangle$. Let d be the metric on $|K|$. Since $n \geq 1$, for any $x, y \in |\overline{\sigma}|, d(x,y) < d(f(x), f(y))$. So $f(\overline{\sigma})$ has at most one fixed point of f. Therefore if σ is a simplex of $\mathrm{sd}^n K$ such that $\mathrm{Fix}(f) \cap \langle\sigma\rangle = \{x\}$, then $\mathrm{st}(\sigma, \mathrm{sd}^n K) \cap \mathrm{Fix}(f) = \{x\}$. The result now follows since K is a finite simplicial complex. ■

It is clear that the fixed point set of f is left invariant by the subdivision operator i.e. $\mathrm{Fix}(\mathrm{sd} f) = \mathrm{Fix}(f)$. Let x be a fixed point of f. For all $p \geq 0$, let $\sigma_{[p]}$ be the carrier of x in $\mathrm{sd}^p K$. Then f maps $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$ into $\overline{\mathrm{st}}(\sigma_{[0]}, K)$. The homology groups of the stars are isomorphic, $H_p(\overline{\mathrm{st}}(\sigma_{[0]}, K)) \xrightarrow{\cong} H_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K))$, the isomorphism being induced by a retraction, $|\overline{\mathrm{st}}(\sigma_{[0]}, K)| \xrightarrow{\theta} |(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)|$. Since $\mathrm{Fix}(\theta f)$ need not be equal to $\mathrm{Fix}(f)$, $L(\theta f)$ does not necessarily define $i(f,x)$, though $f = \theta f$ on $|\overline{\mathrm{st}}(\sigma_{[n+1]}, \mathrm{sd}^{n+1} K)|$ implies that $i(f,x) = i(\theta f, x)$. If θ' is another retraction then at the homology level $\theta_* = \theta'_*$. So the Lefschetz number of the map θf is independent of θ. We define a simplicial map

$$\rho : \mathrm{sd}^n(\overline{\mathrm{st}}(\sigma_{[0]}, K)) \to \overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$$

which is a deformation retraction.

Now x is a fixed point of f and the carrier of x in $\mathrm{sd}^p K, p \geq 0$, is $\sigma_{[p]}$. Hence for all $1 \leq j \leq n$, $\langle\sigma_{[j]}\rangle \subset \langle\sigma_{[j-1]}\rangle$ and $\dim \sigma_{[j]} = \dim \sigma_{[j-1]}$. Fix $s, 0 \leq s \leq n$. Let $\sigma_{[s]} = [v_0, \ldots, v_p]$ and $\sigma_{[s+1]} = [b(\mu_0), \ldots, b(\mu_p)]$ where $\mu_j = [v_0, \ldots, v_j]$. Define

$$\rho_s : \mathrm{sd}(\overline{\mathrm{st}}(\sigma_{[s]}, \mathrm{sd}^s K)) \to \overline{\mathrm{st}}(\sigma_{[s+1]}, \mathrm{sd}^{s+1} K)$$

by

$$\rho_s(b(\tau)) = b(\sigma_s \cup \tau) \quad \text{if } \mu_j \prec \tau \text{ implies that } \mu_{j+1} \prec \tau \ \forall 0 \leq j \leq p$$
$$\rho_s(b(\tau)) = b(\mu_j) \quad \text{if } \exists j \text{ such that } \mu_j \prec \tau \text{ and } \mu_{j+1} \not\prec \tau.$$

For example if $\sigma_{[0]} = v$, a vertex of K, then $\sigma_{[j]} = v$ for all $0 \leq j \leq n$ and $\rho_s(b(\tau)) = b(\tau \cup v)$ for all $\tau \in \overline{\mathrm{st}}(v, \mathrm{sd}^s K)$.

ρ_s is a deformation retraction and we define

$$\rho : \mathrm{sd}^n(\overline{\mathrm{st}}(\sigma_{[0]}, K)) \to \overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$$

to be the composition

$$\rho = \rho_{n-1} \circ \mathrm{sd}\rho_{n-2} \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0.$$

Then ρ is a deformation retraction. Also for any point y of $\mathrm{st}(\sigma_{[0]}, K)$, $\rho(y)$ is an interior point of the carrier of y in K and ρ maps $\overline{\mathrm{st}}(\sigma_{[0]}, K) - \mathrm{st}(\sigma_{[0]}, K)$ on to $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K) - \mathrm{st}(\sigma_{[n]}, \mathrm{sd}^n K)$.

The composition ρf is a self map on $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$. It is clear from the above remarks that the only fixed point of ρf in $\mathrm{st}(\sigma_{[n]}, \mathrm{sd}^n K)$ is x. If we can now get a subcomplex M of $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ which contains all the fixed points of the map ρf on $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ and is such that ρf maps M into itself, then the relative Lefschetz number of ρf for the pair $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M$ gives the index of f at x by a principle analogous to the additive property of fixed point indices [3].

Let y be a point of $\mathrm{st}(\sigma_{[0]}, K)$ and $\sigma_{(0)}(y)$ be the carrier of y in K. Then $\rho(y) \in \sigma_{(0)}(y)$. So a point y of $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K) - \mathrm{st}(\sigma_{[n]}, \mathrm{sd}^n K)$ is a fixed point of ρf only if $f(y) \in \langle \sigma_{(0)}(y) \rangle$. Thus if τ is a simplex of $\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$ which contains a fixed point of ρf then $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$ i.e. f is *expanding* on $\sigma_{[n]} \cup \tau$. So we can think of f as a generalisation of *expanding maps*. Our requirement that a subcomplex M of $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ exists which contains all the fixed points of the map ρf on $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ and which maps into itself by the map ρf is in this sense the requirement that the map f is *expanding directions preserving* at x.

DEFINITION 3.2

The map $f : \mathrm{sd}^n K \to K$ will be called *expanding directions preserving (e.d.p.)* at a fixed point x of f, if there is a subcomplex $M(x) = M$ of $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ which contains all simplices τ of $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$, with the property that, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$, and which further satisfies,

(a) $\tau \in M$ and $f(\tau) \prec \sigma_{[0]}$ implies that $\tau \prec \sigma_{[n]}$.
(b) $\tau \in M$ implies that $\mathrm{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]} \subset M$.

$M(x)$ will be called a *subcomplex at x expanded by the map f*. The map f will be called *expanding directions preserving (e.d.p.)* if it is *e.d.p.* at each of its fixed points.

We assume from now on that f: $\mathrm{sd}^n K \to K$, $n \geq 1$, is e.d.p. and x is a fixed point of f. We denote the carrier of x in $\mathrm{sd}^p K$, $p \geq 0$ by $\sigma_{[p]}$.

Let λ_p: $C_p(\overline{\mathrm{st}}(\sigma_{[0]}, K)) \to C_p(\mathrm{sd}^n\{\overline{\mathrm{st}}(\sigma_{[0]}, K)\})$ be the subdivision operator on the simplicial chain complex of the star of $\sigma_{[0]}$ and

$$\tilde{f}_p = \rho_p \circ \lambda_p \circ f_p : C_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)) \to C_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)).$$

It is clear that Lefschetz number of $\tilde{f}_*$ is the Lefschetz number of the map $|\rho| \circ |f|$.

Observation 3.3. If the support of a chain $c \in C_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K))$ is contained in M, then the support $\|\tilde{f}_p(c)\|$ of the chain $\tilde{f}_p(c)$ is contained in M.

Hence there is a chain homomorphism

$$\tilde{f}_p : C_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M) \to C_p(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M).$$

and we define the Lefschetz number of this chain map on relative homology groups to be the fixed point index of f at x.

$$\boxed{I(f, x) = L(\tilde{f} | (\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M)).}$$

Example 3.4. If f maps $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K)$ into $\mathrm{Lk}(\sigma_{[0]}, K)$, then a choice of M could be, $M = \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$. If in this case we also have that for all simplex τ of

$\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K)$, $\langle \sigma_{[n]} \cup \tau \rangle \not\subset \langle \sigma_{[0]} \cup f(\tau) \rangle$ then we could also choose M to be $\dot{\sigma}_{[n]}$. Thus M is not uniquely determined by f. However if M_1 and M_2 are two subcomplexes at x expanded by f then $M_1 \cap M_2$ is also a subcomplex at x expanded by f and it follows by an application of the Hopf trace theorem that

$$\begin{aligned} L(\tilde{f}|(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M_1)) &= L(\tilde{f}|(\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K), M_1 \cap M_2)) \\ &= L(\tilde{f}|(\overline{st}(\sigma_{[n]}, \mathrm{sd}^n K), M_2)). \end{aligned}$$

On the other extreme if f maps $\mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ into $\overline{\sigma}_{[0]}$, then M has to be $\dot{\sigma}_{[n]}$.

It quite easily follows that if f is e.d.p, so is $\mathrm{sd} f$ and in fact for any fixed point x of f if M is a subcomplex at x expanded by f then $\mathrm{sd}\, M$ is a subcomplex at x expanded by $\mathrm{sd} f$. Also $I(f,x) = I(\mathrm{sd}\, f, x)$. We now obtain the LFPF:

Theorem 3.5. *Let $f : sd^n K \to K$ be an e.d.p map. Then,*

$$L(f) = \sum_{x \in \mathrm{Fix}(f)} I(f,x)$$

Proof. Let $\mathrm{Fix}(f) = \{x_1, \ldots, x_k\}$. Let the carrier of x_j in $\mathrm{sd}^p K, p \geq 0$, be $\sigma_{[p]}(x_j)$ and the subcomplex at x_j expanded by f be $M(x_j)$ for all $1 \leq j \leq k$. Without loss of generality we can assume that

$$\overline{\mathrm{st}}\,(\sigma_{[0]}(x_j), K) \cap \overline{\mathrm{st}}\,(\sigma_{[0]}(x_j), K) = \emptyset, \text{ for all } 1 \leq i \neq j \leq k.$$

For all q-simplex μ of $\mathrm{sd}^n K$, let $\lambda_q f_q(\mu) = \sum n_{\nu\mu} \nu$, where the sum is over all q-simplex $\nu \in \mathrm{sd}^n K$. Then by the Hopf trace theorem [8]

$$L(f) = \sum_{q \geq 0} (-1)^q \left(\sum_{\mu \in \mathrm{sd}^n K} n_{\mu\mu} \right).$$

Let $F = \cup_{j=1}^k \overline{\sigma}_{[n]}(x_j)$ and $F' = \cup_{j=1}^k \overline{\sigma}_{[0]}(x_j)$. Clearly, if $\mu \in \mathrm{sd}^n K - \mathrm{N}(F, \mathrm{sd}^n K)$ then $n_{\mu\mu} = 0$. Hence,

$$L(f) = \sum_{q \geq 0} (-1)^q \Big(\sum \{ n_{\mu\mu} : \mu \in \mathrm{sd}^n K; \langle \mu \rangle \in \mathrm{N}(F, \mathrm{sd}^n K) \} \Big).$$

Also $\overline{N}(F', K) = \cup_{j=1}^k \overline{\mathrm{st}}(\sigma_{[0]}(x_j), K)$. So we can define

$$\rho : \mathrm{sd}^n(\overline{\mathrm{N}}(F', K)) \to \overline{\mathrm{N}}(F, \mathrm{sd}^n K)$$

to be the union of all $\rho(j)$, where,

$$\rho(j) : \mathrm{sd}^n\{\overline{\mathrm{st}}(\sigma_{[0]}(x_j), K)\} \to \overline{\mathrm{st}}(\sigma_{[n]}(x_j), \mathrm{sd}^n K)$$

is the simplicial retraction defined earlier. Let $M = \cup_{j=1}^k M(x_j)$. By the relative Mayer-Vietoris sequence

$$L(\tilde{f}|(\overline{\mathrm{N}}(F, \mathrm{sd}^n K), M)) = \sum_{j=1}^k I(f, x_j)$$

Also as a consequence of the Hopf trace theorem,

$$L(\tilde{f}|(\overline{\mathrm{N}}(F, \mathrm{sd}^n K), M)) = L(\tilde{f}|(\overline{\mathrm{N}}(F, \mathrm{sd}^n K))) - L(\tilde{f}|M)$$

Therefore to arrive at the LFPF it is enough to show that

$$L(f) = L(\tilde{f}|\overline{\mathrm{N}}(F, \mathrm{sd}^n K)) - L(\tilde{f}|M) \tag{1}$$

For all q-simplex μ of $\overline{\mathrm{N}}(F, \mathrm{sd}^n K)$, let $\tilde{f}_q(\mu) = \sum m_{\nu\mu}\nu$, where the sum is over all q-simplices ν of $\overline{\mathrm{N}}(F, \mathrm{sd}^n K)$. Also

$$\tilde{f}_q(\mu) = \rho_* \lambda_q f_q(\mu) = \rho_* \left(\sum n_{\nu\mu}\nu\right) = \sum n_{\nu\mu}\rho(\nu).$$

So for all μ such that $\langle\mu\rangle \subset \mathrm{N}(F, \mathrm{sd}^n K)$, $n_{\mu\mu} = m_{\mu\mu}$. Hence,

$$\begin{aligned}
&\mathrm{Trace}\ \{\tilde{f}_q : C_q(\overline{\mathrm{N}}(F, \mathrm{sd}^n K)) \to C_q(\overline{\mathrm{N}}(F, \mathrm{sd}^n K))\} \\
&\quad = \sum \{n_{\mu\mu} : \langle\mu\rangle \subset \mathrm{N}(F, \mathrm{sd}^n K)\} \\
&\qquad + \sum \left\{ m_{\mu\mu} : \mu \in \bigcup_{j=1}^{k} \mathrm{Lk}(\sigma_{[n]}(x_j), \mathrm{sd}^n K) * \dot{\sigma}_{[n]}(x_j), |\overline{\mu}| \subset \|\tilde{f}_q(\mu)\| \right\}.
\end{aligned}$$

Now for any simplex μ of $\mathrm{Lk}(\sigma_{[n]}(x_j), \mathrm{sd}^n K) * \dot{\sigma}_{[n]}(x_j)$, $|\overline{\mu}| \subset \|\tilde{f}_q(\mu)\|$ implies that $\langle\sigma_{[n]}(x_j) \cup \mu\rangle \subset \langle\sigma_{[0]}(x_j) \cup f(\mu)\rangle$ and hence that μ is a simplex of $M_j \subset M$. Hence eq. (1) holds and the result follows. ■

Remark 3.6. It is also clear from the above discussion that

$$I(f, x_j) = \sum_{q \geq 0} (-1)^q \sum \{n_{\mu\mu} : \langle\mu\rangle \subset \mathrm{st}(\sigma_{[n]}(x_j), \mathrm{sd}^n K), \mu \text{ a q-simplex}\}.$$

This reduces to the definition given by O'Neill [6] for the fixed point index of f on the open neighbourhood $\mathrm{st}(\sigma_{[n]}(x_j), \mathrm{sd}^n K)$ of x_j.

4. Weakly hyperbolic simplicial maps

We show that the class of e.d.p maps is fairly large by showing that if a simplicial map $\mathrm{sd}^n K \to K$, $n \geq 1$ is weakly hyperbolic then it is e.d.p. We also show that a continuous map having an e.d.p simplicial approximation need not be weakly hyperbolic.

In the simplicial category a weakly hyperbolic map can be defined as follows:

DEFINITION 4.1

A simplicial map $f : K' \to K$, from a subdivision K' of a finite simplicial complex K to K, is *weakly hyperbolic* if for every fixed point component C of f there is an open neighbourhood W of C in $|K|$ and an indicator map $t : W \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that $t^{-1}((0,0)) = C$ and for all $x \in W \cap f^{-1}(W)$, if the jth coordinate of $t(x)$ is denoted by $t_j(x)$ then,

$$t_1(f(x)) \geq t_1(x) \text{ and } t_2(f(x)) \leq t_2(x).$$

Let $f : X \to X$ be a map on a connected compact polyhedron. Let $A \subset X$ and W be a neighbourhood of A in X. Define,

$$[A]^{-}_{\overline{W}} = \{y \in \overline{W} : \exists \{y_n\}_{n \geq 0} \subset \overline{W}, \text{ where, } y_0 = y,\ f(y_n) = y_{n-1} \text{ and, } \lim_{n \to \infty} y_n \in A\}.$$

We need the following lemma. Compare ([4, Proposition on p. 12, § 3]).

Lemma 4.2. *Let $\varphi : X \to X$ be a weakly hyperbolic map on a polyhedron and y be a fixed point of φ. Let for any open neighbourhood W of y in X,*

$$[y]_W^+ = \left\{x \in W : \varphi^n(x) \in W, \text{ for all } n \geq 1, \text{ and } \lim_{n\to\infty} \varphi^n(x) = y\right\}$$

Then $[y]_W^+ \cap [y]_W^- = \{y\}$.

Remark 4.3. It is not generally true that $[y]_X^+ \cap [y]_X^- = \{y\}$. For instance, consider the map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $\varphi(e^{i\theta}) = e^{2i\theta}$. Then $\mathrm{Fix}\varphi = \{1\}$. The map is expanding in a neighbourhood of 1 and hence is weakly hyperbolic. But any $e^{i\frac{\pi}{2^n}}, n \in \mathbb{N}$, belongs to $[1]_{\mathbb{S}^1}^+ \cap [1]_{\mathbb{S}^1}^-$.

PROPOSITION 4.4

Let $g : \mathrm{sd}^n K \to K$, $n \geq 1$, be a simplicial map. If for each fixed point x of g, there is a neighbourhood W of x in $|K|$ such that $[x]_{\overline{W}}^- \cap g^{-1}(x) = \{x\}$, then g is e.d.p.

Proof. Let x be a fixed point of g. We show that there is a subcomplex M at x which is expanded by g. We denote the carrier of x in $\mathrm{sd}^p K$ by $\sigma_{[p]}$.

We assume without loss of generality that $W \subset \mathrm{st}(\sigma_{[n]}, \mathrm{sd}^n K)$. Let y be a point of $|\overline{st}(\sigma_{[n]}, \mathrm{sd}^n K)|$ such that there is a sequence $\{y_m\}$ in $|\overline{st}(\sigma_{[n]}, \mathrm{sd}^n K)|$ satisfying $y_0 = y$, $g(y_m) = y_{m-1}$ and $\lim_{m\to\infty} y_m = x$. Also assume that $g(y) = x$. Now g is a piecewise linear map. Therefore there is a sequence $\{z_m\} \subset W$ such that z_m lies on the line segment joining x and y_m and also $g(z_m) = z_{m-1}$, $\lim_{m\to\infty} z_m = x$. Therefore $z_0 \in [x]_W^-$. Since z_0 lies on the line segment joining x and y_0, $g(z_0) = x$, a contradiction. Hence,

$$[x]_{\overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)}^- \cap g^{-1}(x) = \{x\}.$$

For any $\tau \in \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ define

$$L^1(\tau) = \mathrm{sd}^n(\sigma_{[0]} * g(\overline{\tau})) \cap \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$$
$$S^1(\tau) = \mathrm{sd}^n(\sigma_{[0]} * g(\overline{\tau})) \cap \mathrm{st}(\sigma_{[n]}, \mathrm{sd}^n K).$$

Suppose that we have defined $L^k(\tau)$ and $S^k(\tau)$ for $1 \leq k \leq p-1$. Define,

$$L^p(\tau) = \cup\{\overline{\mu} : \overline{\mu} \in L^1(\nu), \text{ for all } \nu \in L^{p-1}(\tau)\}$$
$$S^p(\tau) = \cup\{\overline{\mu} : \overline{\mu} \in S^1(\nu), \text{ for all } \nu \in L^{p-1}(\tau)\}$$

Let $\tau \in \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$, be such that $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup g(\tau) \rangle$. Take any $y \in |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap \overline{\mathrm{st}}(\sigma_{[n]}, \mathrm{sd}^n K)$. Then there is a point y_1 of $\sigma_{[n]} \cup \tau$, such that, $g(y_1) = y$. It can be shown inductively that for all $m \geq 1$, there is a $y_m \in \mathrm{st}(\sigma_{[mn]}, \mathrm{sd}^{mn} K)$, such that, $g(y_m) = y_{m-1}$. Clearly then the sequence $\{y_m\}$ converges to x. Thus $S^1(\tau) \subset [x]_{\overline{\mathrm{st}}(\sigma_{[n]} \mathrm{sd}^n K)}^-$ and

$$g : L^1(\tau) \to \mathrm{Lk}(\sigma_{[0]}, \mathrm{sd}^n K) * \dot{\sigma}_{[0]}.$$

By similar reasoning, $S^p(\tau) \subset [x]_{\overline{\mathrm{st}}(\sigma_{[n]} \mathrm{sd}^n K)}^-$ and,

$$g : L^p(\tau) \to \mathrm{Lk}(\sigma_{[0]}, \mathrm{sd}^n K) * \dot{\sigma}_{[0]}.$$

Clearly $L^p(\tau) = L^{p+1}(\tau)$, if for all $\mu \in L^p(\tau), L^1(\mu) \subset L^p(\tau)$. Since K is finite there is a p such that $L^p(\tau) = L^{p+1}(\tau)$. We denote this $L^p(\tau)$ by $L(\tau)$. Now it easily follows that the

subcomplex M defined by,

$$M = \cup\{L(\tau) : \tau \in \mathrm{Lk}(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}, |\overline{\sigma}_{[n]} * \overline{\tau}| \subset |\overline{\sigma}_{[0]} * g(\overline{\tau})|\}$$

is a subcomplex at x which is expanded by g. ∎

The following is an immediate consequence of Lemma 4.2 and Proposition 4.4.

Theorem 4.5. *A weakly hyperbolic simplicial map is e.d.p.*

Example 4.6. We now give an example of a continuous map which has an e.d.p. simplicial approximation but which is not weakly hyperbolic. Let $X = \mathbb{S}^2$ where we think of $\mathbb{S}^2$ as the suspension $S(\mathbb{S}^1)$. A triangulation of $X = |K|$ is shown in figure 1.

Let $f : X \to X$ be the map $f([z,t]) = [z^2, t]$ for all $z \in \mathbb{S}^1, 0 \leq t \leq 1$. Then $L(f) = 3$. It is clear from figure 1 that $\mathrm{Fix}(f)$ is the subcomplex $[0,u] \cup [0,v]$. A simplicial approximation, $g : \mathrm{sd}\, K \to K$ to f is shown in figure 2. This figure is to be read as follows: A vertex β of $\mathrm{sd}\, K$ is labelled as α in the figure if $g(\beta) = \alpha$.

It is clear from figure 2 that $\mathrm{Fix}\, g = \{0, u, v\}$ and g is e.d.p In fact one can take $M(u) = M(v) = \emptyset$. Hence $I(g,u) = I(g,v) = 1$. Also $M(0) = \mathrm{Lk}(0, \mathrm{sd}\, K)$. So,

$$H_i(\overline{\mathrm{st}}(0, \mathrm{sd}\, K), M(0)) = 0 \text{ for } i \neq 2$$
$$H_2(\overline{\mathrm{st}}(0, \mathrm{sd}\, K), M(0)) \cong \mathbb{Z}$$

the generator of the second homology group being a sum c of all 2 simplices of $\overline{\mathrm{st}}(0, \mathrm{sd}\, K)$ suh that $\partial c = 0$ in $C_1(\overline{\mathrm{st}}(0, \mathrm{sd}\, K), M(0))$. Since $\tilde{g}_*(c) = c$, it follows that $I(g,0) = 1$. Note that

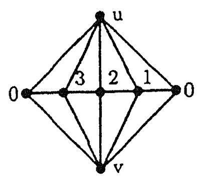

Figure 1. A triangulation of $\mathbb{S}^2$.

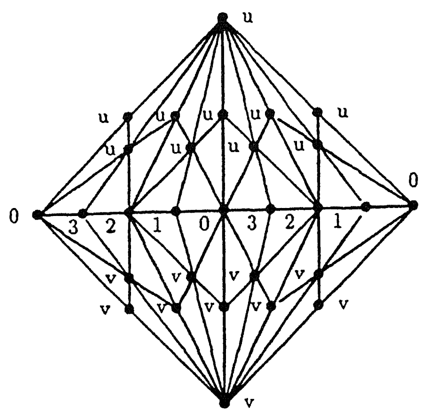

Figure 2. A simplicial approximation to $[z,t] \mapsto [z^2, t]$ on $\mathbb{S}^2$.

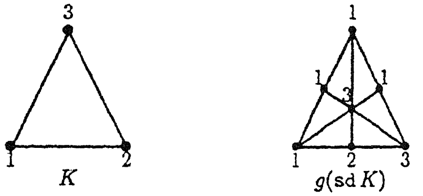

Figure 3. A map which is not e.d.p.

f and g are homotopic maps such that the homotopy has no fixed points on the boundary of a neighbourhood of $\mathrm{Fix}(f)$. Hence $i(f,\{[1,t]: t\in[0,1]\})=i(g,u)+i(g,v)+i(g,0)=3$.

On the other hand f is not weakly hyperbolic. In fact, any neighbourhood W of $\mathrm{Fix}(f)$ will contain $\mathbb{S}^1\times[1-\epsilon,1]$ for some $\epsilon>0$ and then,

$$\mathbb{S}^1\times[1-\epsilon,1]\subset[F]^+_{\overline{W}}\cap[F]^-_{\overline{W}}.$$

Example 4.7. Not all simplicial maps are e.d.p. Let K be the standard 2-simplex with $V(K)=\{1,2,3\}$. Define a simplicial map $g:\mathrm{sd}K\to K$ as shown in figure 3. Then $\mathrm{Fix}\, g=\{1\}$ and g is not e.d.p. at 1.

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Dolbeault cohomology of compact complex homogeneous manifolds

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Abstract. We show that if M is the total space of a holomorphic bundle with base space a simply connected homogeneous projective variety and fibre and structure group a compact complex torus, then the identity component of the automorphism group of M acts trivially on the Dolbeault cohomology of M. We consider a class of compact complex homogeneous spaces W, which we call generalized Hopf manifolds, which are diffeomorphic to $S^1 \times K/L$ where K is a compact connected simple Lie group and L is the semisimple part of the centralizer of a one dimensional torus in K. We compute the Dolbeault cohomology of W. We compute the Picard group of any generalized Hopf manifold and show that every line bundle over a generalized Hopf manifold arises from a representation of its fundamental group.

Keywords. Dolbeault cohomology; complex homogeneous manifolds; generalized Hopf manifolds; automorphism groups; Picard group.

1. Introduction

Let M be a compact complex manifold. It is a deep result of Bochner and Montgomery [7] that the group $\mathrm{Aut}(M)$ of all biholomorphic automorphisms of M is a complex Lie group and the action map is complex analytic. If M is a connected homogeneous complex manifold, then $M = G/H$ for some closed subgroup $H \subset G$, where $G = \mathrm{Aut}_1(M)$ denotes the connected component of $\mathrm{Aut}(M)$ that contains the identity map of M. Furthermore, if M is simply connected then a result of Montgomery [17] says that $M = K/L$ for a maximal compact connected subgroup of G and L a closed connected subgroup of K.

It is well known from the work of Wang [19] that if K is any compact semisimple Lie group and L a closed connected Lie subgroup of K whose semisimple part equals the semisimple part of the centralizer of a toral subgroup, then $M = K/L$ admits a K-invariant complex structure compatible with the usual differentiable structure on M provided it is even dimensional. Conversely, if a compact semisimple Lie group K acts transitively and biholomorphically on a simply connected compact complex manifold M then Wang [19] shows that M can be expressed as K/L where L is a closed Lie subgroup of K such that the semisimple part of L coincides with the semisimple part of the centralizer of a toral subgroup of K. If the second Betti number of M is zero, then M admits uncountably many invariant complex structures. As Wang noted, this class of complex manifolds includes the Calabi–Eckmann manifolds, the Stiefel manifold $V_{n,2k}$ when it is even dimensional, and the product $V_{m,2l} \times V_{n,2k}$ when it is even dimensional. Here $V_{n,k}$ denotes the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n, k < n$.

Recently, Lescure [15] has constructed a compact complex manifold M with a $\mathbb{C}^*$-action such that the induced action on $H^1(M, \mathcal{O}_M)$ is nontrivial. (See Remark 1 in §2

below, and [16].) The earliest example of a (necessarily non-Kähler) manifold which i homogeneous compact complex with the action of $\mathrm{Aut}_1(M)$ on the Dolbeault cohomo logy of M being nontrivial is due to Kodaira. (See [11].) Akhiezher [1] has proved that i M is compact and is a quotient of a connected complex reductive algebraic group G, the the action of G on $H^q(M, \mathcal{O}_M)$, $q \geq 0$, is trivial. (cf. [2].) When M is a simply connecte compact complex homogeneous space, M fibres over a (generalized) flag variety, wit fibre and structure group a compact complex torus T, that is, M is the total space of a *Tit bundle*. (A locally trivial holomorphic bundle $\pi : M \longrightarrow X$ with M compact and X a fla variety is called a Tits bundle if $\pi' : M \longrightarrow X'$ is any other such bundle, then there exists morphism of varieties $q : X \longrightarrow X'$ such that $\pi' = q \circ \pi$. An equivalent condition is that th fibre of π be a parallelizable complex manifold. See [3].)

We shall prove the following

Theorem 1. *Suppose that M is a connected compact complex homogeneous manifol which fibres over a flag variety $X = G/P$, G being a connected complex semisimple Li group and P a parabolic subgroup, with fibre and structure group a complex toru $T(\cong (S^1)^{2k})$. Then $\mathrm{Aut}_1(M)$ acts trivially on $H^{p,q}(M)$ for all $p, q \geq 0$.*

Our proof of the above theorem makes use of the Borel spectral sequence [8] applied t the bundle $p : M \longrightarrow G/P$ with fibre and structure group a complex torus T. Our proo closely follows the argument in the proof of Lemma 9.3, [8] showing that the grou $\mathrm{Aut}_1(M)$ acts trivially on the Dolbeault cohomology of M when M is a Calabi–Eckman manifold.

We shall introduce the notion of generalized Hopf manifolds which are connected t compact complex homogeneous manifolds and which fibre over a projective variety G/I where G is a complex simple Lie group and P a maximal parabolic subgroup of G, wit fibre and structure group a one dimensional complex torus. Since the second Bet number of a generalized Hopf manifold W vanishes, it is non-Kähler. (See theorem 3 We establish the vanishing of $H^{1,0}(W)$ and use this to compute the Dolbeault cohomolog of a generalized Hopf manifold W. As an example we compute the Dolbeault cohomo logy of $S^1 \times V_{n,2}$. We also compute the Dolbeault cohomology of $V_{m,2} \times V_{n,2}$. In the la section we compute the Picard group of W when W is a generalized Hopf manifold.

As the fundamental group of a generalized Hopf manifold is infinite cyclic (theorem 3 the class of generalized Hopf manifolds does not form a subclass of the class of compa complex homogeneous manifolds which were classified by Wang.

2. Action of $\mathrm{Aut}_1(M)$ on Dolbeault cohomology

Let M be a compact connected complex manifold which is the total space of an analyt bundle $p : M \longrightarrow G/P$, where G is a complex semisimple Lie group and P a parabol subgroup, with fibre and structure group a complex torus T. By [6] one knows that ar automorphism of M preserves this fibre bundle structure and so one obtains homomorphism $\phi : \mathrm{Aut}_1(M) \longrightarrow \mathrm{Aut}_1(G/P)$. Again since G/P is simply connected, usir theorem 3 of [6], one sees that the kernel of this homomorphism is T, and so one obtai an exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Aut}_1(M) \xrightarrow{\phi} \mathrm{Aut}_1(G/P).$$

The automorphism group of the projective variety G/P has been studied by Onishchi Tits and others (see [4],[3]). In particular it is known that if G is simple and P is

parabolic subgroup, then $\mathrm{Aut}_1(G/P)$ is the group $\bar{G} = G/Z(G)$ except in three cases. These exceptional cases are $\mathrm{Aut}_1(SO(2n+1)/P_n) = PSO(2n+2)$, $\mathrm{Aut}_1(\mathrm{Sp}(2n)/P_1) = PSL(2n)$, and $\mathrm{Aut}_1(G_2/P_1) = PSO(7)$. (Here we follow the conventions of [9] for indexing maximal parabolic subgroups of a complex simple Lie group.) In fact using theorem 2.2, Chap. 2, Part II of [10], one sees that if X is any of the above three varieties and G is *any* complex Lie group that acts transitively and effectively on X then G has to be one of the two obvious candidates. (For example, when $X = \mathrm{Sp}(2n)/P_1$, one has $G = P\mathrm{Sp}(2n)$ or $PSL(2n)$.)

Suppose $G = \prod_{1\leq i\leq k} G_i$ with each G_i simple, and $P = \prod_{1\leq i\leq k} P_i$, $P_i \subset G_i$ then $\mathrm{Aut}_1(G/P) = \prod_{1\leq i\leq k} \mathrm{Aut}_1(G_i/P_i)$ [6].

Note that any subgroup of $\mathrm{Aut}_1(G/P)$ that acts transitively has to be *semisimple*. Indeed, let $H \subset \mathrm{Aut}_1(G/P)$ be a complex Lie subgroup that acts transitively on the flag variety G/P. Then $G/P \cong H/Q$ for some closed subgroup $Q \subset H$. Since G/P is simply connected and complete, Q must be a parabolic subgroup of H. If $N = \mathrm{rad}(H)$, the radical of H, then $N \subset Q$. Since N is normal in H, the action of N on $X = H/Q$ has to be trivial. Since $H \subset \mathrm{Aut}(G/P)$ the H action on G/P is effective. It follows that $N = 1$ and so H must be semisimple. In particular, the image of $\phi : \mathrm{Aut}_1(M) \longrightarrow \mathrm{Aut}_1(G/P)$ has to be semisimple because the action of $\mathrm{Aut}_1(M)$ on M is transitive. We have proved

PROPOSITION 2

Let M be a compact complex homogeneous manifold which has the structure of a holomorphic principal bundle over a flag variety with fibre and structure group a compact complex torus T. Then $\mathrm{Aut}_1(M)$ is an extension of a semisimple Lie group by the torus T. □

Proof of Theorem 1. Our proof mimics the argument in the proof of Lemma 9.3, [8]. The automorphism group $\mathrm{Aut}_1(M)$ acts on the Borel spectral sequence for $p : M \longrightarrow G/P$. Restricting this action to the group $T \subset \mathrm{Aut}_1(M)$ we see that T acts trivially on $H^{p,q}(T)$ as the fibre T is Kählerian and the action of T on the base space G/P is in fact the identity automorphism. Hence T acts trivially on the E_2 diagram of the Borel spectral sequence. It follows that it acts trivially on E_∞ as well. Complete reducibility of T implies that it acts trivially on the Dolbeault cohomology M.

The same argument shows that for any connected compact subgroup $K \subset \mathrm{Im}(\phi)$ the group $\phi^{-1}(K)$ acts trivially on the Dolbeault cohomology of M. Indeed, since the T action on the Borel spectral sequence is trivial, the action of the group $\phi^{-1}(K)$ on the Borel spectral sequence is induced by the action of the group K on the cohomology of G/P. Since K acts trivially on the cohomology of G/P, we see that $\phi^{-1}(K)$ acts trivially on the E_2-diagram of the Borel spectral sequence. Again complete reducibility for the compact group $\phi^{-1}(K)$ implies that the action of $\phi^{-1}(K)$ on the Dolbeault cohomology of M is trivial.

We have shown that the homomorphism $\mathrm{Aut}_1(M) \longrightarrow GL(H^{p,q}(M))$ factors through ϕ and hence yields a homomorphism $\rho : \mathrm{Im}(\phi) \longrightarrow GL(H^{p,q}(M))$. The kernel of this homomorphism is a normal subgroup of the semisimple Lie group $\mathrm{Im}(\phi)$ which contains all compact subgroups of $\mathrm{Im}(\phi)$. It follows that kernel of ρ must be the whole of $\mathrm{Im}(\phi)$ and hence the action of $\mathrm{Aut}_1(M)$ on $H^{p,q}(M)$ is trivial. This completes the proof. □

Remark. (1) Lescure [16] has shown that if a complex Lie group G acts holomorphically on a compact complex manifold M, then the induced representation of G in the Dolbeault cohomology $H^{p,q}(M) = H^q(M; \Omega^p_M)$ of M is holomorphic. Furthermore, given a positive

integer q, a semisimple complex Lie group G and a finite dimensional irreducible holomorphic representation R of G, Lescure has constructed a compact complex manifold M with a G action on it such that the induced G action on $H^q(M;\mathcal{O}_M)$ contains R as a sub representation. See § 3 of [16].

(2) The automorphism groups of Calabi–Eckmann manifolds have been computed by A Blanchard [6]. This result is inaccurately quoted in [8]. However, the proof of Lemma 9.3 in [8] is valid with the correct description of the automorphism groups as given in [6].

(3) We do not know if the identity component of the automorphism group of a complex manifold M as in theorem 1 is *algebraic*, assuming that T is an abelian variety. When $M = S^1 \times S^{2n-1}, n > 1$, is the Hopf manifold, one knows that $\mathrm{Aut}_1(M) = GL(n,\mathbb{C})/\Gamma =: G$ where Γ is an infinite cyclic discrete central subgroup of $GL(n,\mathbb{C})$. (cf. [6].) Using the isomorphism $SL(n,\mathbb{C}) \times_Z \mathbb{C}^* = GL(n,\mathbb{C})$, where $Z \cong \mathbb{Z}/n$ is the centre of $SL(n,\mathbb{C})$, and the fact that $\Gamma \cap Z \subset GL(n,\mathbb{C})$ is trivial it follows that $G \cong SL(n,\mathbb{C}) \times_Z E$, where E is the elliptic curve $(\mathbb{C}^*)/\Gamma$. It follows that G is algebraic since it is a quotient of the algebraic group $SL(n,\mathbb{C}) \times E$ by a *finite* central subgroup Z. When M is the Calabi–Eckmann manifold $S^{2n-1} \times S^{2m-1}$, $n, m > 1$, one knows [6] that $\mathrm{Aut}_1(M) = GL(n,\mathbb{C}) \times GL(m,\mathbb{C})/\Gamma$ where $\Gamma \cong \mathbb{C}$ is a certain complex analytic subgroup of the centre $D \cong \mathbb{C}^* \times \mathbb{C}^*$ of $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ such that $\Gamma \cap (SL(n,\mathbb{C}) \times SL(m,\mathbb{C}))$ is trivial. Again one shows that $\mathrm{Aut}_1(M) \cong (SL(n,\mathbb{C}) \times SL(m,\mathbb{C})) \times_Z E$ where E is the elliptic curve D/Γ, and Z is the centre of $SL(n,\mathbb{C}) \times SL(m,\mathbb{C})$. Thus $\mathrm{Aut}_1(M)$ is an algebraic group in this case.

3. Generalized Hopf manifolds

Let G be a simply connected complex semisimple Lie group and P be a maximal parabolic subgroup of G containing a fixed Borel subgroup B. We assume that a maximal torus $A \cong (\mathbb{C}^*)^l$ contained in B is fixed. Note that since P is a maximal parabolic, $X = G/P$ is isomorphic to $\bar{G}/\bar{P}$ where $\bar{G}$ is a suitable simple Lie group. So we may (and we do) assume without loss of generality that G itself is simple. Let E denote the total space of the principal $\mathbb{C}^*$ bundle associated to the line bundle $\mathcal{O}_X(-1)$. Let c be any complex number with $|c| > 1$ and $\phi : E \longrightarrow E$ denote the bundle map $e \mapsto e.c$ for any $e \in E$. Then ϕ generates an action of $\mathbb{Z}$ on E which is free and properly discontinuous. The quotient space W_c (or simply W) is a compact connected complex manifold which fibres over X with fibre and structure group the complex torus $T = \mathbb{C}^*/\langle c \rangle \cong \mathbb{C}^*/\mathbb{Z}$ with periods $\{1, \tau\}$ where $\exp(2\pi i\tau) = c$, $\tau \in \mathbb{C}$. Note that $\mathrm{Im}(\tau) \neq 0$ as $|c| > 1$. When $X = SL(n)/P_1 = \mathbb{P}^{n-1}$, one has $E = \mathbb{C}^n \setminus \{0\}$ and W is the Hopf manifold $S^1 \times S^{2n-1}$. For this reason *we call W a generalized Hopf manifold.*

Let V denote the complex vector space which is the dual of the space of all sections of the line bundle $\mathcal{O}_X(1)$. (Here $X = G/P$.) As is well-known, V is the fundamental representation for G with highest weight ϖ, where ϖ is the fundamental weight stabilized by the Weyl group of P. Choosing a basis for V (consisting of weight vectors), one obtains a G-equivariant embedding $f : X \longrightarrow \mathbb{P}(V)$, under which the identity coset is mapped to $\mathbb{C}e$, where $e \in V$ is a highest weight vector. We regard f as an "inclusion". The tautological bundle over $\mathbb{P}(V)$ restricts to the bundle $\mathcal{O}_X(-1)$ and so E is the inverse image of X under the projection $V_0 = V \setminus \{0\} \longrightarrow \mathbb{P}(V)$.

Let K be a maximal compact subgroup of G. We assume that $K \cap A$ is a maximal torus of K. The group K acts on V by restriction. We put a K-invariant hermitian metric on the

complex vector space V. Let $e \in E$ be a highest weight vector of norm 1, and let L be the stabilizer of e. The orbit of e under K is thus isomorphic to K/L. Since the action of K on X is transitive, the projection $E \longrightarrow X$ restricts to a surjective map $p : K/L \longrightarrow X$. The map p is the projection of the sphere bundle associated to the real 2-plane bundle underlying the complex line bundle $\mathcal{O}_X(-1)$ over X. Alternatively $p : K/L \longrightarrow X$ may also be regarded as the principal S^1 bundle obtained by reducing the structure group $\mathbb{C}^* \cong S^1 \times \mathbb{R}^+$ of $E \longrightarrow X$ to the group S^1, where $\mathbb{R}^+$ is the multiplicative group of positive reals. In particular this S^1-bundle is canonically oriented.

Theorem 3. *With notation as above, L is the semisimple part of the centralizer of a subgroup S of K isomorphic to S^1. In particular the space K/L is 2-connected. The space $W = E/\langle c \rangle$ is diffeomorphic to $K/L \times S^1$.*

Proof. In the Serre spectral sequence of the sphere bundle $p : K/L \longrightarrow X$ with fibre and structure group S^1, the "positive" generator of $H^1(S^1; \mathbb{Z})$ transgresses to the first Chern class of $\mathcal{O}_X(-1)$ which is a generator of $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$. It follows that $H^2(K/L; \mathbb{Z}) = 0 = H^1(K/L; \mathbb{Z})$. From the homotopy exact sequence of the sphere bundle $p : K/L \longrightarrow X$, one sees that the fundamental group of K/L is cyclic. Since both $H^1(K/L; \mathbb{Z})$ and $H^2(K/L; \mathbb{Z})$ vanish, it follows that K/L must be simply connected. This implies that in the homotopy exact sequence the map $\pi_2(X) \longrightarrow \pi_1(S^1) \cong \mathbb{Z}$ must be surjective since $\pi_2(X) \cong \mathbb{Z}$. It follows that $\pi_2(X) \longrightarrow \pi_1(S^1)$ is an isomorphism. Therefore K/L is 2-connected.

Since K is connected and K/L is simply connected we conclude that L must be connected. Since K is semisimple, and K/L is 2-connected, it follows that L is also semisimple. The projective variety X is the quotient of K by a subgroup M which equals the centralizer of a toral subgroup $S \cong S^1$. Since $L \subset M$, it follows that L is contained in the centralizer of S. Clearly L must equal the semisimple part of the centralizer of S since L is semisimple and $M/L \cong S^1$.

Write $c = \exp(2\pi i \tau), \tau = \alpha + i\beta$. The map $\Phi : E \longrightarrow \mathbb{R} \times K/L$ defined by $u \mapsto (t, \exp(-2\pi i \alpha t)u/\|u\|)$ where t is the unique real number such that $\exp(2\pi\beta t) = 1/\|u\|$, is a diffeomorphism. Also it can be verified that Φ is $\mathbb{Z}$ equivariant where the $\mathbb{Z}$ action on $\mathbb{R} \times K/L$ is generated by $(t, v) \mapsto (t + 1, v)$. Hence Φ defines a diffeomorphism of W onto $S^1 \times K/L$. □

Conversely, starting with any semisimple Lie group K and a subgroup $L \subset K$ which is the semisimple part of the centralizer of a circle group $S \cong S^1$, one can show that K/L is the total space of a differentiable fibre bundle with base $K/C(S)$, where $C(S)$ is the centralizer of S, having fibre and structure group S^1. Furthermore, the space $X = K/C(S)$ has the structure of a complex projective variety such that K/L is the sphere bundle associated to $\mathcal{O}_X(-1)$. We omit the details.

Example 1. Let X denote the Grassmannian $G_{k,n} = SL(n, \mathbb{C})/P_k$. In this case $V = \Lambda^k(\mathbb{C}^n)$. We put the usual hermitian metric on $\mathbb{C}^n$ which induces a hermitian metric on V. Fixing a standard basis $\{e_i \mid 1 \le i \le n\}$ on $\mathbb{C}^n$ and taking $A \subset SL(n, \mathbb{C})$ to be the group of diagonal matrices, one sees that $\{e_\alpha \mid \alpha \in I_{n,k}\}$ form a basis for V consisting of weight vectors. Here $I_{n,k}$ denotes the set of all sequences $\alpha = 1 \le i_1 < \cdots < i_k \le n$, and $e_\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$. The basis vector corresponding to the highest weight is $e_1 \wedge \cdots \wedge e_k$. One has $K = SU(n)$, $L = SU(k) \times SU(n-k)$, and S is the identity component of the group of

all transformations t where $t(e_i) = ze_i$, $1 \leq i \leq k$, and $t(e_j) = we_j$, $k < j \leq n$, where $z, w \in U(1)$, $z^k w^{(n-k)} = 1$.

Note that in view of theorem 1, $\mathrm{Aut}_1(W)$ acts trivially on the Dolbeault cohomology of W, where W is any generalized Hopf manifold. Our aim is to compute the Dolbeault cohomology of W. One can apply the Borel spectral sequence for the Tits bundle $\pi : W \longrightarrow X$. The only nontrivial differential to determine is $d_2 : {}^{1,0}E_2^{0,1} \cong H^{1,0}(T) \longrightarrow H^{1,1}(X) \cong {}^{1,1}E_2^{2,0}$. Since both the vector spaces involved here are one-dimensional, we see that this differential is an isomorphism if and only if $H^{1,0}(W) = 0$. We shall establish the following vanishing theorem.

Theorem 4. *Let W be any generalized Hopf manifold. Then $H^{1,0}(W) = 0$.*

Before we can prove the above proposition, we need the following lemma.

Lemma 5. Let U be a dense open subset of W and let $s_1, \cdots s_m$ be global sections of the holomorphic tangent bundle $\mathcal{T}W$ such that for every $u \in U$, the space $\mathcal{T}_u W$ is spanned by the vectors $s_1(u), \ldots, s_m(u)$, and that each s_i vanishes somewhere in W. Then $H^{1,0}(W)=0$.

Proof. Suppose that ϕ is a global holomorphic 1-form on W. We must show that ϕ must be the zero form.

Clearly $\phi(s_i) =: f_i$ are global holomorphic functions on W and hence constants. Since s_i vanishes somewhere, the same is true of f_i for each i. It follows that each f_i is the zero function. Since the s_i span the holomorphic tangent space at each point of U, it follows that $\phi|U = 0$. Since U is dense in W, we conclude that ϕ itself must be the zero form. □

Our aim is to construct holomorphic vector fields on W satisfying the hypotheses of the above lemma. This will be achieved by constructing $\pi_1(W)$-invariant vector fields on the universal cover E of W. Indeed, G acts on $E \subset V$ and so every one parameter subgroup of G gives rise to a holomorphic vector field on E. We choose the one parameter subgroups in a convenient and natural manner and verify that for these vector fields the hypotheses of the above lemma are valid.

Recall that G is a *simply connected* simple complex Lie group and that P is a maximal parabolic subgroup of G that contains a fixed Borel subgroup B, $A \cong (\mathbb{C}^*)^l$ is a Cartan subgroup contained in B. We let R (resp. R^+) denote the set of roots (resp. positive roots) of Lie(G), and let Δ be the set of simple roots. Denote by $\{X_\beta\}_{\beta \in R} \cup \{H_\gamma\}_{\gamma \in \Delta}$ the Chevalley basis of Lie(G) corresponding to our choice of B, A. We denote the Weyl group of G by W. The 1-parameter subgroup of G generated by X_β, $\beta \in R$, will be denoted as G_β. Let ϖ be the fundamental weight of G stabilized by the Weyl group of P, and let α denote the corresponding simple root.

Let $\{e_\sigma\}_{\sigma \in I}$ denote a basis consisting of weight vectors of $V = V_\varpi$, the irreducible representation of G with highest weight ϖ, and let $\{p_\sigma\}$ denote the dual basis of V^*. Let Λ_ϖ denote the set of weights of V_ϖ. We shall assume that the indexing set I for the chosen basis elements contains Λ_ϖ, and that for $\lambda \in \Lambda_\varpi$, the weight of the element e_λ is λ. Note that if ϖ is not a miniscule weight, then $I \supsetneq \Lambda_\varpi$, and there is no uniqueness in labelling the weight vectors by a weight $\lambda \in \Lambda_\varpi$ which occurs in V_ϖ with multiplicity more than one. Later we will make a more precise choice of the labelling of weight vectors.

Note that $E \subset V$, the total space of the principal $\mathbb{C}^*$-bundle associated to G/P, is invariant under the action of G on V. We denote the holomorphic vector field generated

by the action of G_β, by abuse of notation, by X_β. Thus, for $v \in V$, $X_\beta(v) = \frac{d}{dt}|_{t=0}$ $(\exp(tX_\beta)(v))$. It is obvious that X_β is $\mathbb{C}^*$-invariant. In particular, it is invariant under the action of $\langle c \rangle \subset \mathbb{C}^*$, and hence defines a holomorphic tangent vector field on $W = E/\langle c \rangle$ which we again denote by X_β, $\beta \in R$.

Note that $X_\beta(e_\sigma)$ is zero unless $\mathrm{wt}(\sigma) + \beta$ is a weight of V_ϖ. Here $\mathrm{wt}(\sigma)$ stands for the weight of e_σ. If $\mathrm{wt}(\sigma) + \beta$ is indeed a weight of V, then $X_\beta(e_\sigma) = \sum k^\tau_{\beta,\sigma} e_\tau$, where $k^\tau_{\beta,\sigma} \in \mathbb{C}$ and the sum runs over all τ such that $\mathrm{wt}(\tau) = \mathrm{wt}(\sigma) + \beta$. It is known that, for $\beta \in R^+$, among all $\tau \in I$ for which $\mathrm{wt}(\tau) = \varpi - \beta$, there is at least one τ such that $k^\tau_{-\beta,\omega}$ is nonzero. See Theorem 6.1, Chap. XVII, [12]. By relabelling the $(\varpi - \beta)$-weight vectors if necessary, we may assume $k^\tau_{-\beta,\varpi} \neq 0$ for $\tau = \varpi - \beta$. A straightforward verification leads to the following expression for the vector field X_β on E in terms of $\{(\partial/\partial p_\sigma)\}$:

$$X_\beta = \sum_\tau \left(\sum_\sigma p_\sigma k^\tau_{\beta,\sigma} \right) (\partial/\partial p_\tau).$$

Let $S \subset R^+$ be the set of complementary roots of G with respect to P. Note that $\#S = \dim G/P$. It is well-known that $\varpi - \beta$ is a weight of $V = V_\varpi$ for each $\beta \in S$.

We put a partial order on the indexing set I of the weight vectors of V by declaring that $\sigma \leq \tau$ if either $\sigma = \tau$ or $\mathrm{wt}(\sigma) < \mathrm{wt}(\tau)$.

Lemma 6. *With the above notation, the set of vector fields* $\{X_{-\beta} \mid \beta \in S\} \cup \{X_\alpha\}$, *span the holomorphic tangent bundle* $\mathcal{T}U$ *of* $U = \{[v] \in W | v \in E,\ p_\varpi(v) \neq 0, p_{\varpi-\alpha}(v) \neq 0\} \subset W$.

Proof. Note that, on E,

$$X_{-\beta} = p_\varpi k^{\varpi-\beta}_{-\beta,\varpi}(\partial/\partial p_{\varpi-\beta}) + \text{other terms},$$

where the "other terms" on the right involve $(\partial/\partial p_\tau)$ with τ either incomparable or strictly less than $\varpi - \beta$ in the partial order. Similarly,

$$X_\alpha = p_{\varpi-\alpha} k^\varpi_{\alpha,\varpi-\alpha}(\partial/\partial p_\varpi) + \text{lower terms}.$$

Recall that $k^{\varpi-\beta}_{-\beta,\varpi} \neq 0$, for $\beta \in S$, and that $k^\varpi_{\alpha,\varpi-\alpha} \neq 0$. This shows that, for $v \in E$ with $p_\varpi(v) \neq 0$ and $p_{\varpi-\alpha}(v) \neq 0$ the tangent vectors $X_{-\beta}(v)$, $\beta \in S$, $X_\alpha(v)$ are related by means of an upper triangular matrix with nonzero diagonal entries to the vectors $(\partial/\partial p_{\varpi-\beta})|_v$, $\beta \in S$, $(\partial/\partial p_\varpi)|_v \in \mathcal{T}_v V$. Therefore $X_{-\beta}(v)$, $\beta \in S$, $X_\alpha(v)$ are linearly independent. Since $\dim_{\mathbb{C}} E = \dim G/P + 1 = \#S + 1$, they span $\mathcal{T}_v E$. Since the vector fields involved are invariant under the action of $\langle c \rangle$ on E, it follows that $\{X_{-\beta} \mid \beta \in S\} \cup \{X_\alpha\}$, regarded as vector fields on W, span the holomorphic tangent bundle of the dense open set $U \subset W$, and the lemma follows. □

Proof of Theorem 4. In view of lemmas 5 and 6, we need only show that each of the vector fields $X_{-\beta}, \beta \in S, X_\alpha$, vanish somewhere on W. One merely notes that $X_{-\beta}([e_\lambda]) = 0, \beta \in S$, where $\lambda = w_0(\varpi)$, w_0 being the longest element of the Weyl group of G, so that $e_\lambda \in I$ is the lowest weight vector. Also, $X_\alpha([e_\varpi]) = 0$. This completes the proof. □

We shall now compute the Dolbeault cohomology of generalized Hopf manifolds W using Borel spectral sequence for the Tits bundle $\pi : W \longrightarrow X$ with fibre and structure group $T = \mathbb{C}^*/\langle c \rangle$. Let $R = H^*(X; \mathbb{C}) = \oplus_{p \geq 0} H^{p,p}(X; \mathbb{C})$, and let A denote the annihilator ideal of $y \in H^2(X; \mathbb{Z}) \subset H^{1,1}(X; \mathbb{C}) \cong \mathbb{C}$. We shall denote the ring $R/\langle y \rangle$ by S. Note that S

is a graded ring coming from the grading on the cohomology algebra of X, and that A is a graded S-module. We denote by $\tilde{A}$ the graded S-algebra which is just $S \oplus A$ as the underlying S-module and the multiplication in $\tilde{A}$ is defined by setting $u.v = 0$ for $u, v \in \tilde{A}$; the gradation on $\tilde{A}$ is as follows: if $u \in A$ is of type (p, p) then its type in $\tilde{A}$ is declared to be $(p+1, p)$. We are now ready to state our theorem.

Theorem 7. *We keep the notation of the previous paragraph. Let W be a generalized Hopf manifold. Then as a bigraded algebra the Dolbeault cohomology of W is isomorphic to the algebra $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$.*

Proof. Since by theorem 4, $H^{1,0}(W) = 0$, the generator $x_{1,0} \in H^{1,0}(T, \mathbb{C}) \cong \mathbb{C}$ transgresses to zy for some *nonzero* complex number z. This shows that in the E_2 diagram, everything in the ideal generated by y is in the image of d_2 and hence $E_\infty^{*,0} = H^*(X; \mathbb{C})/\langle y\rangle =: S \subset H^*_{\bar\partial}(W)$. Furthermore each class $u \otimes x_{1,0}$ with $u \in H^*(X; \mathbb{C})$ such that $uy = 0$ is a permanent cocycle and yields a class in the Dolbeault cohomology of W. By degree consideration the generator $x_{0,1} \in H^{0,1}(T; \mathbb{C}) \cong \mathbb{C}$ transgresses to zero. By the multiplicative property of the spectral sequence, we see that the subalgebra of $H^*_{\bar\partial}(W)$ generated over S by the image of the classes $u \otimes x_{1,0}$ in E_2 as u varies in the annihilator ideal A of y is isomorphic to $\tilde{A}$. Since the Dolbeault cohomology of the fibre is isomorphic to the exterior algebra over $\mathbb{C}$ on the classes $x_{0,1}, x_{1,0}$, it follows that $H^*_{\bar\partial}(W)$ is isomorphic to $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$. □

Example 2. Let $X = Q_n = SO(n, \mathbb{C})/P_1 \cong SO(n, \mathbb{R})/(SO(2, \mathbb{R}) \times SO(n-2, \mathbb{R}))$, $n \geq 5$ which is the complex quadric $\{[z] \in \mathbb{P}^{n-1} | q(z) = \sum_{1 \leq i \leq n} z_i^2 = 0\}$. In this case the $W = E/\langle c\rangle$ is the product $S^1 \times V_{n,2}$, where $V_{n,2}$ is the real Stiefel manifold $SO(n, \mathbb{R})/\ SO(n-2, \mathbb{R})$. Let $\pi : W_n \longrightarrow Q_n$ denote the projection of the associated Tits bundle.

Recall that the cohomology algebra $H^*(Q_n; \mathbb{C})$ is isomorphic to $\mathbb{C}[x, e_t]/\sim$ where $t = [(n-2)/2]$ and the relations are given by $x^{n-1} = 0, e_t = 0$ if n is odd and $e_t^2 = (-1)^t x^{n-2}$ if $n = 2t+2$ is even; x and e_t are of type $(1, 1)$ and (t, t). One can take x to be the first Chern class of $\mathcal{O}_{Q_n}(-1)$ and e_t to be the Euler class of a certain oriented real $n-2$-plane bundle which is complementary to the real 2-plane bundle underlying the complex line bundle $\mathcal{O}_{Q_n}(-1)$; we will refer to x and e_t as the canonical generators of $H^*(Q_n; \mathbb{C})$.

Case (i). Let n be odd. In this case $H^*(Q_n; \mathbb{C})$ is generated by $y \in H^2(Q_n; \mathbb{C}) \cong \mathbb{C}$, and so in the notation of the above theorem, one has $S \cong \mathbb{C}$, $A = H^{2d}(Q_n; \mathbb{C}) = \mathbb{C}y^d$, where $d = \dim_{\mathbb{C}} Q_n = n-2$. Hence $H^*_{\bar\partial}(W_n; \mathbb{C}) = \mathbb{C}[x, u]/\sim$ where the relations are $x^2 = 0$, $u^2 = 0$, $xu = -ux$; x and u are of type $(0, 1)$ and $(d+1, d)$ respectively.

Case (ii). Let $n = 2t+2 > 5$ be even. Then a straightforward computation in $H^*(Q_n; \mathbb{C})$ shows that $S = \mathbb{C}[e_t]$, where $e_t \in H^{n-2}(Q_n; \mathbb{C})$, and $e_t^2 = 0$ in S. Also, $A = \mathbb{C}e_t \oplus \mathbb{C}e_t^2$ and $\tilde{A} = S \oplus A = S[u]/\langle u^2\rangle$, where u is of type $(t+1, t)$. Hence $H^*_{\bar\partial}(W_n; \mathbb{C}) = S[x, u]/\sim = \mathbb{C}[x, e_t, u]/\sim$ where the relations are $x^2 = 0 = u^2 = e_t^2$, $xe_t = e_t x$, $ue_t = e_t u$, $xu = -ux$ and x, e_t, u are of type $(0, 1), (t, t), (t+1, t)$ respectively.

4. Dolbeault cohomology of $V_{m,2} \times V_{n,2}$

In this section we compute the Dolbeault cohomology of the space $W_{m,n} = V_{m,2} \times V_{n,2} = (SO(m, \mathbb{R})/SO(m-2, \mathbb{R})) \times (SO(n, \mathbb{R})/SO(n-2, \mathbb{R})) = (SO(m, \mathbb{R}) \times SO(n, \mathbb{R}))/(SO(m-2, \mathbb{R}) \times SO(n-2, \mathbb{R}))$. Let $E_{m,n} = E_m \times E_n$ denote the total space of the principal

$\mathbb{C}^* \times \mathbb{C}^*$ bundle associated to the holomorphic 2-plane bundle $\mathcal{O}_{Q_m}(-1) \oplus \mathcal{O}_{Q_n}(-1)$ on $Q_m \times Q_n$. Indeed $E_{m,n}$ may be identified with the subspace $\{(z,w) \in \mathbb{C}^m \times \mathbb{C}^n \mid \sum_{1\le i\le m} z_i^2 = 0 = \sum_{1\le j\le n} w_j^2\} \setminus \{0\}$. The action of $\mathbb{C}^* \times \mathbb{C}^*$ on $E_{m,n}$ is then given by $(\lambda,\mu).(z,w) = (\lambda.z, \mu.w)$. Let τ be any complex number such that $\mathrm{Im}(\tau) \neq 0$. Then the map $\alpha \mapsto (\exp(2\pi i\tau\alpha), \exp(2\pi i\alpha))$ is an analytic imbedding of $\mathbb{C}$ into $\mathbb{C}^* \times \mathbb{C}^*$. The group $\mathbb{C}$ acts on $E_{m,n}$ via this imbedding. The quotient of $E_{m,n}$ by this action of $\mathbb{C}$ is nothing but $W_{m,n}$ which gives a complex structure on $W_{m,n}$. The complex structure so obtained depends on τ. One sees that the map $\pi : W_{m,n} \longrightarrow Q_m \times Q_n$ is the projection of an analytic bundle with fibre and structure group the one dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^*/\mathbb{C}$. It is straightforward to see that T is isomorphic to the complex torus with periods $\{1,\tau\}$. Also, the restriction of the T-bundle $W_{m,n} \longrightarrow Q_m \times Q_n$ to $Q_m \times \{[w]\}$, $[w] \in Q_n$ is isomorphic to the T-bundle $W_m \longrightarrow Q_m$.

We denote by y_1, e_s and y_2, e_t the canonical generators of $H^*(Q_m \times Q_n; \mathbb{C}) \cong H^*(Q_m; \mathbb{C}) \otimes H^*(Q_n; \mathbb{C})$.

Lemma 8. *In the Borel spectral sequence of the T-bundle $\pi : W_{m,n} \longrightarrow Q_m \times Q_n$, the generator $x_{1,0} \in H^{1,0}(T)$ transgresses to $u = \alpha y_1 + \beta y_2$, with $\alpha, \beta \in \mathbb{C}^*$. In particular $H^{1,0}(W_{m,n}) = 0$.*

Proof. We need only show that α, β are nonzero. The inclusion map j of the T-bundle $W_m \longrightarrow Q_m$ into $W_{m,n} \longrightarrow Q_m \times Q_n$ induces a map j^* between the Borel spectral sequences associated to the two bundles. Using naturality of the Borel spectral sequence and the fact that transgression commutes with j^* we see that, in view of Example 2, we must have $\alpha \neq 0$. Similarly $\beta \neq 0$. □

Note. The numbers α and β are determined by τ.

Let $5 \le m \le n$, $s = [(m-2)/2]$, $t = [(n-2)/2]$. Let $u_1 = \alpha y_1$, and $u_2 = \beta y_2$, so that $u = u_1 + u_2 \in H^2(Q_m \times Q_n; \mathbb{C})$. Denote by S the ring $H^*(Q_m \times Q_n; \mathbb{C})/\langle u\rangle$. A simple calculation shows that $S \cong \mathbb{C}[u_1]/\langle u_1^{m-1}\rangle[e_s, e_t]/\sim$ where the relations are as follows: (i) $e_s = 0$ (resp. $e_t = 0$) if m (resp. n) is odd, (ii) $u_1 e_s = 0$, $u_1 e_t = 0$, (iii) $e_s^2 e_t^2 = 0$, (iv) $e_s^2 = (-1)^s \alpha^{2-m} u_1^{m-2}$ and, (v) $e_t^2 = 0$ if $m < n$, and $e_t^2 = (-1)^m \alpha^{m-2}\beta^{2-m} e_s^2$ if $m = n$. The elements e_s and e_t are of type (s,s) and (t,t) respectively. The annihilator ideal $A = \mathrm{ann}(u) \subset H^*(Q_m \times Q_n; \mathbb{C})$ is generated, as an algebra over $\mathbb{C}$, by $e_s e_t, e_t u_1^{m-2}, e_s u_2^{n-2}$, $v_0, v_1 := u_1 v_0, \ldots, v_{m-2} := u_1^{m-2} v_0$ where $v_0 = u_2^{n-2} - u_2^{n-3} u_1 + \cdots + (-1)^{m-2} u_2^{n-m} u_1^{m-2}$. Note that $v_0^2 = 0$ if $m < n$ and $v_0^2 = (-1)^m 2t v_{m-2} = (-1)^m 2t u_1^{m-2} u_2^{n-2}$ if $m = n$; $v_i v_j = 0$, if $0 \le i \le j$, $j \ge 1$; and $e_s^2 e_t^2 = (-1)^{s+t} \alpha^{m-2}\beta^{n-2} u_1^{m-2} u_2^{n-2} = (-1)^{s+t}\alpha^{m-2}\beta^{n-2} u_1^{m-2} v_0$. Furthermore, as an S-module the relations in A can be easily obtained from the relations in $H^*(Q_m \times Q_n; \mathbb{C})$. For example, $u_1 e_s e_t = 0$, $e_s.(e_s e_t) = (-1)^s \alpha^{2-m} e_t u_1^{m-2}$, $e_t.v_i = 0$ unless $m = n$ and $i = 0$, in which case one has $e_t.v_0 = (-1)^m e_t u_1^{m-2}$, $u_1.v_i = v_{i+1}$, $0 \le i < m-2$, $u_1.v_{m-2} = 0$.

We define $\tilde{A}$ to be the graded S-algebra $S \oplus A$ where the multiplication is defined by setting $a.b = 0$ for all $a, b \in A$. The gradation in $\tilde{A}$ is got by declaring the bidegree of an element of type (r,r) in A to be $(r+1, r)$. The proof of the following theorem is completely analogous to that of theorem 7 and is therefore omitted.

Theorem 9. *Let $5 \le m \le n$, and let $W_{m,n} = V_{m,2} \times V_{n,2}$. With the above notation the Dolbeault cohomology of $W_{m,n}$ is isomorphic as an algebra to $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$, where x is of type $(0,1)$.* □

Remark. (1). The above method can be used to compute the Dolbeault cohomology of the spaces $S^{2m-1} \times V_{n,2}$, $m \geq 2$ and $V_{n,4}$.

(2). Let $X = G/P$ where G is a complex simple algebraic group and P is a maximal parabolic subgroup. The annihilator ideal of y, the generator of $H^2(X, \mathbb{C}) \cong \mathbb{C}$, is the same as the ideal generated by the cohomology classes defined by *primitive* algebraic cycles. In particular, $\mathrm{ann}(y) \cap H^k(X, \mathbb{C})$ is zero for $k < \dim_{\mathbb{C}} X$.

5. Picard groups

We compute here the Picard groups of generalized Hopf manifolds. Picard groups of Hopf manifolds $S^1 \times S^{2n-1}$ have been computed by Ise [14]. Our main result of this section is

Theorem 10. *Let W be a generalized Hopf manifold. Then* $\mathrm{Pic}^0(W) = \mathrm{Pic}(W) \cong \mathbb{C}^*$. *Every line bundle over W arises from a representation of the fundamental group $\pi_1(W) \cong \mathbb{Z}$ and hence admits an integrable holomorphic connection.*

Proof. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_W^* \longrightarrow 1.$$

In the induced long exact sequence in cohomology one has

$$H^1(W, \mathbb{Z}) \longrightarrow H^1(W, \mathcal{O}_W) \longrightarrow H^1(W, \mathcal{O}_W^*) \longrightarrow H^2(W, \mathbb{Z}).$$

The left most homomorphism is easily seen to be $1-1$, and one has isomorphisms $H^1(W, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(W, \mathbb{Z}) \cong 0$ as can be seen using theorem 3. As $H^2(W, \mathbb{Z}) = 0$, we conclude that $\mathrm{Pic}(W) = \mathrm{Pic}^0(W)$. Also, $H^1(W, \mathcal{O}_W) = H^{0,1}(W) \cong \mathbb{C}$ by using the Borel spectral sequence. By definition one has $H^1(W, \mathcal{O}_W^*) \cong \mathrm{Pic}(W)$. It follows that $\mathrm{Pic}(W) \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$.

Choose a base point in $w \in W$ and fix a generator $t \in \pi_1(W) \cong \pi_1(S^1) \cong \mathbb{Z}$. Let $\alpha \in \mathbb{C}^*$ and consider the homomorphism $\pi_1(W) \longrightarrow \mathbb{C}^*$ which maps t to α. Then $L_\alpha := E \times_{\mathbb{Z}} \mathbb{C}$ is the total space of a line bundle over $W = E/\mathbb{Z}$, where E is the universal cover of W and $\mathbb{Z}$ acts on $\mathbb{C}$ by $n.z = \alpha^n.z$. As is well-known, this leads to an analytic homomorphism of groups $\phi : \mathbb{C}^* \cong \mathrm{Hom}(\pi_1(W), \mathbb{C}^*) \longrightarrow \mathrm{Pic}(W) \cong \mathbb{C}^*$. Let T be the fibre of the Tits bundle associated to W that contains $w \in W$. Choosing α to be in the unit circle $\{z \in \mathbb{C}^* \mid |z| = 1\}$, $\alpha \neq 1$, one sees that $\phi(\alpha)$ restricts to a nontrivial line bundle on T (see §2 of [18]). Hence ϕ is not the trivial homomorphism. By dimension consideration it follows that ϕ is surjective. As is well-known (see [5]), it follows that every line bundle over W admits an integrable holomorphic connection.

Remark. Line bundles over simply connected compact complex homogeneous manifolds have been studied by Ise [13].

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Torus quotients of homogeneous spaces – II

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Abstract. We classify the homogeneous spaces X for which there is a T linearised ample line bundle L on X such that $X_T^{ss}(L)=X_T^s(L)$.

Keywords. Torus; parabolic subgroups; semi-stable points.

1. Introduction

Let k be an algebraically closed field. Let G be a semisimple algebraic group over k, T a maximal torus of G, B a Borel subgroup of G containing T, Q a parabolic subgroup of G containing B and N the normaliser of T in G.

Consider the canonical actions of T and N on G/Q. We had seen in [SSK], that the study of these actions on G/Q when $G = SL_n(k)$, Q is the maximal parabolic subgroup of G associated to the simple root α_2, is related to invariant theory of binary quantics. In the case of $G=SL_n$ and $Q=P_r$ the maximal parabolic subgroup of G associated to the simple root α_r and L_r the line bundle associated to the fundamental weight ϖ_r, we have proved

$$(G/P_r)_T^{ss}(L_r)=(G/P_r)_T^s(L_r)$$

if and only if r and n are coprime (see Theorem 3.3 [SSK]). In this paper, we give a complete generalisation of this result for any semisimple algebraic group G and any parabolic subgroup Q of G containing B. It is easy to reduce to the case when G is a **simple** algebraic group. For a proof, see § 3.

We divide the problem into two cases:

(i) G is a **simple** algebraic group of type A_l and
(ii) G is a **simple** algebraic group not of type A_l.

When G is a **simple** algebraic group of type A_l, the result is the following:

(a) Let $G = SL_n$, T a maximal torus of G, B a Borel subgroup of G containing T, and $Q =\cap_{r\in I}P_r$ be a parabolic subgroup of G containing B. Then, there is a line bundle L on $X=G/Q$ such that $X_T^{ss}(L)= X_T^s(L)$ if and only if the least common multiple of $\{\frac{n}{(r,n)} : r \in I\}$ is n.

When G is a **simple** algebraic group of type other than A_l, the result is the following :

(b) Let G be a **simple** algebraic group of type other than A_l, T a maximal torus of G, B a Borel subgroup of G containing T, Q a parabolic subgroup of G containing B and let $X = G/Q$. Then, there is a line bundle L on X such that $X_T^{ss}(L)= X_T^s(L)$ if and only if $Q = B$. For more details, see Theorem 3.1 and Theorem 3.5.

The layout of this paper is as follows:

The second section consists of notations, conventions and basic theorems. In § 3, we prove results (a) and (b).

2. Notations and conventions

Let k be an algebraically closed field. Let G be a semisimple algebraic group over k, T a maximal torus of G, B a Borel subgroup of G containing T and Q a parabolic subgroup of G containing B.

Let N be the normaliser of T in G, $W = N/T$, the Weyl group . Let Φ be the set of all roots. Let Φ^+ and Φ^- be the sets of positive and negative roots (with respect to B) respectively. Let $\Delta = \{\alpha_1, \alpha_2, \ldots \alpha_l\}$ be the set of all simple roots, where l is the rank of G. Let s_i be the simple reflection with respect to α_i.

Let $\{\varpi_i : i \in \{1, 2 \ldots l\}\}$ be the fundamental weights with respect to B. We denote the maximal parabolic subgroup of G associated to the simple root α_r by P_r.

Let $X(T)$ and $\Gamma(T)$ denote the sets of all characters and one parameter subgroup of T, respectively. Denote the canonical bilinear form by

$$\langle\ ,\ \rangle : X(T) \times \Gamma(T) \longrightarrow \mathbb{Z}.$$

Let $E = \Gamma(T) \bigotimes \mathbb{R}$,

$$\overline{C(B)} = \{\lambda \in E : \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Since the pairing $\langle ., . \rangle$ is nondegenerate, there is a finite set of one parameter subgroups $\{\lambda_i : i \in \{1, 2 \cdots l\}\}$ such that

$$\langle \alpha_i, \lambda_j \rangle = n\delta_{i,j},$$

where n is the order of the group of characters of the centre of the group G.

For a general semisimple algebraic group G, let $w_0(G)$ denote the longest element of the Weyl group of G.

Denote the global sections of a line bundle L over X by $H^0(X, L)$, and denote by $p(id)$ a lowest weight vector of the G module $H^0(G/B, L)$ where L is a line bundle associated to a dominant weight. Let L_r denote the line bundle associated to the fundamental weight ϖ_r. For any dominant weight λ, we denote the line bundle on G/Q associated to λ by L_λ. Let $X(\tau)$ denote the Schubert variety defined by τ in W.

We use the notations for semistable points, stable points, numerical functions, etc. as in [GIT].

We recall some important results of [CSS I] here which will be used in § 3.

Lemma 2.1. [CSS I] *Let G be a semisimple algebraic group, T a maximal torus of G, B a Borel subgroup of G containing T, and $\overline{C(B)}$ be as in this section.*

(a). *Let L be the line bundle defined by the character $\chi \in X(T)$. Then if $x \in G/B$ is represented by bwB, $b \in B$, $w \in W$ represented by an element of N in the Bruhat decomposition of G and λ is a 1-PS of T which lies in $\overline{C(B)}$, we have*

$$\mu^L(x, \lambda) = -\langle w(\chi), \lambda \rangle.$$

(b). *Given any set S of finite number of non-trivial 1-PS λ of T, there is an ample line bundle L on G/B such that*

$$\mu^L(x, \lambda) \neq 0$$

for all $x \in G/B$, $\lambda \in S$.

3. Torus quotients

In this section, we classify all the parabolic subgroups Q of general semisimple algebraic groups G containing B for which there is a G linearised line bundle L on G/Q such that

$$(G/Q)^{ss}_T(L) = (G/Q)^s_T(L), \tag{3.0.1}$$

when set of semistable points $(G/Q)^{ss}_T(L)$ is nonempty.

We first reduce the classification problem (3.0.1) to the case when G is a simply connected simple algebraic group. Next, we deal the problem (3.0.1) in two cases: that G is a simple group of type A_l and G is (simple) not of type A_l. So, now we prove the reduction to the case when G is simple.

Proof of the reduction to the case G is simple. Let G be any semisimple algebraic group, let T, B, Q as in § 2. Let $\pi : \widehat{G} \longrightarrow G$ be a simply connected covering of G and let $\widehat{T}$, $\widehat{B}$ and $\widehat{Q}$ be the pullbacks of T, B and Q respectively in $\widehat{G}$.

Now, consider the action of $\widehat{T}$ on G/Q through the homomorphism $\pi|\widehat{T} : \widehat{T} \longrightarrow T$. Since the homomorphism $\pi|\widehat{T} : \widehat{T} \longrightarrow T$ is surjective, for any G linearised line bundle L on G/Q

$$(G/Q)^{ss}_{\widehat{T}}(L) = (G/Q)^{ss}_T(L). \tag{3.0.2}$$

Once again by the definition of the action of $\widehat{T}$ on G/Q the $\widehat{T}$ orbit of any point $x \in G/Q$ is just the T orbit of x since the homomorphism $\pi|\widehat{T} : \widehat{T} \longrightarrow T$ is surjective.

(3.0.3) Hence the $\widehat{T}$ orbit of a point $x \in G/Q$ is closed in $(G/Q)^{ss}_{\widehat{T}}(L) = (G/Q)^{ss}_T(L)$ if and only if the corresponding T orbit of x is closed in $(G/Q)^{ss}_T(L)$.

(3.0.4) Also, since the kernel of the homomorphism $\pi|\widehat{T} : \widehat{T} \longrightarrow T$ is finite, the isotropy of $x \in G/Q$ in $\widehat{T} = (\pi|\widehat{T})^{-1}$(*isotropy of x in T*) is finite if and only if the isotropy of x in T is finite.

Therefore, from the above observations (3.0.2), (3.0.3) and (3.0.4) for any G linearised line bundle L on G/Q, we have,

$$(G/Q)^s_{\widehat{T}}(L) = (G/Q)^s_T(L). \tag{3.0.5}$$

Now the morphism $\pi_Q : \widehat{G}/\widehat{Q} \longrightarrow G/Q$ induced by $\pi : \widehat{G} \longrightarrow G$ is a $\widehat{T}$ equivariant isomorphism (where the action of $\widehat{T}$ on G/Q is the one induced by the homomorphism $\pi|\widehat{T}$).

(3.0.6) Therefore for any T linearised line bundle L on G/Q, $\pi^*_Q(L)$ is a $\widehat{T}$ linearised line bundle on $\widehat{G}/\widehat{Q}$.

Hence, from the above observations (3.0.2), (3.0.5) and (3.0.6) for our problem (3.0.1), we can assume G is simply connected. It is well known that G (being simply connected) is isomorphic to $\Pi^m_{i=1} G_i$ with each G_i simply connected and simple group (p.17 [C]). We fix such an isomorphism and **we do not carry this isomorphism throughout the argument to avoid cumbersome notations**, and so let us take $G = \Pi^m_{i=1} G_i$, $T = \Pi^m_{i=1} T_i$ and $Q = \Pi^m_{i=1} Q_i$, where each T_i is a maximal torus of G_i and each Q_i is a parabolic subgroup of G_i. Hence the morphism $\phi : G/Q \longrightarrow \Pi_i G_i/Q_i$ defined by

$$\phi((g_1, g_2, \cdots g_m)Q) = (g_1Q_1, g_2Q_2, \cdots g_mQ_m) \tag{3.0.7}$$

is a $\{1\} \times \{1\} \times \cdots T_i \times \{1\} \cdots \times \{1\}$ equivariant isomorphism, where T_i is in the i^{th} place.

(3.0.8) Also, the homomorphism $\psi^\wedge : \Pi^m_{i=1} \Gamma(T_i) \longrightarrow \Gamma(T)$ defined by $\psi^\wedge((\lambda^1, \lambda^2, \ldots, \lambda^m)) = \lambda$, where $\lambda(t) = (\lambda^1(t), \lambda^2(t), \ldots, \lambda^m(t))$ *for* $t \in G_m$ is an isomorphism. Let ψ be the inverse of ϕ and let $\psi_i : G/Q \longrightarrow G_i/Q_i$ be the canonical projection.

Now, from the observations (3.0.7) and (3.0.8), let $M = \sum_{i=1}^{m} \psi_i^*(M_i)$ (with each M_i a G_i linearised line bundle on G_i/Q_i) be an arbitrary G linearised line bundle on G/Q, let $\lambda = \psi((\lambda^1, \lambda^2, \ldots \lambda^m))$ (with each $\lambda^i \in \Gamma(T_i)$) be an arbitrary one parameter subgroup of T and let $x \in G/Q$ be arbitrary. Hence, by Lemma 2.1 of [CSS I], we have

$$\mu^M(x, \lambda) = \sum_{i=1}^{m} \mu^{M_i}(\psi_i(x), \lambda^i),$$

and hence for a fixed M and x (but arbitrary) $\mu^M(x, \lambda) \geq 0$ for all $\lambda \in \Gamma(T)$ if and only if $\mu^{M_i}(\psi_i(x), \lambda) \geq 0$ for all $\lambda \in \Gamma(T_i)$ and for all $i \in \{1, 2 \cdots m\}$. Therefore by Theorem 2.1 of [GIT], a point $x \in (G/Q)_T^{ss}(M)$ if and only if $\psi_i(x) \in (G_i/Q_i)_{T_i}^{ss}(M_i)$. Also, by a similar argument, a point $x \in (G/Q)_T^{s}(M)$ if and only if $\psi_i(x) \in (G_i/Q_i)_{T_i}^{s}(M_i)$.

Thus, the problem (3.0.1) is reduced to the case of a simply connected simple algebraic group.

We now solve problem (3.0.1) for simple group G of type A_l, that is $G = SL_n, n = l + 1$. We denote the greatest common divisor of two positive integers r and n by (r, n).

We prove the following theorem.

Theorem 3.1. *Let $G = SL_n$, T a maximal torus of G, B a Borel subgroup of G containing T. Let $Q = \cap_{r \in I} P_r$ be a parabolic subgroup of G containing B. Then, there is a line bundle L on $X = G/Q$, such that $X^{ss}(L) = X^s(L)$ if and only if the least common multiple of $\{\frac{n}{(r,n)} : r \in I\}$ is n.*

Proof. Let $X = G/Q$ and let $\pi_r : X \longrightarrow G/P_r$ be the canonical projection. First, let us assume that the least common multiple of $\{\frac{n}{(r,n)} : r \in I\}$ is n.

Now, by Lemma 2.1 of [CSS I]

$$\mu^{\pi_r^*(L_r)}(x, \lambda) = \mu^{L_r}(\pi_r(x), \lambda)$$

for all $x \in X$ and $\lambda \in \Gamma(T)$. Therefore, by line 22 of p. 5 of [SSK], we have

$$\mu^{\pi_r^*(L_r)}(x, \lambda_s) = -[r(n - s) - n.m_s] \tag{3.1.1}$$

for $r \leq s$ and from line 29 of p. 5 of [SSK], we have

$$\mu^{\pi_r^*(L_r)}(x, \lambda_s) = -[(n - r)s - n.m_s] \tag{3.1.2}$$

for $r > s$. Now, since the least common multiple of the integers $\{\frac{n}{(r,n)} : r \in I\}$ is n, for any $s \in \{1, 2, \ldots, l\}$ there is a $r \in I$ such that $\frac{n}{(r,n)}$ does not divide s. Therefore by the observations (3.1.1) and (3.1.2) for every $s \in \{1, 2, \ldots, l\}$ there is a $r \in I$ such that $\mu^{\pi_r^*(L_r)}(x, \lambda_s) \neq 0$ for every $x \in X$. Since $\{\mu^{\pi_r^*(L_r)}(x, \lambda_s) : r \in I, x \in X, s \in \{1, 2, \ldots, l\}\}$ is a finite set of nonzero integers, we can choose a set of positive integers $\{c_r : r \in I\}$ such that

$$\mu^{\sum_{r \in I} c_r \pi_r^*(L_r)}(x, \lambda_s) \neq 0$$

for all $s \in \{1, 2, \ldots, l\}$ and $x \in X$. Let $L = \sum_{r \in I} c_r \pi_r^*(L_r)$. By an argument as in page 4 of [SSK], we have

$$X^{ss}(L) = X^s(L).$$

Conversely, let the least common multiple of the integers $\{\frac{n}{(r,n)} : r \in I\}$ be less than n. Let q be the least common multiple of $\{\frac{n}{(r,n)} : r \in I\}$. Since q divides n and $\frac{n}{(r,n)}$ divides q

for every $r \in I$, $\frac{n}{q}$ is an integer dividing r. Hence, write $n = q.d$ and $r = m_r.d$ with $d, m_r \in \mathbb{N}$. Since $q < n$, we must have $d > 1$. Now, since d is a common divisor of all $r \in I$ and n, there is a $w \in W$ (independent of $r \in I$) as in the proof of the Theorem 3.3 (p. 6 of [SSK]) and a point $x \in BwQ/Q$ such that

$$\pi_r(x) \in (G/P_r)^{ss}(L_r) \text{ and } \mu^{L_r}(\pi_r(x), \lambda_q) = 0.$$

Now, let $L = \sum_{r \in I} a_r \pi_r^*(L_r)$ with $a_r \in \mathbb{Z}_{\geq 0}$ for all $r \in I$. Since the point $\pi_r(x) \in (G/P_r)^{ss}$ (L_r), $\mu^{L_r}(\pi_r(x), \lambda) \geq 0$ for all one parameter subgroup λ of T and hence

$$\mu^L(x, \lambda) = \sum_{r \in I} a_r \mu^{\pi_r^*(L_r)}(x, \lambda) \geq 0$$

for all one parameter subgroups of T. Therefore, by Theorem 2.1 of [GIT], we see that $x \in X^{ss}(L)$. Now, by Lemma 2.1 of [CSS I] we have

$$\mu^L(x, \lambda_q) = \sum_{r \in I} a_r \mu^{\pi_r^*(L_r)}(x, \lambda_q) = \sum_{r \in I} 0 = 0.$$

Thus, by Theorem 2.1 of [GIT]

$$x \in X^{ss}(L) - X^s(L).$$

Hence the Theorem.

Now, let G be a simple algebraic group of rank l and of type other than A_l. Let T, B be as in § 2 and Q a parabolic subgroup of G containing B. Let $X = G/Q$. For $j \in \{1, 2 \ldots l\}$, we denote the semisimple part of the Levi component of the parabolic subgroup P_j by G_j, $T_j = G_j \cap T$, $w_0(G_j)$ denotes the longest element of the Weyl group W_j of G_j with respect to T_j. Let V_j be the $\mathbb{Q}$ vector subspace of $V = X(T) \otimes \mathbb{Q}$ spanned by $\Delta - \{\alpha_j\}$. Then, we have

Lemma 3.2. *Let G be as above. Then for any* $i \in \{1, 2, \ldots, l\}$, *there is a* $w_i \in W$ *and a* $j(i) \in \{1, 2, \ldots, l\}$ *such that*

(1) $w_i(\varpi_r) \in V_{j(i)}$ *for any* $r \neq i$ *and*
(2) $\tau \in W_{j(i)}$ *implies* $\tau w_i \leq w_0(G_{j(i)}) w_i$.

Proof. (1) We write the elements w_i and the integers $j(i)$ for each of the groups, case by case, such that $w_i(\varpi_r) = \varpi_r - (\sum_{k=1}^{l} x_{r,k}\alpha_k)$, where $x_{r,j(i)}$ is equal to the coefficient of $\alpha_{j(i)}$ in the expression of ϖ_r for all $r \neq i$.

Case (I): G is of type B_l

Subcase(1): $i = 1$.
$w_i = s_2 s_3 \cdots s_{l-1} s_l s_{l-1} s_{l-2} \cdots s_3 s_2$ and $j(i) = 2$.
Using the formula $s_k(\mu) = \mu - \langle \mu, \alpha_k \rangle \alpha_k$, it is easy to see that

$$w_i(\varpi_r) = \varpi_r - \left(2 \sum_{k \neq 1} \alpha_k\right)$$

for $1 < r < l$ and $w_i(\varpi_r) = \varpi_r - \sum_{k \neq 1} \alpha_k$ for $r = l$.

Subcase (2): $2 \leq i \leq l-2$.
Take $w_i = (s_2 s_1)(s_3 s_2) \cdots (s_i s_{i-1})(s_{i+1} \cdots s_{l-1} s_l s_{l-1} \cdots s_{i+1})$ and $j(i) = 2$.

One can check that if $r < i$, then

$$w_i(\varpi_r) = \varpi_r - \left(\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right)$$

and if $l > r > i$, then

$$w_i(\varpi_r) = \varpi_r - \left(2\sum_{k=2}^{l} \alpha_k\right).$$

If $r = l$, then

$$w_i(\varpi_r) = \varpi_r - \left(\sum_{k=2}^{l} \alpha_k\right).$$

Subcase (3): $i = l - 1$.
Take $w_i = (\prod_{k=1}^{l-2}(s_{k+1}s_k))s_l$ and $j(i) = 2$.
If $r \leq l - 2$, then

$$w_i(\varpi_r) = \varpi_r - \left(\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right).$$

If $r = l$,

$$w_i(\varpi_r) = \varpi_r - \left(\sum_{k=2}^{l} \alpha_k\right).$$

Subcase (4): $i = l$
Take $w_i = \prod_{k=1}^{l-1} s_k$ and $j(i) = 1$.

$$w_i(\varpi_r) = \varpi_r - \left[\sum_{k=1}^{r} \alpha_k\right]$$

for $1 \leq r \leq l - 1$.

Case (II)$G = C_l$

The choice of w_i and $j(i)$ to be the same as in the case of type B_l, but we will write tl explicitly so that computing the weights $w_i(\varpi_r)$ will be easy.

Subcase (1): $i = 1$.
$w_i = (\prod_{k=2}^{l} s_k)(\prod_{k=1}^{l-2} s_{l-k})$ and $j(i) = 2$.

$$w_i(\varpi_r) = \varpi_r - \left[2\sum_{k=2}^{l-1} \alpha_k + \alpha_l\right]$$

for $r \neq 1$.
Subcase (2): $2 \leq i \leq l - 2$.
$w_i = \prod_{k=1}^{i-1}(s_{k+1}s_k)(\prod_{k=i+1}^{l} s_k)(\prod_{k=1}^{l-(i+1)} s_{l-k})$ and $j(i) = 2$ for $r < i$

$$w_i(\varpi_r) = \varpi_r - \left[\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right]$$

and

$$w_i(\varpi_r) = \varpi_r - \left(2\left[\sum_{k=2}^{l-1}\alpha_k\right] + \alpha_l\right)$$

for $i < r < l$ and

$$w_i(\varpi_l) = \varpi_r - \left[2\sum_{k=2}^{l-1}\alpha_k + \alpha_l\right].$$

Subcase (3): $i = l - 1$.
$w_i = (\prod_{k=1}^{l-2}(s_{k+1}s_k))s_l$ and $j(i) = 2$.

$$w_i(\varpi_1) = \varpi_1 - (\alpha_1 + \alpha_2)$$

and

$$w_i(\varpi_r) = \varpi_r - \left(\alpha_1 + 2\left(\sum_{k=2}^{r}\alpha_k\right) + \alpha_{r+1}\right)$$

for $1 < r < l - 1$ and

$$w_i(\varpi_l) = \varpi_l - \left(2\sum_{k=2}^{l-1}\alpha_k + \alpha_l\right).$$

Subcase (4): $i = l$.
$w_i = \prod_{k=1}^{l-1} s_k$ and $j(i) = 1$.

$$w_i(\varpi_r) = \varpi_r - \left[\sum_{k=1}^{r}\alpha_k\right].$$

Case (III): $G = D_l$

Subcase (1): $i = 1$.
$w_i = (\prod_{k=2}^{l} s_k)(\prod_{k=2}^{l-2} s_{l-k})$ and $j(i) = 2$

$$w_i(\varpi_r) = \varpi_r - \left[2\left(\sum_{k=2}^{l-2}\alpha_k\right) + \alpha_{l-1} + \alpha_l\right]$$

for $1 < r < l - 1$,

$$w_i(\varpi_{l-1}) = \varpi_{l-1} - \left(\sum_{k=2}^{l-1}\alpha_k\right)$$

and

$$w_i(\varpi_l) = \varpi_l - \left[\sum_{k=2}^{l-2}\alpha_k + \alpha_l\right].$$

Subcase (2): $2 \leq i \leq l - 3$
$w_i = \prod_{k=1}^{i-1}(s_{k+1}s_k)(\prod_{k=i+1}^{l} s_k)(\prod_{k=2}^{l-(i+1)} s_{l-k})$ and $j(i) = 2$.

$$w_i(\varpi_1) = \varpi_1 - (\alpha_1 + \alpha_2)$$

$$w_i(\varpi_r) = \varpi_r - \left[\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right]$$

for $2 \leq r \leq i-1$,

$$w_i(\varpi_r) = \varpi_r - \left[2\left(\sum_{k=2}^{l-2} \alpha_k\right) + \alpha_{l-1} + \alpha_l\right]$$

for $i < r < l-1$,

$$w_i(\varpi_l - 1) = \varpi_l - 1 - \left[\sum_{k=2}^{l-1} \alpha_k\right]$$

and

$$w_i(\varpi_l) = \varpi_l - \left[\sum_{k=2}^{l-2} \alpha_k + \alpha_l\right].$$

Subcase (3): $i = l-2$
$w_i = \left(\prod_{k=1}^{l-3}(s_{k+1}s_k)\right)(s_{l-1}s_l)$ and $j(i) = 2$

$$w_i(\varpi_1) = \varpi_1 - [\alpha_1 + \alpha_2],$$

$$w_i(\varpi_r) = \varpi_r - \left[\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right]$$

for $1 < r < l-2$,

$$w_i(\varpi_l - 1) = \varpi_{l-1} - \left[\sum_{k=2}^{l-1} \alpha_k\right]$$

and

$$w_i(\varpi_l) = \varpi_l - \left[\sum_{k=2}^{l-2} \alpha_k + \alpha_l\right].$$

Subcase (4): $i = l-1$
$w_i = \prod_{k=1}^{l-3}(s_{k+1}s_k)(s_l s_{l-2})$ and $j(i) = 2$

$$w_i(\varpi_1) = \varpi_1 - [\alpha_1 + \alpha_2]$$

$$w_i(\varpi_r) = \varpi_r - \left[\alpha_1 + 2\left(\sum_{k=2}^{r} \alpha_k\right) + \alpha_{r+1}\right]$$

for $1 < r < l-2$

$$w_i(\varpi_{l-2}) = \varpi_{l-2} - \left[\alpha_1 + 2\left(\sum_{k=2}^{l-2} \alpha_k\right) + \alpha_l\right]$$

and

$$w_i(\varpi_l) = \varpi_l - \left[\sum_{k=2}^{l-2} \alpha_k + \alpha_l\right].$$

Subcase (5): $i = l$

$w_i = \prod_{k=1}^{l-2}(s_{k+1}s_k)$ and $j(i) = 2$. For $1 \leq r \leq l-2$, the extremal weights $w_i(\varpi_r)$ are the same as in the case of $i = l-1$, and

$$w_i(\varpi_{l-1}) = \varpi_{l-1} - \left[\sum_{k=2}^{l-1} \alpha_k\right].$$

Case (4): $G = E_6$

Subcase(1): $i = 1$

$w_1 = s_2s_4s_3s_5s_4s_2s_6s_5s_4s_3$ and $j(i) = 2$.

$$w_1(\varpi_2) = \varpi_2 - [2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5]$$
$$w_1(\varpi_3) = \varpi_3 - [2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_1(\varpi_4) = \varpi_4 - [3\alpha_2 + 2\alpha_3 + 4\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_1(\varpi_5) = \varpi_5 - [2\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_1(\varpi_6) = \varpi_6 - [\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6].$$

Subcase (2): $i = 2$

$w_2 = s_2s_4s_5s_3s_4s_1s_6s_3s_5s_4$ and $j(i) = 2$.

$$w_2(\varpi_1) = \varpi_1 - \left[\sum_{k=1}^{4} \alpha_k\right]$$
$$w_2(\varpi_3) = \varpi_3 - [\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5]$$
$$w_2(\varpi_4) = \varpi_4 - [\alpha_1 + 3\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_2(\varpi_5) = \varpi_5 - [2\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_2(\varpi_6) = w_1(\varpi_6).$$

Subcase (3): $i = 3$

$w_3 = s_2s_4s_5s_3s_4s_1s_2s_6s_5s_4$ and $j(i) = 2$.

$$w_3(\varpi_1) = w_2(\varpi_1)$$
$$w_3(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 5, 6$

$$w_3(\varpi_4) = \varpi_4 - [3\alpha_2 + 2\alpha_3 + 4\alpha_4 + 2\alpha_5 + \alpha_6].$$

Subcase (4): $i = 4$

$w_4 = s_2s_4s_5s_3s_4s_1s_2s_6s_3s_5$ and $j(i) = 2$.

$$w_4(\varpi_1) = w_2(\varpi_1)$$
$$w_4(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 5, 6$

$$w_4(\varpi_3) = \varpi_3 - [\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5].$$

Subcase (5): $i = 5$

$w_5 = s_2s_4s_5s_3s_4s_1s_2s_6s_3s_4$ and $j(i) = 2$.

$$w_5(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 6$,

$$w_5(\varpi_1) = w_2(\varpi_1)$$
$$w_5(\varpi_3) = w_4(\varpi_3)$$

and

$$w_5(\varpi_4) = \varpi_4 - [\alpha_1 + 3\alpha_2 + 2\alpha_3 + 4\alpha_4 + 2\alpha_5].$$

Subcase (6): $i = 6$

$w_6 = s_2s_4s_5s_3s_4s_2s_1s_3s_4s_5$ and $j(i) = 2$.

$$w_6(\varpi_1) = w_2(\varpi_1),$$
$$w_6(\varpi_2) = w_1(\varpi_2),$$
$$w_6(\varpi_3) = w_4(\varpi_3),$$
$$w_6(\varpi_4) = \varpi_4 - [\alpha_1 + 3\alpha_2 + 2\alpha_3 + 4\alpha_4 + 2\alpha_5]$$

and

$$w_6(\varpi_5) = \varpi_5 - [\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5].$$

Case (5): $G = E_7$

Subcase (1): $i = 1$

$w_1 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_5s_2s_4s_3$ and $j(i) = 1$.

$$w_1(\varpi_2) = \varpi_2 - [2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_1(\varpi_3) = \varpi_3 - [3\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7]$$
$$w_1(\varpi_4) = \varpi_4 - [4\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + \alpha_7]$$
$$w_1(\varpi_5) = \varpi_5 - [3\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7]$$
$$w_1(\varpi_6) = \varpi_6 - [2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7]$$
$$w_1(\varpi_7) = \varpi_7 - [\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7].$$

Subcase (2): $i = 2$

$w_2 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_5s_1s_3s_4$ and $j(i) = 1$.

$$w_2(\varpi_1) = \varpi_1 - [2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5]$$
$$w_2(\varpi_3) = \varpi_3 - [3\alpha_1 + 2\alpha_2 + 4\alpha_3 + 4\alpha_4 + 2\alpha_5 + \alpha_6]$$
$$w_2(\varpi_4) = \varpi_4 - [4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7]$$
$$w_2(\varpi_r) = w_1(\varpi_r)$$

for all $r = 5, 6, 7$.

Subcase (3): $i = 3$

$w_3 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_5s_2s_4s_1$ and $j(i) = 1$.

$$w_3(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 4, 5, 6, 7$ and

$$w_3(\varpi_1) = w_2(\varpi_1).$$

Subcase (4) $i = 4$

$w_4 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_5s_2s_1s_3$ and $j(i) = 1$.

$$w_4(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 5, 6, 7$ and

$$w_4(\varpi_r) = w_2(\varpi_r)$$

for $r = 1, 3$.

Subcase (5): $i = 5$

$w_5 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_2s_1s_3s_4$ and $j(i) = 1$.

$$w_5(\varpi_r) = w_2(\varpi_r)$$

for $r = 1, 3$ and

$$w_5(\varpi_4) = \varpi_4 - [4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6]$$

and

$$w_5(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 6, 7$.

Subcase (6): $i = 6$

$w_6 = s_1s_3s_4s_5s_6s_7s_2s_4s_5s_3s_4s_2s_1s_3s_4s_5$ and $j(i) = 1$.

$$w_6(\varpi_r) = w_2(\varpi_r)$$

for $r = 1, 3$,

$$w_6(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 7$,

$$w_6(\varpi_4) = w_5(\varpi_4)$$

and

$$w_6(\varpi_5) = \varpi_5 - [3\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 2\alpha_6].$$

Subcase (7): $i = 7$

$w_7 = s_1s_3s_4s_5s_6s_2s_4s_5s_3s_4s_2s_1s_3s_4s_5s_6$ and $j(i) = 1$.

$$w_7(\varpi_r) = w_6(\varpi_r)$$

for all $r = 1, 2, 3, 4, 5$ and

$$w_7(\varpi_6) = \varpi_6 - [2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6].$$

<u>Case (6): $G = E_8$</u>

Subcase (1): $i = 1$

$w_1 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_2s_8s_7s_6s_5s_4s_3$ and $j(i) = 8$.

$$\begin{aligned}
w_1(\varpi_2) &= \varpi_2 - [2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 3\alpha_8],\\
w_1(\varpi_3) &= \varpi_3 - [3\alpha_1 + 3\alpha_2 + 6\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 5\alpha_7 + 4\alpha_8],\\
w_1(\varpi_4) &= \varpi_4 - [4\alpha_1 + 5\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 7\alpha_7 + 6\alpha_8],\\
w_1(\varpi_5) &= \varpi_5 - [3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 9\alpha_4 + 8\alpha_5 + 7\alpha_6 + 6\alpha_7 + 5\alpha_8],\\
w_1(\varpi_6) &= \varpi_6 - [2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 6\alpha_5 + 6\alpha_6 + 5\alpha_7 + 4\alpha_8],\\
w_1(\varpi_7) &= \varpi_7 - [\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 4\alpha_6 + 4\alpha_7 + 3\alpha_8]
\end{aligned}$$

and

$$w_1(\varpi_8) = \varpi_8 - [\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + 2\alpha_8].$$

Subcase (2): $i = 2$

$w_2 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_8s_7s_6s_5s_4$ and $j(i) = 8$.

$$\begin{aligned} w_2(\varpi_1) &= \varpi_1 - [2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + 2\alpha_8], \\ w_2(\varpi_3) &= \varpi_3 - [3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 5\alpha_6 + 4\alpha_7 + 4\alpha_8], \\ w_2(\varpi_4) &= \varpi_4 - [4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 8\alpha_6 + 7\alpha_7 + 6\alpha_8] \end{aligned}$$

and

$$w_2(\varpi_r) = w_1(\varpi_r)$$

for all $r = 5, 6, 7, 8$.

Subcase (3): $i = 3$

$w_3 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_2s_8s_7s_6s_5s_4$ and $j(i) = 8$.

$$w_3(\varpi_r) = w_1(\varpi_r)$$

for $r = 2, 4, 5, 6, 7, 8$ and

$$w_3(\varpi_1) = w_2(\varpi_1).$$

Subcase (4): $i = 4$

$w_4 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_2s_8s_7s_6s_5$ and $j(i) = 8$.

$$w_4(\varpi_r) = w_3(\varpi_r)$$

for all $r = 1, 2, 5, 6, 7, 8$ and

$$w_4(\varpi_3) = w_2(\varpi_3).$$

Subcase (5): $i = 5$

$w_5 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_2s_4s_8s_7s_6$ and $j(i) = 8$.

$$w_5(\varpi_r) = w_4(\varpi_r)$$

for all $r = 1, 2, 3, 6, 7, 8$ and

$$w_5(\varpi_4) = \varpi_4 - [4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 6\alpha_8].$$

Subcase (6): $i = 6$

$w_6 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_2s_4s_5s_8s_7$ and $j(i) = 8$.

$$w_6(\varpi_r) = w_5(\varpi_r)$$

for $r = 1, 2, 3, 4, 7, 8$ and

$$w_6(\varpi_5) = \varpi_5 - [3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 10\alpha_4 + 9\alpha_5 + 7\alpha_6 + 5\alpha_7 + 5\alpha_8].$$

Subcase (7): $i = 7$

$w_7 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_2s_4s_5s_8s_6$ and $j(i) = 8$.

$$w_7(\varpi_r) = w_6(\varpi_r)$$

for all $r = 1, 2, 3, 4, 5, 8$ and

$$w_7(\varpi_6) = \varpi_6 - [2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 4\alpha_7 + 4\alpha_8].$$

Subcase (8): $i = 8$

$w_8 = s_8s_7s_6s_5s_4s_2s_3s_4s_1s_5s_3s_4s_2s_6s_5s_7s_6s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7$ and $j(i) = 8$.

$$w_8(\varpi_r) = w_7(\varpi_r)$$

for $r = 1, 2, 3, 4, 5, 6$ and

$$w_8(\varpi_7) = \varpi_7 - [2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 3\alpha_8].$$

Case (7) : $G = F_4$

Subcase(1): $i = 1$

$w_1 = s_1s_2s_3s_4s_2s_3s_2$ and $j(i) = 1$

$$w_1(\varpi_2) = \varpi_2 - [3\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4]$$
$$w_1(\varpi_3) = \varpi_3 - [2\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4]$$
$$w_1(\varpi_4) = \varpi_4 - \left[\sum_{k=1}^{4} \alpha_k\right].$$

Subcase (2): $i = 2$

$w_2 = s_1s_2s_3s_4s_2s_3s_1$ and $j(i) = 1$

$$w_2(\varpi_1) = \varpi_1 - [2(\alpha_1 + \alpha_2 + \alpha_3)]$$
$$w_2(\varpi_r) = w_1(\varpi_r)$$

for $r = 3, 4$.

Subcase (3): $i = 3$

$w_3 = s_4s_3s_2s_1s_3s_2s_4$ and $j(i) = 4$

$$w_3(\varpi_1) = \varpi_1 - [\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4]$$
$$w_3(\varpi_2) = \varpi_2 - [\alpha_1 + 3\alpha_2 + 4\alpha_3 + 4\alpha_4]$$
$$w_3(\varpi_4) = \varpi_4 - [\alpha_2 + 2\alpha_3 + 2\alpha_4].$$

Subcase (4): $i = 4$

$w_4 = s_4s_3s_2s_1s_3s_2s_3$ and $j(i) = 4$

$$w_4(\varpi_r) = w_3(\varpi_r)$$

for $r = 1, 2$ and

$$w_4(\varpi_3) = \varpi_3 - [\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4].$$

Case (8): $G = G_2$

Subcase(1): $i = 1$

$w_1 = s_1s_2$ and $j(i) = 1$.

$$w_1(\varpi_2) = \varpi_2 - [\alpha_1 + \alpha_2].$$

Subcase (2): $i = 2$

$w_2 = s_2s_1$ and $j(i) = 2$.

$$w_2(\varpi_1) = \varpi_1 - [\alpha_1 + 3\alpha_2].$$

(2) Now, let $w_i = \prod_{j=1}^{m} s_{i_j}$ be the reduced expression that we have written for w_i. It is easy to see that for any $k \in \{2, 3 \cdots m\}$, the coefficient of $\alpha_{j(i)}$ in the expression of the root

$\prod_{j=1}^{k-1} s_{i_j}(\alpha_{i_k})$ is a positive integer and hence it is a positive root. Since the only simple reflection $s_{j(i)}$ taking $\alpha_{j(i)}$ to a negative root does not occur in any reduced expression of $w_0(G_{j(i)})$, $w_0(G_{j(i)})\cdot(\prod_{j=1}^{k-1} s_{i_j})(\alpha_{i_k})$ is also a positive root for any $k \in \{2, 3 \cdots m\}$. Using this argument, and the fact that $l(w.s_i) = l(w) + 1$ if and only if $w(\alpha_i)$ is a positive root, it is easy to see that

$$l(w_0(G_{j(i)})).w_i = l(w_0(G_{j(i)})) + l(w_i). \tag{3.2.1}$$

Now, let $\tau \in W_{j(i)}$ be arbitrary. Then, there is a $\tau' \in W_{j(i)}$ such that, $\tau'\tau = w_0(G_{j(i)})$ and $l(\tau') + l(\tau) = l(w_0(G_{j(i)}))$. Therefore by(3.2.1),

$$l(w_0(G_{j(i)})w_i) = [l(\tau') + l(\tau)] + l(w_i).$$

Hence, for any $\tau \in W_{j(i)}$, we must have $\tau w_i \leq w_0(G_{j(i)})w_i$.

Hence the Lemma.

Note 3.3. With notations as above, there is no nonzero $W_{j(i)}$ invariant vector in $V_{j(i)}$.

Proof. If $\mu \in V_{j(i)}$ is $W_{j(i)}$ invariant, then

$$\mu = s_\alpha(\mu) = \mu - (\mu, \alpha^\vee)\alpha$$

for all $\alpha \in \Delta - \{\alpha_{j(i)}\}$. Since the pairing $(.,.)$ restricted to $V_{j(i)} \times V_{j(i)}$ is nondegenerate, $\mu = 0$.

Let $j(i)$, w_i be as in Lemma 3.2. For each $i \in \{1, 2 \cdots l\}$, set $\tau_i = w_0(G_{j(i)})w_i$. Let $\lambda_{j(i)}$ be the one parameter subgroup of T such that

$$\langle \alpha_k, \lambda_{j(i)} \rangle = n\delta_{k,j(i)},$$

where n is the order of the finite group of characters of the center of the group G.

Then, we have

COROLLARY 3.4

The weights $\{\tau w_i(\varpi_r) : \tau \in W_{j(i)}\}$ are weights of the T module $H^0(X(\tau_i), L_r|X(\tau_i))$ such that the sum $\sum_{\tau \in W_{j(i)}} \tau w_i(\varpi_r)$ is zero and $\langle \tau_i(\varpi_r), \lambda_{j(i)} \rangle = 0$.

Proof. By Lemma 3.2, for any $r \neq i$ one has, $w_i(\varpi_r) \in V_{j(i)}$. Since $V_{j(i)}$ is a $W_{j(i)}$ submodule of V, $\tau w_i(\varpi_r) \in V_{j(i)}$ for any $\tau \in W_{j(i)}$. In particular, $\tau_i(\varpi_r) \in V_{j(i)}$. Now since $V_{j(i)}$ is orthogonal to $\lambda_{j(i)}$,

$$\langle \tau_i(\varpi_r), \lambda_{j(i)} \rangle = 0.$$

Also, by Lemma 3.2 $\tau w_i \leq \tau_i$ for any $\tau \in W_{j(i)}$. Hence, the set of weights $\{\tau w_i(\varpi_r)$: $\tau \in W_{j(i)}\}$ is contained in the set of all the weights of the T module $H^0(X(\tau_i), L_r|X(\tau_i))$. Now, $\sum_{\tau \in W_{j(i)}} \tau w_i(\varpi_r)$ is a $W_{j(i)}$ invariant vector in $V_{j(i)}$. Therefore, by the above note $\sum_{\tau \in W_{j(i)}} \tau w_i(\varpi_r) = 0$.

Summarising the above discussions, we have

Theorem 3.5. *Let G be a* **simple** *algebraic group of type not equal to A_l, T a maximal torus of G, B a Borel subgroup of G containing T, Q a parabolic subgroup of G containing B and let $X = G/Q$. Then, there is a line bundle L on X such that $X_T^{ss}(L) = X_T^s(L)$ if and only if $Q = B$.*

Proof. Let $Q = B$. Then by Proposition 3.1 of [SSK], there is a G linearised line bundle L on X such that $X_T^{ss}(L) = X_T^s(L)$.

Conversely, let $Q \neq B$ be a parabolic subgroup of G properly containing B. Then $Q = \cap_{r\in I} P_r$, where $I \not\subseteq \{1, 2, \ldots, l\}$. Hence, choose $i \in \{1, 2, \ldots, l\} - I$. Now, let L be an effective line bundle. Then $L = \sum_{r\in I} c_r L_r$ with $c_r \in \mathbb{Z}_{\geq 0}$ for all $r \in I$ and at least one c_r is positive.

Let i be as above, and τ_i, $j(i)$ be as in the Corollary 3.4. Now, let $U = B(\tau_i Q)/Q$, $N_{j(i)}$ be the order of the group $W_{j(i)}$ and p be a lowest weight vector of the module $H^0(G/Q, L)$. By Corollary 3.4, $\sigma = \prod_{\tau\in W_{j(i)}} p(\tau w_i)$ is a T invariant section of the line bundle $L^{\otimes N_{j(i)}}$. By Lemma 3.2 (2), we have $\tau w_i \leq w_0(G_{j(i)}) w_i = \tau_i$ for every $\tau \in W_{j(i)}$ and hence for any $\tau \in W_{j(i)}$, $p(\tau w_i)$ does not vanish identically on $X(\tau_i)$. Therefore, σ does not vanish identically on $X(\tau_i)$ (since $X(\tau_i)$ is irreducible). Thus the set $X(\tau_i)_T^{ss}(L|X(\tau_i))$ is nonempty. Since U is an open subset of $X(\tau_i)$ and $X(\tau_i)$ is irreducible, $U \bigcap X(\tau_i)_T^{ss}(L|X(\tau_i))$ is nonempty. On the other hand, let $\pi: G/B \longrightarrow G/Q$ be the canonical projection and let $x \in U$ be arbitrary. Choose a point $y \in B(\tau_i B)/B$ such that $\pi(y) = x$. Then, by Lemma 2.1 of [CSS I] we have

$$\mu^L(x, \lambda_{j(i)}) = \mu^{\pi^*(L)}(y, \lambda_{j(i)}).$$

Hence, by Lemma 2.1 and Corollary 3.4 we have

$$\mu^L(x, \lambda_{j(i)}) = -\sum_{r\in I} c_r \langle \tau_i(\varpi_r), \lambda_{j(i)} \rangle = 0$$

for $x \in U$. Thus, by Theorem 2.1 of [GIT]

$$X(\tau_i)_T^{ss}(L|X(\tau_i)) \neq X(\tau_i)_T^s(L|X(\tau_i)).$$

Hence the Theorem.

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Note added in proof

We give here a better proof of Theorem (3.1) and Theorem (3.5), suggested by Professor C. De Concini. (We wish to thank Professor De Concini for the proofs.)

We have

Lemma 4.1. *Let T be a maximal torus of a simply connected simple algebraic group G. Let $B \subset G$ be a Borel subgroup of G containing T. Let Q be a parabolic subgroup of G containing B. Let L be an ample line bundle on G/Q associated to the dominant dominant weight $\chi \in X(T)$. Then, with respect to L, $G/Q^{ss} = G/Q^{s}$ if and only if for all fundamental weights ϖ (with respect to B) and for all elements w in the Weyl group W, we have $\langle w(\varpi), \chi\rangle \neq 0$.*

Proof. Assume that for all fundamental weights ϖ and for all elements w in the Weyl group W, we have $\langle w\varpi, \chi\rangle \neq 0$. Let $x \in G/Q^{ss}$. Let λ be a one parameter subgroup. There exists a $w \in W$ such that $w(\lambda)$ is dominant. Thus $w(\lambda)=\sum_{i=1}^{l} n_i\varpi_i$ with $\{\varpi_1, \ldots, \varpi_l\}$ denoting the set of fundamental weights and n_i non negative integers. We deduce that $\mu^L(wx, w\chi) = \mu^L(x, \chi) \geq 0$. But if wx lies in the Schubert variety indexed by the element $u \in W$ (if Q is not a Borel we of course take the shortest representative) we know $\mu^L(wx, w\chi) = -\langle u\lambda, w\chi\rangle = -\sum_{i=1}^{l} n_i\langle u\lambda, \varpi_i\rangle$. Now $\langle u\lambda, w\varpi_i\rangle \geq 0$ for each i since $wx \in G/Q^{ss}$. On the other hand $\langle u\lambda, w\varpi_i\rangle = \langle\lambda, u^{-1}\varpi_i\rangle \neq 0$ by assumption. Therefore, we have $x \in G/Q^{s}$.

Assume now that there exists a fundamental weight ϖ and an element w in the Weyl group W with $\langle w\varpi, \chi\rangle = 0$. Eventually multiplying $w\varpi$ by a positive integer, associate to it a one parameter subgroup $\phi : G_m \to T$. Set $H \subset G$ equal to the centralizer of $\phi(G_m)$. Since ϖ is fundamental, it follows that H is of semisimple rank $r-1$ (it is indeed the Levi factor of the parabolic subgroup associated to $w\varpi$). Notice that $H \cap Q$ is a parabolic in H. Consider $H/(Q \cap H) \subset G/Q$. The restriction L_H of L to $H/(Q \cap H)$ inherits a T linearization and the one parameter subgroup $\phi(G_m)$ which is the connected component of the center of H acts trivially on L_H (and hence on $H^0(H/(Q \cap H), L_H^r)$ for each r) since by our hypothesis that $\langle w\varpi, \lambda\rangle = 0$ it acts trivially on the fiber above Q. Hence the H action on $H^0(H/(Q \cap H), L_H^r)$ factors through the semisimple group $H/\phi(G_m)$. But then, we can find a non zero section $\bar{s} \in H^0(H/(Q \cap H), L_H^r)^T = H^0(H/(Q \cap H), L_H^r)^{T/\phi(G_m)}$ for some r (for example, take the product of all extremal weight vectors in the irreducible H module $H^0(H/(P \cap H), L_H)$. Taking a representative $s \in H^0(G/Q, L^r)^T$ for $\bar{s}$ we get a non zero invariant which vanishes at some point $x \in H/(P \cap H)$. Since $\phi(G_m)$ fixes x, we deduce that $x \in G/Q^{ss}$ but $x \notin G/Q^{s}$.

In type A_{n-1}, we recall that a vector which is W conjugated to a multiple of the fundamental weight ϖ_j is a vector in $\mathbb{Z}^n$ with the sum of its coordinates equal to zero and with j equal positive entries and $n-j$ equal negative entries.

Lemma 4.2. *For each $i, j = 1, \ldots, n-1$ there exists a $w \in W$ with $\langle\varpi_i, w\varpi_j\rangle = 0$ if and only if n divides ij.*

Proof. In order to solve our problem, we need to find non zero integers x, y, z with $x > 0$, $0 < z < i$ such that

$$jx + (n-j)y = 0 \text{ and } zx + (i-z)y = 0.$$

Eliminating x and substituting we get $nz = ij$, so our condition is necessary.

On the other hand assume that n divides ij. Set $z = \frac{ij}{n}$, $x = n-j$, $y = -j$. This is clearly a solution of our system.

Using the Lemma (4.1) and Lemma (4.2), we now prove Theorem (4.3).

Theorem 4.3. *Let P_i be the maximal parabolic (containing a chosen Borel) associated to ϖ_i. Given $I \subset \{1, \ldots, n-1\}$, set $Q_I = \cap_{i \in I} P_i$. Then, there exists an ample line bundle L*

on G/Q_I such that, with respect to L, $G/Q_I^{ss} = G/Q_I^s$ if and only if, setting (n, I) to be the ideal in $\mathbb{Z}$ generated by n and the elements in I, $(n, I) = (1)$.

Proof. From Lemma (4.1) and Lemma (4.2), if we set $I = \{i_1, \ldots, i_h\}$ we have that an L with the above properties exists if and only if there is no $j < n$ with n dividing $i_s j$ for each $s = 1, \ldots, h$. Assume $(n, i_1, \ldots, i_h) = 1$, and such a j exists. Let p be a prime such that p^t, $t > 0$, is the highest power of p dividing n. There must exist a i_s such that p does not divide i_s. This implies p^t divides j, hence n divides j contrary to the fact that $n > j$.

Conversely, assume that p divides $(n, i_1, \ldots, i_h)$. Then set $j = \frac{n}{p}$. We have $ji_s = n\frac{i_s}{p}$ as desired.

Using Lemma (4.1), we now prove Theorem (4.4):

Theorem 4.4. *If G is a simple group not of type A, then if $G/Q^{ss} = G/Q^s$, Q is necessarily a Borel subgroup.*

Proof. If Q is not a Borel subgroup, then necessarily given an ample line bundle L on G/Q the associated dominant weight is not regular. Thus there exists a root α (indeed a simple root) such that $\langle \alpha, \lambda \rangle = 0$. But now since we are not in type A, each root is W conjugate to a multiple of a fundamental weight (indeed to a fundamental weight except in type C). So the assumptions of our Lemma are fulfilled and we are done.

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Parabolic ample bundles III: Numerically effective vector bundles

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Abstract. In this continuation of [Bi2] and [BN], we define numerically effective vector bundles in the parabolic category. Some properties of the usual numerically effective vector bundles are shown to be valid in the more general context of numerically effective parabolic vector bundles.

Keywords. Parabolic structure; numerically effective; numerically flat.

1. Introduction

Parabolic bundles were introduced in [MS]. It is now known that various notions related to the usual vector bundles actually extend to the context of parabolic vector bundles. In [Bi2] the notion of ampleness of a vector bundle was extended to the parabolic category. In [Bi2] and [BN], various results on usual ample vector bundles were generalized to parabolic ample bundles.

The notion of numerically effectiveness of a vector bundle is very closely related to the notion of ampleness. Defining the notion of numerically effectiveness of a parabolic vector bundle, we generalize some known results on usual numerically effective vector bundles to the more general context of parabolic bundles.

2. Some properties of numerically effective vector bundles

Let X be a connected smooth projective variety over $\mathbb{C}$. For a vector bundle E over X, the projective bundle over X consisting of all hyperplanes in the fibers of E will be denoted by $\mathbb{P}E$. The tautological relative ample line bundle over $\mathbb{P}E$ will be denoted by $\mathcal{O}_{\mathbb{P}E}(1)$. We recall the definition of a numerically effective vector bundle.

DEFINITION 2.1

A line bundle L over X is called *numerically effective* (abbreviated as *nef*) if for any morphism $f : C \longrightarrow X$, where C is a connected smooth projective curve, the inequality

$$\deg(f^*L) \geq 0$$

is valid.

More generally, a vector bundle E over X is called *numerically effective* if the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ over $\mathbb{P}E$ is nef in the above sense.

For a vector bundle V over a connected smooth curve C, let $d_{\min}(V)$ denote the degree of the final piece of the graded object for the Harder–Narasimhan filtration of V. In other

words, if

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_l \subset V_{l+1} = V \tag{2.2}$$

is the Harder–Narasimhan filtration of V, then $d_{\min}(V) := \deg(V/V_l)$.

PROPOSITION 2.3

A vector bundle E over X is nef if and only if for any morphism f from a curve C to X, as in Definition 2.1, the inequality

$$d_{\min}(f^*E) \geq 0$$

is valid.

Proof. Let $f : C \longrightarrow X$ be a morphism from a smooth curve.

If $d_{\min}(f^*E) \geq 0$, then it is easy to see that $\deg(L) \geq 0$, where L is any quotient line bundle of f^*E. Indeed, if

$$0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l \subset E'_{l+1} = f^*E \tag{2.4}$$

is the Harder–Narasimhan filtration of f^*E and $\deg(V/V_l) \geq 0$, then for any line bundle L' over C with $\deg(L') < 0$, the following is valid

$$H^0(C, \mathrm{Hom}(E'_{i+1}/E'_i, L')) = 0$$

for all $i \in [0, l]$. This implies that L' cannot be a quotient line bundle of f^*E.

The above observation that $\deg(L) \geq 0$ immediately yields that the vector bundle E is nef [Vi, Proposition 2.9].

To prove the converse, assume that E is nef. Take f as above, and let a filtration as in (2.4) be the Harder–Narasimhan filtration of f^*E. Let r be the rank of f^*E/E'_l.

Since E is nef, the pullback f^*E is also nef, and hence we have that $\wedge^r f^*E$ is nef [Vi, Corollary 2.20]. But the line bundle $\wedge^r(f^*E/E'_l)$ is a quotient of $\wedge^r f^*E$. Thus the inequality $d_{\min}(f^*E) = \deg(\wedge^r(f^*E/E'_l)) \geq 0$ is valid. This completes the proof of the proposition. □

Fix a polarization $\mathcal{L}$ on X. Any vector bundle V over X admits a unique Harder–Narasimhan filtration as in (2.2) [Ko, Ch. V, Theorem 7.15]. As before, define

$$d_{\min}(V) := \deg(V/V_l) := \int_X c_1(V/V_l) \cup c_1(\mathcal{L})^{d-1}$$

where $d = \dim_{\mathbb{C}} X$.

Theorem 2.5. *If E is a nef vector bundle over X, then $d_{\min}(E) \geq 0$. If $\dim_{\mathbb{C}} X = 1$, then any vector bundle E over X, with $d_{\min}(E) \geq 0$, is nef.*

Proof. Let $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = E$ be the Harder–Narasimhan filtration of E.

Let $f : C \longrightarrow X$ be a smooth irreducible complete intersection curve such that the restriction of all E_{i+1}/E_i, $0 \leq i \leq l$, to C is a semistable vector bundle. The existence of such a curve C is ensured by the main theorem of [MR] which says that a vector bundle over a smooth projective variety is semistable if and only if its restriction to the general

complete intersection curve of sufficiently large degree is semistable. Thus the restriction

$$0 = E_0 \subset E_1|_C \subset E_2|_C \subset \cdots \subset E_l|_C \subset E_{l+1}|_C = f^*E$$

of the above filtration of E to C is actually the Harder–Narasimhan filtration of f^*E. Indeed, if $F \subset f^*E$ is the maximal semistable subbundle, then $\mu(F) \geq \mu(E_1|_C)$, and hence F does not admit any nonzero homomorphism to the quotient vector bundle $(E_{i+1}|_C)/(E_i|_C)$ for any $i \geq 1$. Thus F must coincide with $E_1|_C$. Now the above claim is established by using induction on the length of the Harder–Narasimhan filtration of the vector bundle E.

If E is nef, then from Proposition 2.3 it follows that $d_{\min}(f^*E) \geq 0$. This, in turn, implies that $d_{\min}(E) \geq 0$.

Now let X be a Riemann surface, and let E be a vector bundle over X with $d_{\min}(E) \geq 0$. This implies that $d_{\min}(S^k(E)) \geq 0$, where $S^k(E)$ is the k-fold symmetric tensor power of E.

Let L be a line bundle over X with $\deg(L) > 0$. A consequence of the inequality $d_{\min}(S^k(E)) \geq 0$ is that any quotient of $S^k(E)$ is of nonnegative degree. Hence any quotient of $S^k(E) \otimes L$ is of strictly positive degree. Now from a theorem of [Ha] we conclude that $S^k(E) \otimes L$ is ample. The theorem of [Ha] in question says that a vector bundle over a complete curve is ample if and only if its degree and also the degrees of all its quotient bundles are all strictly positive.

For a nonconstant morphism of curves $f : C \longrightarrow X$, if V is an ample vector bundle over X, then f^*V is ample. So from [Vi, Proposition 2.9] we conclude that E must be nef. This completes the proof of the theorem. □

We remark that the second part of Theorem 2.5 can also be deduced using Proposition 2.3.

A vector bundle E over a projective manifold X is called *numerically flat* if both E and its dual E^* are nef [DPS, Definition 1.17].

PROPOSITION 2.6

A vector bundle E is numerically flat if and only if E is semistable with $c_1(E) = 0 = c_2(E)$.

Proof. Let E be a numerically flat vector bundle over X. Since the Harder–Narasimhan filtration of E^* is simply the dual of the Harder–Narasimhan filtration of E, the conditions obtained from Theorem 2.5, namely $d_{\min}(E) \geq 0$ and $d_{\min}(E^*) \geq 0$, immediately imply that E is semistable with $c_1(E) = 0$. That $c_2(E) = 0$ follows, of course, from Corollary 1.19 of [DPS].

Conversely, let E be a semistable vector bundle over X with $c_1(E) = 0 = c_2(E)$. From [Si, Theorem 2, page 39] we know that E admits a filtration

$$F_1 \subset F_2 \subset \cdots \subset F_k \subset F_{k+1} = E$$

and a flat connection ∇ on E which preserves each F_i and the induced connection on each F_{i+1}/F_i is a unitary flat connection.

Thus for any map $f : C \longrightarrow X$ from a curve C, the vector bundle f^*E over C has a flat connection, namely $f^*\nabla$, such that on each f^*F_{i+1}/f^*F_i it induces a unitary flat connection. This implies that f^*E is a semistable vector bundle over the curve C with $\deg(f^*E) = 0$. Now from the second part of Theorem 2.5 we conclude that f^*E is nef. Thus E must be nef. The same argument shows that E^* is nef. This completes the proof of the proposition. □

In the next section we will define the notion of nefness in the context of parabolic sheaves.

3. Parabolic nef bundles

Let D be an effective divisor on X. Let E_* be a parabolic vector bundle over X with parabolic structure over D. In [Bi2] parabolic ample bundles were defined; this generalizes the notion of ample vector bundles to the parabolic context.

DEFINITION 3.1

A parabolic vector bundle E_* is called *parabolic nef* if there is an ample line bundle L over X such that $S^k(E_*)\otimes L$ is parabolic ample for every k, where $S^k(E_*)$ denotes the k-fold parabolic symmetric tensor power of the parabolic bundle E_*. (See [Bi2] for the definition of parabolic tensor product.)

If the parabolic structure of E_* is trivial, i.e., zero is the only parabolic weight, then from Proposition 2.9 of [Vi] it follows that E_* is parabolic nef if and only if the underlying vector bundle is nef in the usual sense.

Henceforth, we will assume that the parabolic divisor D on X is a normal crossing divisor. By this we mean that D is reduced, each irreducible component of D is smooth, and furthermore, the irreducible components intersect transversally.

The parabolic structure of a parabolic bundle E_* is defined as follows: for each irreducible component D_i of the parabolic divisor D, a filtration by coherent subsheaves of the vector bundle $E|_{D_i}$ over D_i is given, together with a system of parabolic weights corresponding to the filtration [MS], [Bi1].

We will henceforth consider only those parabolic bundles E_* for which the filtration over any D_i, defining the quasi-parabolic structure, is by subbundles of $E|_{D_i}$.

Let E_* be a parabolic vector bundle with rational parabolic weights. Then there is a Galois covering map

$$p : Y \longrightarrow X$$

and an orbifold vector bundle V on Y, such that the parabolic bundle E_* is obtained by taking invariants of the direct image of the twists of V using the irreducible components of D [Bi1].

PROPOSITION 3.2

A parabolic vector bundle E_* with rational parabolic weights is parabolic nef if and only if the underlying vector bundle for the corresponding orbifold vector bundle V on Y is nef in the usual sense.

Proof. Let L be an ample line bundle over X. Since the above covering map p is finite, the line bundle p^*L over Y is also ample.

Assume that the vector bundle V is nef. So $S^k(V)\otimes p^*L$ is ample for sufficiently large k [Vi, Proposition 2.9(c)]. Since the orbifold bundle $S^k(V)$ corresponds to the parabolic bundle $S^k(E_*)$ [Bi2], and furthermore, from the definition of parabolic amplitude it is immediate that the parabolic bundle corresponding to an orbifold bundle whose underlying vector bundle is ample, is actually parabolic ample, we conclude that E_* is parabolic nef.

Now assume that E_* is parabolic nef. Lemma 4.6 of [Bi2] says that $S^k(V) \otimes L$ is ample if $S^k(E_*) \otimes L$ is parabolic ample. So we conclude that V must be nef. This completes the proof of the proposition. □

As the tensor product of a nef vector bundle and an ample vector bundle is ample, the above proposition has the following corollary:

COROLLARY 3.3

Let E_* and F_* be two parabolic vector bundles with rational parabolic weights and with parabolic structure over a normal crossing divisor D. Assume that E_* is parabolic nef and F_* is parabolic ample. Then the parabolic tensor product $E_* \otimes F_*$ is parabolic ample.

Fix a polarization over X to define the parabolic degree of a parabolic bundle. A parabolic vector bundle admits a canonical filtration of parabolic subsheaves with each subsequent quotient parabolic semistable, which is a natural generalization of the Harder–Narasimhan filtration to the parabolic context. Following the definition of $d_{\min}$ in § 2, we make the following definition.

For a parabolic sheaf E_*, define $d^{\mathrm{par}}_{\min}(E_*)$ to be the parabolic degree of the minimal parabolic semistable subquotient of E_*, or in other words, $d^{\mathrm{par}}_{\min}(E_*)$ is the parabolic degree of the final piece of the graded object for the Harder–Narasimhan filtration of E_*.

Now, as for Corollary 3.3, the Proposition 3.2 combines with Theorem 2.5 to give the following corollary:

COROLLARY 3.4

Let E_* be a parabolic vector bundle with rational parabolic weights. If E_* is parabolic nef, then $d^{\mathrm{par}}_{\min}(E_*) \geq 0$. If $\dim X = 1$, then the converse is also true; namely, if the inequality $d^{\mathrm{par}}_{\min}(E_*) \geq 0$ is valid, then E_* must be parabolic nef.

A parabolic vector bundle E_* will be called *numerically flat* if both E_* and its parabolic dual E_*^* are parabolic nef.

Let E_* be a numerically flat parabolic bundle over X with rational parabolic weights and with parabolic structure over a normal crossing divisor D. Let $V \longrightarrow Y$ be the orbifold bundle corresponding to E_* for a suitable Galois covering map

$$p : Y \longrightarrow X$$

with Galois group G. Proposition 3.2 says that V is numerically flat, i.e., both V and V^* are nef. Now Proposition 2.6 says that V is semistable with $c_1(V) = 0 = c_2(V)$.

Since V is semistable, from [Bi1] it follows that E_* is parabolic semistable. Since the first and the second Chern class of V vanish, from [Bi3] it follows that the first and the second parabolic Chern class of E_* vanish.

Conversely, if E_* is parabolic semistable with its first and the second parabolic Chern class zero, then from [Bi1] and [Bi3] we know that the corresponding orbifold bundle V is semistable with the first and the second Chern class of V being zero. So Proposition 2.6 yields that V is numerically flat. Now Proposition 3.2 says that the parabolic bundle E_* is numerically flat.

Thus we have proved the following theorem:

Theorem 3.5. *A parabolic bundle E_* with rational parabolic weights is numerically flat if and only if E_* is parabolic semistable with vanishing first and second parabolic Chern classes.*

Using [Bi3], from the above theorem it is easy to deduce that a parabolic vector bundle E_* over X, with rational parabolic weights, is numerically flat if and only if the following condition is valid: the underlying vector bundle E for the parabolic vector bundle E_* has a filtration by subbundles of E such that each subsequent quotient vector bundle equipped with the induced parabolic structure, induced by E_*, corresponds to a unitary representation of the fundamental group of the complement $X - D$, where D is the divisor on X over which the parabolic structure of E_* is defined. The above statement can also be deduced using Proposition 3.2 together with [DPS, Theorem 1.18].

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On the existence of automorphisms with simple Lebesgue spectrum

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Abstract. It is shown that if T is a measure preserving automorphism on a probability space $(\Omega, \mathcal{B}, m)$ which admits a random variable X_0 with mean zero such that the stochastic sequence $X_0 \circ T^n, n \in \mathbb{Z}$ is orthonormal and spans $L_0^2(\Omega, \mathcal{B}, m)$, then for any integer $k \neq 0$, the random variables $X \circ T^{nk}, n \in \mathbb{Z}$ generate $\mathcal{B}$ modulo m.

Keywords. Dynamical system; simple Lebesgue spectrum; Walsh function.

1. Introduction

The purpose of this note is to show that if there is an automorphism with simple Lebesgue spectrum, then the probability measure of the corresponding stationary processes must be supported in a small subset of $\mathbb{C}^{\mathbb{Z}}$, in some precise sense explained below.

Let $\mathbb{C}$ denote the set of complex numbers with its Borel σ-algebra. Let $\Omega_0 = \mathbb{C}^{\mathbb{Z}}$ with the product σ-algebra $\mathcal{B}_0$ and let T be the left shift

$$T(\omega_n)_{n=-\infty}^{\infty} = (y_n)_{n=-\infty}^{\infty}, \quad y_n = \omega_{n+1}, \quad n \in \mathbb{Z}.$$

Suppose there exists a probability measure m on Ω_0 invariant under T and such that:

(i) the coordinate functions X_n, $n \in \mathbb{Z}$, $X_n(\omega) = \omega_n$, $\omega \in \Omega_0$, all have mean zero, they are square integrable and mutually orthogonal,

(ii) X_n, $n \in \mathbb{Z}$, span $L_0^2(\Omega_0, \mathcal{B}_0, m)$, the linear subspace of functions in $L^2(\Omega_0, \mathcal{B}_0, m)$ with mean zero.

Write:

$$X = \prod_{n=-\infty}^{\infty} \mathbb{C}_{2n+1}, \quad Y = \prod_{n=-\infty}^{\infty} \mathbb{C}_{2n}, \quad \mathbb{C}_{2n} = \mathbb{C}_{2n+1} = \mathbb{C},$$

and view Ω_0 as the product $X \times Y$. If $\omega = (\omega_n)_{n=-\infty}^{\infty} \in \Omega_0$, then

$$\zeta = (\omega_{2n+1})_{n=-\infty}^{\infty} \in X, \quad \eta = (\omega_{2n})_{n=-\infty}^{\infty} \in Y$$

and we identify (ζ, η) with ω.

Let m_1 and m_2 denote the projections of m on X and Y respectively. Define $T_1 : X \longrightarrow Y$, $T_2 : Y \longrightarrow X$ as follows:

$$T_1(\omega_{2n+1})_{n=-\infty}^{\infty} = (y_{2n})_{n=-\infty}^{\infty}, \quad y_{2n} = \omega_{2n+1},$$
$$T_2(\omega_{2n})_{n=-\infty}^{\infty} = (y_{2n+1})_{n=-\infty}^{\infty}, \quad y_{2n+1} = \omega_{2n+2}.$$

Then

$$T(\zeta,\eta)=(T_2\eta,T_1\zeta);\quad (\zeta,\eta)\in X\times Y.$$

It is shown that m (if one such exists) is supported on a set S in Ω_0 such that the projection maps onto X and Y, when restricted to S, are one-to-one. It follows that T^2, $T_1\circ T_2$, $T_2\circ T_1$ are isomorphic and the σ-algebra generated by the variables X_{2n}, $n\in\mathbb{Z}$, is, modulo m-null sets, equal to the Borel σ-algebra of Ω_0.

For the case when $\Omega_0=\{-1,+1\}^{\mathbb{Z}}$, it is shown that such a measure, if it exists, admits a support which is wandering under the natural action of the subgroup of Ω_0 consisting of sequences $\omega_n, n\in\mathbb{Z}$, with all but finitely many $\omega_n=+1$.

2. Orthogonal decompositions

DEFINITION 2.1

Let $(X,\mathcal{B}_X),(Y,\mathcal{B}_Y)$ be standard Borel spaces in the sense that each is isomorphic to the unit interval equipped with its Borel σ-algebra. We say that a probability measure m on $(X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y)$ is *good*, if for every complex valued measurable function f on $X\times Y$ there exist complex valued measurable functions u on X and v on Y such that

$$f(x,y)=u(x)+v(y)\ m\text{–a.e.}\tag{1}$$

The set of measure zero where (1) fails to hold may depend on f.

Let

$$L_0^2(X\times Y,m)=\{f\in L^2(X\times Y,m): E(f)=0\},$$

where $E(f)=\int_{X\times Y}f\,\mathrm{d}m$, denotes the expected value of f. We say that m is *very good* if every function $f\in L_0^2(X\times Y,m)$ can be expressed in the form (1) with $u\in L_0^2(X,m_1)$ and $v\in L_0^2(Y,m_2)$ satisfying $E(u\cdot\overline{v})=0$, where m_1, m_2 denote the projections of m on X and Y respectively (also called *marginal measures*).

2.2. In this paper we will be concerned with very good measures although good measures seem relevant for study of measure preserving automorphism whose associated unitary operators have multiplicity one. Theorem 2.4 gives a necessary and sufficient condition on m under which it is a very good measure. The second half of the proof of this theorem is obtained with the assistance of the referee, improving an earlier weaker result.

DEFINITION 2.3

A measurable subset $S\subseteq X\times Y$ is called *a measurable couple*, if $S=G\cup H$, where G is the graph of a measurable function g defined on the measurable subset $A\subseteq X$ into Y, H is the graph of a measurable function h defined on the measurable subset $B\subseteq Y$ into X and

$$g(A)\cap B=h(B)\cap A=\emptyset.$$

Theorem 2.4. *A probability measure m on $(X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y)$ is very good if and only if it is supported on a measurable couple.*

Proof. Assume that the measure m is supported on a measurable couple $S=G\cup H$. Let $f\in L_0^2\ (X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y,m)$. Without loss of generality assume that $m(H)\neq 0$. Then,

since m is supported on a measurable couple, $m(H) = m_1(X \setminus A) = m_2(B) \neq 0$. Since $E(f) = 0$, we see that $\int_G f + \int_H f = 0$. Write $\int_G f = a$. Define

$$u(x) = \begin{cases} f(x, g(x)) & \text{if } x \in A, \\ -\frac{a}{m_1(X \setminus A)} & \text{if } x \in X \setminus A, \end{cases}$$

$$v(y) = \begin{cases} f(h(y), y) + \frac{a}{m_1(X \setminus A)} & \text{if } y \in B, \\ 0 & \text{if } y \in Y \setminus B. \end{cases}$$

(In case $G = \emptyset$, $u(x) = 0$ for all $x \in X$.) We note that (1) holds for all $(x, y) \in G \cup H$ and

$$Eu = Ev = E(u \cdot \overline{v}) = 0.$$

Assume now that m is very good. Let Π_1, Π_2 denote the projections of $X \times Y$ onto X and Y respectively and let $\mathcal{B}_X, \mathcal{B}_Y$ denote also the σ-algebras $\Pi_1^{-1}(\mathcal{B}_X)$ and $\Pi_2^{-1}(\mathcal{B}_Y)$ respectively. Write

$$E^X f = E(f \mid \mathcal{B}_X), \;\; E^Y f = E(f \mid \mathcal{B}_Y).$$

If $f \in L^2(X \times Y, m)$ and $E^X f = 0$, then $f(x, y) = v(y)$ a.e.

To see this note that if $E^X(f) = 0$, then $E(f) = 0$, and since m is very good, we can write

$$f(x, y) = u(x) + v(y) \; m - \text{a.e.}$$

with

$$E(u) = E(v) = E(u \cdot \overline{v}) = 0.$$

Since $E^X(f) = 0$ and u is $\mathcal{B}_X$-measurable

$$E(\overline{u} \cdot f) = E(\overline{u} E^X(f)) = 0 = E(\mid u \mid^2) + E(\overline{u} \cdot v),$$

which implies that $u(x) = 0 \; m_1$ – almost everywhere and then $f(x, y) = v(y) \; m$ – a.e.. So f is $\mathcal{B}_Y$-measurable. Similarly, if $g \in L^2(X \times Y, m)$ satisfies $E^Y g = 0$, then $g(x, y) = u(x)$ m – almost everywhere.

If $f, g \in L^\infty(X \times Y, m)$ and $E^X(f) = E^Y(g) = 0$, then $f \cdot g = 0 \; m$ – a.e.

To see this, let $h = f \cdot g$. Since $E^Y g = 0$ and f is $\mathcal{B}_Y$-measurable we have

$$E(h) = E(f \cdot E^Y(g)) = 0.$$

Since h is bounded, it is square integrable. Since m is very good we can write

$$h(x, y) = t(x) + s(y) \; m - \text{a.e.}$$

with $E(t) = E(s) = E(t \cdot \overline{s}) = 0$.

Again, since $E^X(f) = 0$ and t, g are $\mathcal{B}_X$-measurable,

$$E(\overline{t} \cdot h) = E(\overline{t} \cdot g \cdot E^X f) = 0,$$

so that $E(\mid t \mid^2) = 0$. Similarly $E(\mid s \mid^2) = 0$ and $h(x, y) = 0 \; m$ – a.e.

We conclude that f vanishes where g does not and that g vanishes where f does not.

Now take $\phi \in L^\infty(Y, m_2)$ and $\psi \in L^\infty(X, m_1)$, identified with $\phi \circ \Pi_2$ and $\psi \circ \Pi_1$. Letting $f = \phi - E^X(\phi)$, $g = \psi - E^Y(\psi)$, it follows that there exists a measurable set K in $X \times Y$ such that

$$\phi(y) = (E^X \phi)(x) \text{ for } m \text{ - almost all } (x, y) \in K,$$

$$\psi(x) = (E^Y \psi)(y) \text{ for } m \text{ - almost all } (x, y) \in X \times Y \setminus K.$$

Since $m|_K$ is supported on $G = \{(x,y) : \phi(y) = E^X\phi(x)\}$ and $m|_{X\times Y\setminus K}$ is supported on $H = \{(x,y) : \psi(x) = E^Y\psi(y)\}$, if we choose for ϕ, ψ one-to-one Borel maps of Y and X onto $[0,1]$, then G and H are measurable graphs of functions defined on subsets of X and Y respectively. Thus m is supported by a union of measurable graphs.

Moreover, it is supported by a couple. Indeed, fix a one-to-one bounded function ψ and consider a sequence of bounded functions ϕ_n, $n \in \mathbb{N}$, which is dense in $L^2(Y, m_2)$. Let K be the intersection of the sets K_n corresponding to the pair ϕ_n, ψ. Each $f_n = \phi_n - E^X\phi_n$ satisfies $E^X f_n = 0$ and thus is equal a.e. to a function of y. Hence $K_n = \{(x,y) : f_n(y) = 0\}$, $(\mathrm{mod}\, m)$ and $K = X \times B, (\mathrm{mod}\, m)$ with $B = \bigcap_{i=1}^{\infty}\{y : f_n(y) = 0\}$. Since $\phi_n, n \in \mathbb{N}$, are dense in $L^2(Y, m_2)$,

$$\phi(y) = E^X(\phi)(x) \text{ for } m - \text{almost all } (x,y) \in X \times B$$

holds for every $\phi \in L^2(Y, m_2)$.

Let then ϕ be the indicator function of $X \times B$. We find $1 = E^X\phi\; m$ - a.e. on $X \times B$. Let $A = \{x \in X : (E^X\phi)(x) = 1\}$. Since $E^X(\phi) = 1$ on $X \times B$, we see that the part of m on $A \times Y$ is concentrated on $A \times B$, whence $m(A \times (Y \setminus B)) = 0$. Since for $(x,y) \in (X \setminus A) \times B$, $E^X\phi(x) \neq 1 = \phi(x,y)$, we see that $m((X \setminus A) \times B) = 0$. Since the restrictions of m to $A \times B(\subset K)$ and $(X \setminus A) \times (Y \setminus B)(\subset X \times Y \setminus K)$ are supported on graphs, the theorem follows.

3. Connection with dynamics, twisted joining

3.1. Let T_1 be a Borel isomorphism from X onto Y and let T_2 be a Borel isomorphism from Y onto X. Define the Borel automorphism T of $X \times Y$ by

$$T(x,y) = (T_2 y, T_1 x), \quad (x,y) \in X \times Y.$$

We assume that T preserves the measure m on $\mathcal{B}_X \otimes \mathcal{B}_Y$ and that T^2 is ergodic. It follows that $m_2 \circ T_1 = m_1$ and $m_1 \circ T_2 = m_2$. Further $T_2 \circ T_1 : X \longrightarrow X$ preserves the measure m_1 and $T_1 \circ T_2 : Y \longrightarrow Y$ preserves the measure m_2.

It is obvious that

$$T^2(x,y) = (T_2 \circ T_1(x), T_1 \circ T_2(y)), \quad (x,y) \in X \times Y,$$

and the automorphisms T, $T_1 \circ T_2$, $T_2 \circ T_1$ are ergodic on the respective spaces.

Theorem 3.2. *Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be standard Borel spaces in the sense that each is Borel isomorphic to the unit interval with its Borel σ-algebra. Let $(\Omega, \mathcal{B}) = (X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y)$ and m be a probability measure on $\mathcal{B}$. Let $T_1 : X \longrightarrow Y$, $T_2 : Y \longrightarrow X$ be Borel automorphisms such that $T : (x,y) \longrightarrow (T_2 y, T_1 x)$ is measure preserving and T^2 is ergodic.*

Assume that m is very good. Then m is supported on the graph of a one-to-one measurable function on a subset of X (and hence also on a subset of Y). The automorphisms T^2, $T_1 \circ T_2$ and $T_2 \circ T_1$ are isomorphic.

Proof. Since m is very good, it is supported on a measurable couple, say $S = G \cup H$, where G is the graph of a measurable function g defined on a measurable subset $A \subset X$, and H is the graph of a measurable function h defined on a measurable subset $B \subset Y$. Without loss of generality assume that $m(G) > 0$. Let $P_x(\cdot), x \in X$, denote the regular conditional probability measure with respect the σ-algebra $\Pi_1^{-1}(\mathcal{B}_X)$, also written as $\mathcal{B}_X$ (by abuse of notation). For each $x \in X$, $P_x(\cdot)$ is a probability measure supported on

$\{x\}\times Y$, such that for any $A\in\mathcal{B}_X\otimes\mathcal{B}_Y$, $P_{(\cdot)}(A)$ is $\mathcal{B}_X$-measurable and

$$m(A)=\int_X P_x(A)\,dm_1.$$

From the construction of $P_x(\cdot), x\in X$, it is easy to see that the $G_1=\{(x,y): P_x\{(x,y)\}=1\}$ is T^2 invariant. It is also the graph of a measurable function on $\Pi_1 G_1$. Clearly, $G\subset G_1$ and $m(G)>0$ by assumption, so, by ergodicity of T^2, $m(G_1)=1$. Moreover $m(TG_1)=1$ and TG_1 is the graph of a measurable function on a measurable subset Y. Clearly m is supported on $G_1\cap TG_1$, the graph of a one-to-one measurable function on a measurable subset of X (hence also the graph of a measurable function on measurable subset of Y). The projection map Π_1 is a measure preserving isomorphism of $(X\times Y,\mathcal{B}_X\otimes\mathcal{B}_Y,m)$ and $(X,\mathcal{B}_X,m_1)$. Further $T^2=\Pi_1^{-1}\circ T_2\circ T_1\circ\Pi_1$. Similarly, T^2 and $T_1\circ T_2$ are isomorphic. The theorem is proved.

4. Application to the problem of simple Lebesgue spectrum

4.1. Now we apply Theorem 3.2 to the problem stated in the introduction. With the notation therein, every function in $L_0^2(\Omega_0,m)$ is an orthogonal sum of functions in $L_0^2(X,m_1)$ and $L_0^2(Y,m_2)$. Indeed, if $f\in L_0^2(\Omega_0,m)$ has the expansion $f=\sum_{n=-\infty}^{\infty}c_nX_n$, then we can set $u=\sum_{n=-\infty}^{\infty}c_{2n+1}X_{2n+1}$ and $v=\sum_{n=-\infty}^{\infty}c_{2n}X_{2n}$. So m is very good. By Theorem 3.2 m is supported on a set S in Ω_0 on which the projections maps in X and Y are one-to-one and so T^2, $T_1\circ T_2$, $T_2\circ T_1$ are isomorphic.

F Parreau has asked if σ-algebra generated by $X_{kn}, n\in\mathbb{Z}$, is equal to the Borel σ-algebra of Ω_0 (modulo m-null sets). This is indeed the case, as Theorem 3.2 has the following generalisation (Theorem 4.3), proved by a similar method.

4.2. Let $(X_i,\mathcal{B}_i)$, $1\le i\le k$ be, as before, standard Borel spaces. Let

$$\Omega_0=\prod_{i=1}^{k}X_i,\quad \mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2\otimes\cdots\otimes\mathcal{B}_k.$$

Call a probability measure m on $\mathcal{B}$ k – *very good*, if every $f\in L_0^2(X,m)$ can be written in the form

$$f(x_1,x_2,\ldots,x_k)=u_1(x_1)+u_2(x_2)+\cdots+u_k(x_k)$$

with $E(u_i)=0$ for all $1\le i\le k$ and $E(u_i\cdot\overline{u_j})=0$ for all $i\ne j$.

Let $T_i: X_i\longleftrightarrow X_{i+1}$, $1\le i\le k$, be Borel automorphisms, where $X_{k+1}=X_1$. Define T on X by

$$T(x_1,x_2,\ldots,x_k)=(T_kx_k,T_1x_1,T_2x_2,\ldots,T_{k-1}x_{k-1}).$$

Theorem 4.3. *Assume that m is a T-invariant and k-very good probability measure on $\mathcal{B}$ such that T^k is ergodic with respect to m. Then m is supported on a set $S\subset\Omega_0$ on which the projection maps $\Pi_1,\Pi_2,\ldots,\Pi_k$ into $X_1,X_2,\ldots,X_k$ respectively are one-to-one.*

Remark 4.4. It is also clear that if A and B are disjoint subsets of $\mathbb{Z}$, $A\cup B=\mathbb{Z}$ and if $X=\prod_{i\in A}\mathbb{C}_i$, $Y=\prod_{i\in B}\mathbb{C}_i$, where $\mathbb{C}_i=\mathbb{C}$ for all $i\in\mathbb{Z}$, then m, if one such exists, is supported on a measurable couple in $X\times Y$. It seems plausible that such an m is supported on a set in Ω_0 on which projections into the coordinate spaces is one-to-one, in

which case the projection of m on any of the coordinate spaces of Ω_0 cannot be discrete, or even admit a discrete component.

5. Walsh functions

5.1. Let $(\Omega, \mathcal{F}, P)$ be a non-atomic probability space. Does there exist a one-to-one and onto measure preserving transformation $T: \Omega \longrightarrow \Omega$ and an event $A \in \mathcal{F}$, $P(A) = \frac{1}{2}$, such that if

$$X(\omega) = \begin{cases} +1 & \text{if } \omega \in A, \\ -1 & \text{if } \omega \in \Omega \setminus A, \end{cases}$$

then the random variables $X \circ T^n$, $n \in \mathbb{Z}$, are pairwise independent and span $L_0^2(\Omega, \mathcal{F}, P)$?

5.2. Let us reformulate the above question differently breaking it into two parts.

Let $\tilde{\Omega} = \prod_{k \in \mathbb{Z}} \{-1, +1\}_k$, $\{-1, +1\}_k = \{-1, +1\}$, equipped with the usual product topology and the resulting Borel structure $\tilde{\mathcal{B}}$; an element $\tilde{\omega} \in \tilde{\Omega}$ is a bilateral sequence $\{\omega_k\}_{k \in \mathbb{Z}}$ of $+1$ and -1.

Does there exist a probability measure μ on $\tilde{\mathcal{B}}$ such that:

(i) the coordinates X_k, $k \in \mathbb{Z}$, are pairwise independent and they span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$?
(ii) moreover, can we choose the measure μ to be invariant under the left shift in $\tilde{\Omega}$?

5.3. The first question has a positive answer provided by the family of Walsh functions defined below.

Expand a real number $x \in [0, 1]$ in its binary form $x = 0.x_1 x_2 \ldots x_k \ldots$, which is made unique by insisting that if there are two such expansions, we choose the one with infinitely many ones. Define

$$R_k(x) = 2x_k - 1, \; k \in \mathbb{Z},$$

equivalently

$$R_k(x) = \begin{cases} +1 & \text{if } x_k = 1, \\ -1 & \text{if } x_k = 0. \end{cases}$$

These are called *Rademacher functions*. They are independent, and since $\int_0^1 R_k(x) \mathrm{d}x = 0$, they are also orthogonal, but they do not span $L_0^2[0, 1]$. However, the collection of all distinct finite products

$$W_{i_1, i_2, \ldots, i_k} = R_{i_1} R_{i_2} \cdots R_{i_k}; \quad i_1 < i_2 < \cdots < i_k,$$

called the *Walsh functions*, is mutually orthogonal and span $L_0^2[0, 1]$. Since they assume only two distinct values, they are also pairwise independent.

5.4. There is another way of viewing Walsh functions. Consider $\tilde{\Omega}$ as a compact group with coordinatewise multiplication and let h denote the normalised Haar measure on $\tilde{\Omega}$. If to each coordinate space $\{-1, +1\}$ we give uniform probability distribution, then h is the product of these measures. With respect to the measure h, the coordinate functions X_k, $k \in \mathbb{Z}$, correspond to the Rademacher functions.

The finite products $X_{i_1} X_{i_2} \cdots X_{i_k}$, $i_1 < i_2 < \cdots < i_k$, which is the collection of non-trivial continuous characters of $\tilde{\Omega}$, correspond to the Walsh functions.

Let f_k, $k \in \mathbb{Z}$, be an enumeration of Walsh functions $W_{i_1,i_2,\dots,i_k}$ on $[0,1]$. Write:

$$\psi(x) = \{f_k(x)\}_{k\in\mathbb{Z}},\ x \in [0,1].$$

The unit interval is mapped by ψ in a one-to-one Borel manner into $\tilde{\Omega}$. Let $\mu_W(A) = \lambda \circ \psi^{-1}(A)$, $A \in \tilde{\mathcal{B}}$, where λ denotes the Lebesgue measure on $[0,1]$. The coordinate functions X_k, $k \in \mathbb{Z}$, are pairwise independent and span $\mathcal{L}_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu_W)$. This gives the affirmative answer to the first question. We shall call μ_W the measure induced by an enumeration of Walsh functions. It was pointed out by F Parreau to one of us that such a μ_W is not invariant under the shift. In the sequel we will give a description of μ_W, from which this will follow.

5.5. The second question remains unsolved. A positive answer to it will solve the problem of the simple Lebesgue spectrum affirmatively. At this point we mention that an example (or rather a family of examples) of mixing rank one transformation, due to D Ornstein ([1]), allows us to construct a strictly stationary processes $\{f_k\}_{k\in\mathbb{Z}}$ such that $\int_{\tilde{\Omega}} f_k f_0 \mathrm{d}m \longrightarrow 0$ as $k \to \infty$, while $\{f_k;\ k \in \mathbb{Z}\}$ span $L_0^2(\tilde{\Omega}, \mathcal{F}, m)$. Ornstein's example is deep and has not so far been modified or improved to yield a transformation with simple Lebesgue spectrum.

5.6. Let $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ consisting of those $\omega \in \tilde{\Omega}$, which have only finitely many -1; it is a countable dense subgroup of $\tilde{\Omega}$. The action of $\tilde{\Omega}_0$ on $\tilde{\Omega}$, $\omega \longrightarrow \omega\omega_0$, $\omega \in \tilde{\Omega}$, $\omega_0 \in \tilde{\Omega}_0$, is uniquely ergodic, the Haar measure h being the only probability measure invariant under the $\tilde{\Omega}_0$ action. Other product measures are quasi-invariant and ergodic under this action and there are many other measures with respect to which this action is non-singular and ergodic. The theorem below shows that all these measures are singular to any measure μ for which the coordinate functions are orthogonal and span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$.

Theorem 5.7. *If μ is a probability measure on $\tilde{\Omega}$ such that the coordinate functions are pairwise independent and span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$, then there is a Borel set E which supports μ and which is wandering under the $\tilde{\Omega}_0$ action, i.e., $\omega_0 E, \omega_0 \in \tilde{\Omega}_0$, are pairwise disjoint. In the case when μ is given by an enumeration of Walsh functions, μ is the Haar measure on a closed subgroup of $\tilde{\Omega}$.*

Proof. We begin with an observation of J B Robertson [2]. If the coordinate functions X_k, $k \in \mathbb{Z}$, are pairwise independent, then they span $L_0^2(\tilde{\Omega}, \tilde{\mathcal{B}}, \mu)$ if and only if for all i, j,

$$X_i X_j = \sum_{k=-\infty}^{\infty} c_k^{i,j} X_k, \quad \sum_{k=-\infty}^{\infty} | c_k^{i,j} |^2 = 1.$$

Further

$$c_k^{i,j} = \int_{\tilde{\Omega}} X_i X_j X_k \,\mathrm{d}\mu \longrightarrow 0,$$

if any one of i, j, k, tends to $+\infty$.

(Note that $X_i X_j$ is of absolute value one, hence its L^2-norm is 1, and the sum over k of the squares of $c_k^{i,j}$ is one.)

The sum $\sum_{k=-\infty}^{\infty} c_k^{i,j} X_k$, converges in L^2, hence (by diagonal method) there exists an increasing sequence N_l, $l \in \mathbb{N}$, of natural numbers such that for all i, j,

$$X_iX_j(\omega) = \lim_{l\to\infty} \sum_{k=-N_l}^{N_l} c_k^{i,j} X_k(\omega)$$

for almost all $\omega \in \tilde{\Omega}$ with respect to μ. Let

$$E_{i,j} = \left\{\omega\colon X_iX_j(\omega) = \lim_{l\longrightarrow\infty} \sum_{k=-N_l}^{N_l} c_k^{i,j} X_k(\omega)\right\},$$

$$E = \bigcap_{-\infty<i,j<\infty} E_{i,j},$$

which is a support of μ. Let $\omega_0 \in \tilde{\Omega}_0$ with -1 at places $i_1, i_2, \ldots, i_p$ and $+1$ at the remaining places. Take an $i \notin \{i_1, i_2, \ldots, i_p\}$ and $j \in \{i_1, i_2, \ldots, i_p\}$, then for all $\omega \in E$

$$X_iX_j(\omega) = c_{i_1}^{i,j} X_{i_1}(\omega) + \ldots + c_{i_p}^{i,j} X_{i_p}(\omega) + \lim_{l\to\infty} \sum_{-N_l \le k \le N_l}^{*} c_k^{i,j} X_k(\omega),$$

where $\sum^*$ indicates that the terms $c_{i_1}^{i,j} X_{i_1}, \ldots, c_{i_p}^{i,j} X_{i_p}$ are deleted from the sum. Assume that for some $\omega_0 \in \tilde{\Omega}_0$, $\omega_0 E \cap E \neq \emptyset$. Then there exists $\omega = \{\omega_k\}_{k\in\mathbb{Z}} \in E$, such that $\omega_0\omega \in E$. We have

$$X_i(\omega) = \omega_i, \quad X_i(\omega_0\omega) = \omega_i, \quad X_j(\omega) = \omega_j, \quad X_j(\omega_0\omega) = -\omega_j,$$

so that

$$\omega_i\omega_j = c_{i_1}^{i,j}\omega_{i_1} + \cdots + c_{i_p}^{i,j}\omega_{i_p} + \lim_{l\to\infty} \sum_{-N_l \le k \le N_l}^{*} c_k^{i,j}\omega_k,$$

$$-\omega_i\omega_j = -c_{i_1}^{i,j}\omega_{i_1} - \cdots - c_{i_p}^{i,j}\omega_{i_p} + \lim_{l\to\infty} \sum_{-N_l \le k \le N_l}^{*} c_k^{i,j}\omega_k,$$

whence,

$$\omega_i\omega_j = c_{i_1}^{i,j}\omega_{i_1} + \cdots + c_{i_p}^{i,j}\omega_{i_p}.$$

This holds for all $i \notin \{i_1, i_2, \ldots, i_p\}$. Letting $i \longrightarrow \infty$, since $c_{i_1}^{i,j}, \ldots, c_{i_p}^{i,j} \longrightarrow 0$, the right hand side of the above equality tends to zero, while the left hand side remains one in absolute value. The contradiction shows that $\omega_0\omega \notin E$, whence $\omega_0 E \cap E = \emptyset$.

Suppose now that μ_W is obtained by an enumeration of Walsh functions. In this case X_iX_j, $i \neq j$, is some X_l. Write $l = g(i, j)$. Then

$$X_iX_j = X_{g(i,j)},$$

so that in the expansion

$$X_iX_j = \sum_k c_k^{i,j} X_k$$

all $c_k^{i,j} = 0$ except for $k = g(i, j)$, in which case $c_k^{i,j} = 1$.

Now $E_{i,j} = \{\omega : \omega_i\omega_j = \omega_{g(i,j)}\}$. The sets $E_{i,j}$ are closed subgroups of $\tilde{\Omega}$. The same is true for the set $E = \bigcap_{-\infty<i,j<\infty} E_{i,j}$. The characters $X_{i_1}X_{i_2}, \ldots, X_{i_k}$ of $\tilde{\Omega}$ are also the characters of E, but they need not be distinct characters. In particular X_iX_j and $X_{g(i,j)}$ agree on E. Further $X_{i_1}X_{i_2}, \ldots, X_{i_k}$ is either equal to some X_p or equal to one. We have

$\int_E X_{i_1} X_{i_2}, \ldots, X_{i_k} \, d\mu_W$ equal to 0 in the first case and equal to 1 in the second, which also holds if μ_W is replaced by the normalised Haar measure on E. Thus μ_W is the normalised Haar measure on E and the theorem is proved.

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When is $f(x,y) = u(x) + v(y)$?

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Abstract. Let X and Y be arbitrary non-empty sets and let S a non-empty subset of $X \times Y$. We give necessary and sufficient conditions on S which ensure that every real valued function on S is the sum of a function on X and a function on Y.

Keywords. Subsets of cartesian products; good set.

1. Introduction

In this note we characterise subsets S of $X \times Y$ with the property that every complex valued function f on S can be expressed in the form

$$f(x,y) = u(x) + v(y), \quad (x,y) \in S, \tag{1}$$

where u and v are functions on X and Y respectively. The question is motivated by the preceding paper [1] where similar subsets occur as supports of measures associated of certain stochastic processes of multiplicity one.

2. Good sets

DEFINITION 2.1

We say that a subset $\emptyset \neq S \subseteq X \times Y$ is *good* if every complex valued function f on S can be expressed in the form (1).

It is obvious that any non-empty subset of a good set has also this property, but there exist sets which are not good and such that all proper subsets of it are good.

The purpose of this section is to describe good subsets of $X \times Y$, when X and Y are finite.

2.2. Let $\Pi_1 : X \times Y \longrightarrow X$ and $\Pi_2 : X \times Y \longrightarrow Y$ be the projections on X and Y respectively. If S is good, then any function $f : S \longrightarrow \mathbb{C}$, $f = u + v$, is completely determined by the values of u on $\Pi_1 S$ and v on $\Pi_2 S$. Therefore it is not a severe restriction on a good set S, if we assume in addition that $\Pi_1 S = X$ and $\Pi_2 S = Y$. This assumption will be made whenever necessary.

2.3. Assume that X and Y are *finite* with m and n elements respectively. We begin with the observation that a good set must be "thin" in the sense that it can have at most $m + n - 1$ points.

Let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and $S = \{s_1, s_2, \ldots, s_k\}$, where

$$s_1 = (x_{i_1}, y_{j_1}),\ s_2 = (x_{i_2}, y_{j_2}), \ldots, s_k = (x_{i_k}, y_{j_k}).$$

We consider the $k \times (m+n)$-matrix M (called *the matrix of S*) with rows $M_p,\ 1 \leq p \leq k$, given by

$$M_p = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 occurs at the places i_p and $m + j_p$, corresponding to the subscripts in the pair $s_p = (x_{i_p}, y_{j_p})$. Since S is good,

$$f(s_p) = f(x_{i_p}, y_{j_p}) = u(x_{i_p}) + v(y_{j_p}), \quad 1 \leq p \leq k.$$

We put

$$u(x_{i_1}) = \zeta_{i_1}, \ldots, u(x_{i_m}) = \zeta_{i_m}, \quad v(y_{j_1}) = \eta_{j_1}, \ldots, v(y_{j_n}) = \eta_{j_n}.$$

The relation (1) gives us k equalities

$$\zeta_{i_p} + \eta_{j_p} = f(s_p), \quad 1 \leq p \leq k.$$

In other words, the column vector $(\zeta_1, \ldots, \zeta_m, \eta_1, \ldots, \eta_n)^t \in \mathbb{C}^{m+n}$ is a solution of the matrix equation

$$M\vec{z} = \vec{\alpha}, \tag{2}$$

where $\vec{\alpha} = (f(s_1), f(s_2), \ldots, f(s_k))^t \in \mathbb{C}^k$.

Since S is good, we know that (2) has solution for every $\vec{\alpha}$. Since M has $m+n$ columns and since the vector $(\underbrace{1, 1, \ldots, 1}_{\text{m times}}, \underbrace{-1, \ldots, -1}_{\text{n times}})^t$ is a solution of the homogeneous equation $M\vec{z} = \vec{0}$, we see that its rank is at most $m+n-1$. Clearly k cannot exceed the rank of M.

On the other hand the set $S = \{\{x_1\} \times Y\} \cup \{X \times \{y_1\}\}$, the union of two "axes", is a good subset of $X \times Y$ of cardinality $m+n-1$. We have proved:

PROPOSITION 2.4

Let X, Y, $S \subseteq X \times Y$ be finite sets with m, n and k elements respectively; $\Pi_1 S = X$, $\Pi_2 S = Y$. Then S is good if and only if $k \leq m+n-1$ and the matrix M of S defined above has rank k. There always exists a good set of cardinality $k \leq m+n-1$.

DEFINITION 2.5

Let $S \subseteq X \times Y$. (We do not assume that X and Y are finite and S is not assumed to be good.) We say that a point $s = (x_0, y_0) \in S$ is *isolated in the vertical direction* (resp. *isolated in the horizontal direction*) if $(\{x_0\} \times Y) \cap S$ (resp. $(X \times \{y_0\}) \cap S$) is a singleton.

2.6. Let $S \subseteq X \times Y$ be an *arbitrary* subset of the cardinality $\leq m+n-1$ with $\Pi_1 S = X$ and $\Pi_2 S = Y$, $|X| = m$, $|Y| = n$, where $|A|$ denotes the cardinality of the set A. Every column of the matrix M associated to S is a non-zero vector. Each row has exactly two ones in it. Since the number of columns is $m+n$, there are at least two columns with exactly one 1 in each of them. Suppose that the jth column has exactly one 1 which occurs in the ith row. This means that s_i is isolated in the vertical direction if $1 \leq j \leq m$, and in the horizontal direction if $m+1 \leq j \leq m+n$. We cancel from M the ith row and the jth column to obtain a matrix $\tilde{M}$ and we drop from S the element s_i to

obtain a set S_1. We cancel from $\tilde{M}$ all columns which consist only of zero's and write M_1 for the matrix thus obtained. It is easy to see that M_1 is the matrix associated to S_1. If the number of rows in M_1 is greater or equal to the number of columns in M_1, then S_1 is not a good set and *a fortiori* S is not a good set. Otherwise, the number of rows in M_1 is smaller than the number of columns in M_1 and we can apply the above procedure to M_1. We obtain a reduced matrix M_2 and the smaller set S_2, of which M_2 is the associated matrix. If this process of reduction stops at a stage $l < k$, in the sense that the number of rows in M_l is greater or equal to the number of columns in M_l, then the set S is not good. If the process continues up to stage k (equal to the number of points in S), then S is good and the rank of the matrix M is equal to k. Thus we have obtained:

PROPOSITION 2.7

A subset $S \subseteq X \times Y$, where X, Y are two finite sets of cardinality m and n respectively, is good if and only if the process of reduction of the matrix M continues up to k steps, where k is the number of elements in S. Equivalently, if and only if the number of rows in M_i is smaller than the number of columns in M_i for each $1 \leq i \leq k$.

3. Graphs, couples and their unions

The sets X and Y are no more assumed to be finite in what follows, except when this assumption is explicitly stated.

PROPOSITION 3.1

If S is the graph of a function $g : E \longrightarrow Y$, where $E \subseteq X$, then S is good. Similarly, if S is the graph of a function $h : F \longrightarrow X$, where $F \subseteq Y$, then S is good.

We have a more general result:

PROPOSITION 3.2

If $S = G \cup H$, where G is the graph of a function $g : E \longrightarrow Y$, $E \subseteq X$, H is the graph of a function $h : F \longrightarrow X \setminus E$, $F \subseteq Y \setminus g(E)$, then S is good.

Proof. For any complex valued function f on S, we define

$$u(x) = \begin{cases} f(x, g(x)) & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E, \end{cases}$$
$$v(y) = \begin{cases} f(h(y), y) & \text{if } y \in F, \\ 0 & \text{if } y \in Y \setminus F, \end{cases}$$

so that (1) is satisfied.

This suggests the following:

DEFINITION 3.3

Let g be a function defined on a subset $E \subseteq X$ into Y and h be a function defined on a subset $F \subseteq Y$ into X. Let G and H be the graphs of g and h respectively. We say that the set $S = G \cup H$ is *a couple* if

$$g(E) \cap F = \emptyset, \ h(F) \cap E = \emptyset.$$

Each couple is a good set. Let us observe also that not every union of two graphs is a couple; for example, if g and h are onto, then $G \cup H$ is not a couple. Moreover, a good set need not be a couple, for example the triplet $\{(0,0),(0,1),(1,0)\}$ is a good set in $\{0,1\} \times \{0,1\}$ which is not a couple.

3.4. We define

$$G = \{(x,y) \in S: \ (x,y) \text{ is isolated in the vertical direction}\},$$
$$H = \{(x,y) \in S: \ (x,y) \text{ is isolated in the horizontal direction}\}.$$

Note that $G \cup H = (G \setminus (G \cap H)) \cup H$ and the latter can be seen as a couple, since $\Pi_1(G \setminus (G \cap H)) \cap \Pi_1 H = \emptyset$ and $\Pi_2(G \setminus (G \cap H)) \cap \Pi_2 H = \emptyset$.

Define

$$S_1 = S \setminus (G \cup H).$$

Let G_1, H_1 be obtained from S_1 in the same manner as G and H are obtained from S. Proceeding thus we get

$$S_2 = S_1 \setminus (G_1 \cup H_1), \ldots, S_{n+1} = S_n \setminus (G_n \cup H_n), \ldots$$

We note that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$. It is easy to see that each $G_i \cup H_i$ is a couple, being equal to $(G_i \setminus (G_i \cap H_i)) \cup H_i$.

A natural generalisation of Proposition 2.7 is the following:

PROPOSITION 3.5

If $\bigcap_{n=1}^{\infty} S_n = \emptyset$, then S is good. If S is good and finite, then $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

DEFINITION 3.6

Two couples $S = G_1 \cup H_1$, $S_1 = G_2 \cup H_2$ are said to be *separated*, if the sets $\Pi_1 G_1$, $\Pi_1 H_1$, $\Pi_1 G_2$, $\Pi_1 H_2$, are mutually disjoint and the same is true for the sets $\Pi_2 G_1$, $\Pi_2 H_1$, $\Pi_2 G_2$, $\Pi_2 H_2$.

In this case $S_2 = (G_1 \cup H_1) \cup (G_2 \cup H_2)$ is a couple too. More generally, it is clear that:

PROPOSITION 3.7

An arbitrary union of pairwise separated couples is a couple, hence a good set.

4. Links, linked and uniquely linked sets, loops

4.1. If S is finite and good, then at least one of the sets G or H defined in 3.4 is non-empty. The following example shows that this need not be true, if S is infinite.

Let $X = Y = \mathbb{Z}$ and $S = \{(n, n-1): \ n \in \mathbb{Z}\} \cup \{(n,n): \ n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$. No point of S is isolated in either direction. However, S is good. For, let f be any complex valued function on S. We define $u(0) = c$, where c is an arbitrary constant. This forces $v(0) = f(0,0) - c$. Having defined $v(0)$, we see that $u(1) = f(1,0) - v(0)$, $v(1) = f(1,1) - u(1)$. Proceeding thus we see that u and v are uniquely determined as soon as we fix the value of $u(0)$.

This example suggests a method of describing good subsets of $X \times Y$, which is valid also when X or Y or both are infinite.

DEFINITION 4.2

Consider two arbitrary points $(x,y), (z,w) \in S \subseteq X \times Y$ (S not necessarily good or finite). We say that (x,y), (z,w) are *linked*, if there exists a sequence $\{(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\}$ of points in S such that:

(i) $(x_1,y_1) = (x,y)$, $(x_n,y_n) = (z,w)$;

(ii) for any $1 \le i \le n-1$ *exactly one* of the following equalities holds:

$$x_i = x_{i+1},\ y_i = y_{i+1};$$

(iii) if $x_i = x_{i+1}$, then $y_{i+1} = y_{i+2}$, $1 \le i \le n-2$, and if $y_i = y_{i+1}$, then $x_{i+1} = x_{i+2}$; equivalently, it is not possible to have $x_i = x_{i+1} = x_{i+2}$ or $y_i = y_{i+1} = y_{i+2}$ for some $1 \le i \le n-2$.

The sequence $\{(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\}$ is then called *a link* (of *length n*) joining (x,y) to (z,w) and we write $(x,y)L(z,w)$.

It is easy to see that the relation L is reflexive and symmetric and a verification shows that it is also transitive, so an equivalence relation.

DEFINITION 4.3

An equivalence class under the relation L is called *a linked component* of S. If $(x,y) \in S$, then the equivalence class to which (x,y) belongs is called the *linked component* of (x,y).

4.4. Let $(x_0,y_0) \in S \subseteq X \times Y$. The linked component of (x_0,y_0) is obtained as a union $\bigcup_{n=1}^{\infty} Q_n$, where

$$Q_1 = (X \times \{y_0\}) \cap S,\ P_1 = \Pi_1 Q_1,$$
$$Q_2 = (\Pi_1^{-1} P_1) \cap S,\ P_2 = \Pi_2 Q_2,$$
$$Q_3 = (\Pi_2^{-1} P_2) \cap S,\ P_3 = \Pi_1 Q_3,$$

and so on. If n is odd, we have

$$P_n = \Pi_1 Q_n,\ Q_{n+1} = (\Pi_1^{-1} P_n) \cap S,\ P_{n+1} = \Pi_2 Q_{n+1},\ Q_{n+2} = (\Pi_2^{-1} P_{n+1}) \cap S, \dots$$

A similar description is obtained if we start from the sets

$$\tilde{Q}_1 = (\{x_0\} \times B) \cap S,\ \tilde{P}_1 = \Pi_2 \tilde{Q}_1.$$

4.5. Suppose that X and Y are standard Borel spaces and that $X \times Y$ is furnished with the product Borel structure. If $S \subseteq X \times Y$ is a Borel set, then each linked component of S is a countable union of analytic sets, hence the equivalence relation L decomposes S into at least analytic sets. We do not know whether the linked components are always Borel, or, if the partition into linked components is countably generated by Borel sets.

DEFINITION 4.6

Two points $(x,y), (z,w) \in S \subseteq X \times Y$ are said to be *uniquely linked*, if there is a unique link joining (x,y) to (z,w).

Theorem 4.7. *Let Q be linked component of S. Then the following properties are equivalent:*

(i) *any two points of Q are uniquely linked;*
(ii) *some two points of Q are uniquely linked;*
(iii) *for some $(x,y)\in Q$ the singleton $\{(x,y)\}$ is the only link joining (x,y) to itself.*

Proof. Left to the reader.

DEFINITION 4.8

A linked component of $S\subseteq X\times Y$ is said to be *uniquely linked* if any two points in it are uniquely linked.

The set S of 4.1 is uniquely linked.

DEFINITION 4.9

A non-trivial link joining (x,y) to itself is called *a loop*; by trivial link joining (x,y) to itself we mean the link consisting of the singleton $\{(x,y)\}$.

It is clear that a linked component is uniquely linked, if it has no loops. The four point set forming the vertices of a rectangle is a loop.

Theorem 4.10. *Assume that $S\subseteq X\times Y$ is linked. Then S is good if and only if it is uniquely linked.*

Proof. Assume that S is uniquely linked and let f be complex valued function on S. Let $(x_0,y_0)\in S$ and define $u(x_0)=c$, where c is a constant. This forces $v(y_0)=f(x_0,y_0)-u(x_0)$. We will now show that $u(x)$ and $v(y)$ can be defined unambiguously for all $(x,y)\in S$, so that (1) holds. Assume that we have defined $u(x)$ and $v(y)$ for all $(x,y)\in S$, which can be joined to (x_0,y_0) by a link of length n. Let $(z,w)\in S$ which is joined to (x_0,y_0) by a link of length $n+1$ and let $\{(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n),(x_{n+1},y_{n+1})\}$ be this link. Since $(x_1,y_1)=(x_0,y_0)$ is joined to (x_n,y_n) by a link of length n, by the induction hypothesis $u(x_n)$ and $v(y_n)$ are correctly defined. If $x_n=x_{n+1}$, then $u(x_{n+1})$ is also defined and $v(y_{n+1})=f(x_{n+1},y_{n+1})-u(x_{n+1})$. Note that $v(y_{n+1})$ is unambiguously defined, for (i) since S is uniquely linked, y_{n+1} cannot occur in $\{y_1,y_2,\dots,y_n\}$, (ii) no point (x,y_{n+1}) can be joined to (x_0,y_0) by a link of length $\leq n$, for if $x\neq x_{n+1}$, then $\{(x_1,y_1),(x_2,y_2),\dots,(x_{n+1}y_{n+1}),(x,y_{n+1})\}$ is the unique link (of length $>n$) joining (x_1,y_1) to (x,y_{n+1}). On the other hand, if $y_n=y_{n+1}$, then $v(y_{n+1})$, hence also $u(x_{n+1})$ is correctly defined. We set u and v equal to zero on $X\setminus\Pi_1(S)$ and $Y\setminus\Pi_2(S)$ respectively and conclude (1).

We note that u and v are uniquely determined on $\Pi_1(S)$ and $\Pi_2(S)$ up to an additive constant, since the assignment of value to $u(x_0)$ completely determines u and v on these sets.

Assume now that the set S, which is good and linked, is not uniquely linked. Then S admits a loop $\{(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\}$, $(x_1,y_1)=(x_n,y_n)$, which we can assume to be of shortest length. Then either $x_1=x_2\neq x_{n-1}$ or $y_1=y_2\neq y_{n-1}$. In either case, since S is good, any function f on S satisfies

$$f(x_1,y_1)-f(x_2,y_2)+\cdots-f(x_{n-1},y_{n-1})=0.$$

Since there are functions on S for which this fails, the theorem follows.

COROLLARY 4.11

A subset $S\subseteq X\times Y$ is good if and only if each linked component of S is uniquely linked, and also if and only if every finite subset of S is good.

Proof. Since S is good if and only if there is no link in S which is a loop, the corollary follows.

Remark 4.12. Assume that S is good. Then the decomposition $f = u + v$ is unique up to additive functions of x and y respectively, which are constant on the linked components. In other words, if $f = u + v = u_1 + v_1$, then $u - u_1$ and $v - v_1$ are constant on the linked components.

Theorem 4.13. *If a subset $S \subseteq X \times Y$ is uniquely linked, then S is of the form $G \cup H$, where G is the graph of a function g on a subset of X and H is the graph of a function h on a subset of Y.*

Proof. Fix $(x_0, y_0) \in S$. Assume that $(\{x_0\} \times Y) \cap S = \{(x_0, y_0)\}$, for simplicity. Let

$$G = \{(x,y)\colon (x,y) \text{ is joined to } (x_0,y_0) \text{ by a link of even length}\},$$
$$H = \{(x,y)\colon (x,y) \text{ is joined to } (x_0,y_0) \text{ by the link of odd length}\}.$$

We note that $S = G \cup H$ and $G \cap H = \emptyset$, since S is uniquely linked. We shall show that G is the graph of a function g on $\Pi_1 G$.

Let $(u,v), (w,z) \in G$, $(u,v) \neq (w,z)$. We show that $u \neq w$. Let $\{(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)\}$ be a link joining (x_0,y_0) to (w,z). Note that y_{n-1} must be equal to y_n, since the link is of even length. If $u = w$, then $\{(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)(= (w,z) = (u,z)), (u,v)\}$ is a link of odd length joining (x_0,y_0) to (u,v), contrary to the assumption that $(u,v) \in G$. Thus G is the graph of a function g on $\Pi_1 G$ defined by $g(x) = y$, if $(x,y) \in G$. Similarly H is the graph of a function h on $\Pi_2 H$ defined by $h(y) = x$, if $(x,y) \in H$.

We now remove the assumption that $(\{x_0\} \times Y) \cap S = \{(x_0,y_0)\}$. Let G_1 denote all those points $(x,y) \in S$, which can be joined to (x_0,y_0) by a link $\{(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)\}, (x_1,y_1) = (x_0,y_0)$, of odd length and such that $x_1 = x_2$; let G_2 denote all those points $(x,y) \in S$, which can be joined to (x_0,y_0) by a link of even length and such that $y_1 = y_2$. Similarly we define H_1 and H_2. These four sets are mutually disjoint. If $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$, then $S = G \cup H$ and as before we can show that G and H are graphs of functions on subsets of X and Y respectively. The theorem is proved.

COROLLARY 4.14

If $S \subseteq X \times Y$ is good, then S is a union of two graphs G and H of functions defined on subsets of X and Y respectively.

Proof. Let $S = \cup_\alpha S_\alpha$ be the partition of S into uniquely linked components. Note that $\Pi_1 S_\alpha \cap \Pi_1 S_\beta = \emptyset$, $\Pi_2 S_\alpha \cap \Pi_2 S_\beta = \emptyset$, if $\alpha \neq \beta$. Since each $S_\alpha = G_\alpha \cup H_\alpha$, where G_α is the graph of a function g_α on $\Pi_1 G_\alpha$ and H_α is the graph of a function h_α on $\Pi_2 H_\alpha$, we see that $S = G \cup H$, $G = \cup_\alpha G_\alpha$, $H = \cup_\alpha H_\alpha$. Moreover, G and H are graphs of functions on $\Pi_1 G$ and $\Pi_2 H$ respectively.

PROPOSITION 4.15

Let $C_i = G_i \cup H_i, i \in I$, be an indexed family of couples, where the indexing set I is totally ordered such that for any $i \in I$, $C_i \cap (\Pi_1^{-1}\Pi_1 G_j \cup \Pi_2^{-1}\Pi_2 H_j) = \emptyset$ for all $j < i$. Then $\cup_{i\in I} C_i$ is a good set.

Proof. Assume, in order to arrive at a contradiction, that $S = \cup_{i\in I}(G_i \cup H_i)$ is not good. Then S admits a loop, say $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, which is of shortest possible length. Since $\cup_{i\in I} C_i = S$ and since there are only finitely many points in the loop, there is an index p such that $G_p \cup H_p$ contains a point from this loop, but no C_i, $i < p$, contains a point of this loop. Since $C_p \cap (\Pi_1^{-1}\Pi_1 G_j \cup \Pi_2^{-1}\Pi_2 H_j) = \emptyset$ for all $j < p$, we can replace X and Y by $X \setminus \cup_{j<p}\Pi_1 G_j$, and $Y \setminus \cup_{j<p}\Pi_2 H_j$. Without loss of generality assume that $(x_1, y_1) \in G_p$. Since G_p is the graph of a function on a subset of X, each point of it isolated in the vertical direction and so we conclude that $x_2 \neq x_1$, $y_1 = y_2$, $x_{n-1} \neq x_1, y_{n-1} = y_1$. But then $(x_2, y_2), (x_3, y_3), \ldots, (x_{n-1}, y_{n-1}), (x_2, y_2)$ is a loop in S of a smaller length if $x_{n-1} \neq x_2$; otherwise $(x_2, y_2), (x_3, y_3), \ldots, (x_{n-1}, y_{n-1})$ is a loop of smaller length in S. The result follows.

It is natural to ask if the good measure as defined in 2.1 of the preceding paper [1] is supported on a good set.

References

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New integral mean estimates for polynomials

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Abstract. In this paper we prove some L^P inequalities for polynomials, where p is any positive number. They are related to earlier inequalities due to A Zygmund, N G De Bruijn, V V Arestov, etc. A generalization of a polynomial inequality concerning self-inversive polynomials, is also obtained.

Keywords. Polynomials; integral mean estimates; inequalities in the complex domain; self-inversive polynomials.

1. Introduction and statement of results

Let $\mathbb{F}_n$ be the class of polynomials $f(z)=\sum_{j=0}^{n} a_j z^j$ of degree at most n. For $f\in \mathbb{F}_n$ define

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \,\mathrm{d}\theta\right\}^{1/p}, \quad 0<p<\infty$$

and

$$\|f\|_\infty = \max_{|z|=1} |f(z)|.$$

If $f \in \mathbb{F}_n$, then according to a famous result known as Bernstein's inequality (for reference see [11] or [17])

$$\|f'\|_\infty \leq n\|f\|_\infty. \tag{1}$$

Concerning the maximum modulus of $f \in \mathbb{F}_n$ on a large circle $|z|= R > 1$, we have

$$\|f(Rz)\|_\infty \leq R^n\|f\|_\infty. \tag{2}$$

Inequality (2) is a simple deduction from the maximum modulus principle (see [16, p. 346] or [12, p. 158, problem III 269]).

Inequalities (1) and (2) can be obtained by letting $p \to \infty$ in the inequalities

$$\|f'\|_p \leq n\|f\|_p, \quad p \geq 1 \tag{3}$$

and

$$\|f(Rz)\|_p \leq R^n\|f\|_p R > 1, \quad p > 0 \tag{4}$$

respectively. Inequality (3) is due to Zygmund [19] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $f(e^{i\theta})$, whereas inequality (4) is a simple consequence of a result of Hardy [9] (see also [13, Th. 5.5]). Since inequality (3) was deduced from M Riesz's interpolation formula [15] by means of

Minkowski's inequality, it was not clear whether the restriction on p was indeed essential. This question was open for a long time. Finally, Arestov [2] proved that (3) remains true for $0 < p < 1$.

If $f(z) \neq 0$ for $|z| < 1$, then for each $p > 0$, inequality (3) can be replaced by [7, 10, 14]

$$\|f'\|_p \leq n \frac{\|f\|_p}{\|1+z^n\|_p} \tag{5}$$

whereas in this case inequality (4) can be replaced by [1, 6, 14]

$$\|f(Rz)\|_p \leq \frac{\|R^n z^n + 1\|_p}{\|1+z^n\|_p} \|f\|_p, \quad \text{for each } p > 0. \tag{6}$$

Recently the authors [5] have investigated the dependence of $\|f(Rz) - f(z)\|_p$ on $\|f\|_p$ and proved that if $f \in \mathbb{F}_n$, then for every $p \geq 1$ and $R \geq 1$,

$$\|f(Rz) - f(z)\|_p \leq (R^n - 1)\|f\|_p. \tag{7}$$

For other results of the same nature see [3]. Here we first show that the inequality (7) holds for each $p > 0$. In fact we prove

Theorem 1. *If $f \in \mathbb{F}_n$, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq (R^n - 1)\|f\|_p. \tag{8}$$

The result is the best possible and equality holds for $f(z) = \alpha z^n$, $\alpha \neq 0$.

Remark 1. Dividing both sides of (8) by $R - 1$ and letting $R \to 1$, we obtain inequality (3) for each $p > 0$.

If $f \neq \mathbb{F}_n$ and $f(z) \neq 0$ for $|z| < 1$, then (8) can be sharpened. In this case we prove the following result which is a generalization of inequality (5).

Theorem 2. *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z| < 1$, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq \frac{(R^n - 1)}{\|1+z^n\|_p} \|f\|_p. \tag{9}$$

The result is best possible and equality holds for $f(z) = az^n + b$, $|a| = |b|$.

Remark 2. Dividing both sides of (9) by $R - 1$ and letting $R \to 1$, we obtain inequality (4) for each $p > 0$.

A polynomial $f \in \mathbb{F}_n$ is said to be self-inversive if $f(z) = Q(z)$ where $Q(z) = z^n \overline{f(1/\bar{z})}$. It is known [4, 8] that if $f \in \mathbb{F}_n$ is self-inversive polynomial, then for every $p \geq 1$,

$$\|f'\|_p \leq n \frac{\|f\|_p}{\|1+z^n\|_p}. \tag{10}$$

Finally we present the following result which extends (10) to $p \in (0, 1)$.

Theorem 3. *If $f \in \mathbb{F}_n$ is self-inversive polynomial, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq \frac{(R^n - 1)}{\|1+z^n\|_p} \|f\|_p. \tag{11}$$

The result is sharp and equality holds for $f(z) = z^n + 1$.

Remark 3. Dividing both sides of (11) by $R-1$ and letting $R \to 1$, we obtain inequality (10) for each $p > 0$.

2. Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 1. *If $f \in \mathbb{F}_n$ and $f(z)$ has all its zeros in $|z| \le k$ where $k \le 1$, then for every $R > 1$,*

$$|f(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |f(z)| \quad \textit{for } |z| = 1.$$

Proof of Lemma 1. Since all the zeros of $f(z)$ lie in $|z| \le k \le 1$, we write

$$f(z) = C\prod_{j=1}^{n}(z - r_j e^{i\theta_j})$$

where $r_j \le k$, $j = 1, 2, \ldots, n$, so that, for each $\theta, 0 \le \theta < 2\pi$ and $R > 1$, it can be easily verified that

$$\begin{aligned}\left|\frac{f(Re^{i\theta})}{f(e^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| \\ &\ge \prod_{j=1}^{n}\left(\frac{R+r_j}{1+r_j}\right) \ge \prod_{j=1}^{n}\left(\frac{R+k}{1+k}\right) \\ &= \left(\frac{R+k}{1+k}\right)^n.\end{aligned}$$

This implies

$$|f(Re^{i\theta})| \ge \left(\frac{R+k}{1+k}\right)^n |f(e^{i\theta})| \quad \text{for } R > 1, \quad 0 \le \theta < 2\pi$$

Hence

$$|f(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |f(z)| \quad \text{for } |z| = 1 \quad \text{and } R > 1.$$

This completes the proof of Lemma 1.

Lemma 2. *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z| < 1$, then*

$$|f(Rz) - f(z)| \le |Q(Rz) - Q(z)| \quad \textit{for } |z| \ge 1 \quad \textit{and } R \ge 1 \tag{12}$$

where $Q(z) = z^n f(1/\bar{z})$.

Proof of Lemma 2. For $R = 1$, there is nothing to prove. Henceforth we assume that $R > 1$. Since the polynomial $f(z)$ has all its zeros in $|z| \ge 1$, therefore, for every complex number β such that $|\beta| > 1$ the polynomial $g(z) = f(z) - \beta Q(z)$, where $Q(z) = z^n \overline{f(1/\bar{z})}$, has all its zeros in $|z| \le 1$. Applying Lemma 1 to the polynomial $g(z)$ with $k = 1$, we get

$$|g(Rz)| \ge \left(\frac{R+1}{2}\right)^n |g(z)| \quad \text{for } |z| = 1 \quad \text{and } R > 1. \tag{13}$$

Clearly $g(Re^{i\theta}) \neq 0$ for every $R > 1$ and $0 \leq \theta < 2\pi$, which implies

$$|g(Rz)| > 0 \quad \text{for } |z| = 1 \quad \text{and } R > 1. \tag{14}$$

Now for points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are not the zeros of $g(z)$, we get from (13)

$$|g(Re^{i\theta})| > |g(e^{i\theta})| \quad \text{for every } R > 1. \tag{15}$$

Since by (14), the inequality (15) is trivially true for those points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are the zeros of $g(z)$, it follows that

$$|g(z)| < |g(Rz)| \quad \text{for } |z| = 1 \quad \text{and } R > 1.$$

Using Rouche's theorem and noting that all the zeros of $g(Rz)$ lie in $|z| \leq (1/R) < 1$, we conclude that the polynomial

$$\begin{aligned} h(z) &= (g(Rz) - g(z)) \\ &= (f(Rz) - f(z) - \beta(Q(Rz) - Q(z)) \end{aligned} \tag{16}$$

has all its zeros in $|z| < 1$ for every β with $|\beta| > 1$ and $R > 1$. This implies

$$|f(Rz) - f(z)| \leq |Q(Rz) - Q(z)| \quad \text{for } |z| \geq 1 \quad \text{and } R > 1. \tag{17}$$

If inequality (17) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|f(Rz_0) - f(z_0)| > |Q(Rz_0) - Q(z_0)|.$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1$, it follows (as in the case of $g(z)$) that all the zeros of $Q(Rz) - Q(z)$ lie in $|z| < 1$ for every $R > 1$. Hence $Q(Rz_0) - Q(z_0) \neq 0$ with $|z_0| \geq 1$. We take

$$\beta = \frac{f(Rz_0) - f(z_0)}{Q(Rz_0) - Q(z_0)}$$

so that β is a well defined real or complex number with $|\beta| > 1$ and with this choice of β, from (16) we get

$$h(z_0) = 0 \quad \text{where } |z_0| \geq 1.$$

This is clearly a contradiction to the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus

$$|f(Rz) - f(z)| \leq |Q(Rz) - Q(z)| \quad \text{for } |z| \geq 1 \quad \text{and } R > 1.$$

This proves Lemma 2.

Next we describe a result of Arestov.

$$\text{For} \quad \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1} \quad \text{and} \quad f(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{F}_n,$$

we define

$$\Lambda_\gamma f(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

(i) $f(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(ii) $f(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 3 [2, Theorem 4]. *Let $\phi(x)=\psi(\log x)$ where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for all $f \in \mathbb{F}_n$ and each admissible operator A_γ,*

$$\int_0^{2\pi} \phi(|A_\gamma f(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(C(\gamma,n)|f(e^{i\theta})|)d\theta,$$

where $C(\gamma,n)=\max(|\gamma_0|,|\gamma_n|)$.

In particular, Lemma 2 applies with $\phi: x \to x^p$ for every $p \in (0,\infty)$. Therefore, we have

$$\|A_\gamma f\|_p \leq C(\gamma,n)\|f\|_p, \quad 0<p<\infty.$$

Next we use Lemma 3 to prove the following interesting result.

Lemma 4. *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z|<1$, then for each $p>0$, $R \geq 1$ and α real,*

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^n f(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p. \tag{18}$$

The result is the best possible and equality holds for $f(z)=az^n+b$, $|a|=|b|$.

Proof of Lemma 4. The result is obvious for $R=1$, so we assume $R>1$. First we show that for every $R>1$ and α real, all the zeros of the polynomial

$$R(z)=\sum_{j=0}^{n}\binom{n}{j}\{(R^j-1)+e^{i\alpha}(R^{n-j}-1)\}z^j$$

lie on the unit circle. Let

$$\begin{aligned} H(z) &= \sum_{j=0}^{n}\binom{n}{j}(R^j-1)z^j \\ &= (Rz+1)^n-(z+1)^n. \end{aligned}$$

The zeros $z_k, k=1,2,\ldots,n$ of $H(z)$ are given by

$$z_k=\frac{1-e^{(2k\pi i/n)}}{e^{(2k\pi i/n)}-R}.$$

Since $R>1$, it can be easily seen that $|z_k|<1, k=1,2\ldots,n$. Hence all the zeros of $H(z)$ lie in $|z|<1$ for every $R>1$. If now

$$\begin{aligned} G(z) &= z^n\overline{H(1/\overline{z})} = z^n H(1/z) \\ &= \sum_{j=0}^{n}\binom{n}{j}(R^{n-j}-1)z^{n-j} \\ &= \sum_{j=0}^{n}\binom{n}{j}(R^{n-j}-1)z^j, \end{aligned}$$

then all the zeros of $G(z)$ lie in $|z|>1$ and it follows that (see [12, Prob. 26, P.108]) the polynomial

$$\begin{aligned} R(z) &= H(z) + e^{i\alpha} z^n \overline{H(1/\bar{z})} \\ &= \sum_{j=0}^{n} \binom{n}{j} \{(R^j - 1) + e^{i\alpha}(R^{n-j} - 1)\} z^j \end{aligned}$$

has all its zeros on the circle $|z| = 1$ for every $R > 1$ and α real. Now by hypothesis $f(z)$ has all its zeros in $|z| \geq 1$ therefore, by Szegö's convolution theorem [18], if $f \in \mathbb{F}_n$, then

$$\begin{aligned} \Lambda f(z) &= (f(Rz) - f(z)) + e^{i\alpha}(R^n f(z/R) - f(z)) \\ &= (R^n - 1)a_n z^n + \{(R^{n-1} - 1) + e^{i\alpha}(R - 1)\} a_{n-1} z^{n-1} \\ &\quad + \cdots + \{(R - 1) + e^{i\alpha}(R^{n-1} - 1)\} a_1 z + (R^n - 1) a_0, \end{aligned}$$

does not vanish in $|z| < 1$ for every $R > 1$ and α real. Therefore, Λ is an admissible operator. Applying Lemma 3, we obtain for each $p > 0$, $R > 1$ and α real,

$$\begin{aligned} &\int_0^{2\pi} |(f(Re^{i\theta}) - f(e^{i\theta})) + e^{i\alpha}(R^n f(e^{i\theta}/R) - f(e^{i\theta}))|^p \, d\theta \\ &\leq (R^n - 1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta. \end{aligned}$$

which is equivalent to the desired result and this completes the proof of Lemma 4.

Lemma 5. *If $f \in \mathbb{F}_n$, then for every $p > 0$, $R \geq 1$ and α real,*

$$\|(f(Rz) - f(z)) + e^{i\alpha}(R^n f(z/R) - f(z))\|_p \leq (R^n - 1)\|f\|_p. \tag{19}$$

The result is the best possible and equality holds for $f(z) = az^n + b$, $|a| = |b|$.

Proof of Lemma 5. The result is trivial for $R = 1$. Henceforth we assume $R > 1$. Since $f(z)$ is a polynomial of degree at most n, we can write

$$f(z) = f_1(z) f_2(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (z - z_j), \quad k \geq 1$$

where all the zeros of $f_1(z)$ lie in $|z| \geq 1$ and all the zeros of $f_2(z)$ lie in $|z| < 1$. First we suppose that $f_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $f_1(z)$ lie in $|z| > 1$. Let $Q_2(z) = z^{n-k} \overline{f_2(1/\bar{z})}$, then all the zeros of $Q_2(z)$ lie in $|z| > 1$ and $|Q_2(z)| = |f_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = f_1(z) Q_2(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (1 - z\bar{z}_j),$$

then all the zeros of $F(z)$ lie in $|z| > 1$ and for $|z| = 1$,

$$|F(z)| = |f_1(z)||Q_2(z)| = |f_1(z)||f_2(z)| = |f(z)|. \tag{20}$$

Since $f(z)/F(z)$ is not a constant, by maximum modulus principle

$$|f(z)| < |F(z)| \quad \text{for } |z| = r < 1.$$

Applying Rouche's theorem, it follows that the polynomial $G(z) = f(z) + \lambda F(z)$ does not vanish in $|z| \leq 1$ for every λ with $|\lambda| > 1$. Using Szegö's convolution theorem [18], it

follows as in the proof of Lemma 4 that the polynomial

$$(G(Rz)-G(z))+e^{i\alpha}(R^nG(z/R)-G(z))$$

does not vanish in $|z|\leq 1$ for every $R>1$ and α real. Replacing $G(z)$ by $f(z)+\lambda F(z)$, it follows that the polynomial

$$\begin{aligned} T(z)= &\{(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))\} \\ &+\lambda\{(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))\} \end{aligned} \tag{21}$$

does not vanish in $|z|\leq 1$ for every λ with $|\lambda|>1$. This implies

$$\begin{aligned} &|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))| \\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))| \end{aligned} \tag{22}$$

for $|z|\leq 1, R>1$ and α real. If inequality (22) is not true, then there is a point $z=z_0$ with $|z_0|\leq 1$ such that

$$\begin{aligned} &|(f(Rz_0)-f(z_0))+e^{i\alpha}(R^nf(z_0/R)-f(z_0))| \\ &\qquad> |(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))|. \end{aligned}$$

Since all the zeros of polynomial $F(z)$ lie in $|z|>1$, it follows that all the zeros of the polynomial

$$(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))$$

lie in $|z|>1$ for every $R>1$ and α real. Hence

$$(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))\neq 0 \quad \text{with } |z_0|\leq 1.$$

We take

$$\lambda=\frac{(f(Rz_0)-f(z_0))+e^{i\alpha}(R^nf(z_0/R)-f(z_0))}{(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))}$$

so that λ is a well defined real or complex number with $|\lambda|>1$ and with this choice of λ from (21), we get

$$T(z_0)=0 \quad \text{where } |z_0|\leq 1.$$

This is clearly a contradiction to the fact that $T(z)$ does not vanish in $|z|\leq 1$. Thus for every $R>1$ and α real, we have

$$\begin{aligned} &|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))| \\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))| \end{aligned}$$

for $|z|\leq 1$, which in particular gives for $|z|=1$,

$$\begin{aligned} &|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))| \\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))|. \end{aligned}$$

Hence for each $p>0$ and $0\leq\theta<2\pi$, we obtain

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}) + e^{i\alpha}(R^nf(e^{i\theta}/R)-f(e^{i\theta/R}))|^p \, d\theta$$
$$\leq \int_0^{2\pi} |(F(Re^{i\theta})-F(e^{i\theta}))+e^{i\alpha}(R^nF(e^{i\theta}/R)-F(e^{i\theta}))|^p \, d\theta.$$

Using Lemma 4 and (20), it follows that for each $p>0$ and $0\leq\theta<2\pi$,

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta})) + e^{i\alpha}(R^nf(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta$$
$$\leq (R^n-1)^p \int_0^{2\pi} |F(e^{i\theta})|^p \, d\theta,$$
$$= (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.$$

This implies for each $p>0,\ R>1$ and α real,

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p. \tag{23}$$

Now if $f_1(z)$ has a zero on $|z|=1$, then applying (23) to the polynomial $f^*(z)=f_1(tz)f_2(z)$ where $t<1$, we get for each $p>0, R>1$ and α real,

$$\|(f^*(Rz)-f^*(z))+e^{i\alpha}(R^nf^*(z/R)-f^*(z))\|_p \leq (R^n-1)\|f^*\|_p. \tag{24}$$

Letting $t\to 1$ in (24) and using continuity, we obtain for each $p>0$, $R>1$ and α real,

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p.$$

This completes the proof of Lemma 5.

3. Proofs of the Theorems

Proof of Theorem 1. Since $f\in\mathbb{F}_n$, by Lemma 5, we have for each $p>0, R\geq 1$, and α real

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))+e^{i\alpha}(R^nf(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta$$
$$\leq (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta, \tag{25}$$

Integrating both sides of (25) w.r.t α from 0 to 2π, we get for each $p>0$,

$$\int_0^{2\pi}\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))+e^{i\alpha}(R^nf(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta \, d\alpha$$
$$\leq 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.$$

Now using in (25) the fact that for any $p>0$,

$$\int_0^{2\pi} |a+be^{i\alpha}|^p \, d\alpha \geq 2\pi \max\{|a|^p, |b|^p\},$$

(see [7, in eq. (19)]) we obtain

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))|^p \,d\theta \le (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \,d\theta,$$

which implies

$$||f(Rz)-f(z)||_p \le (R^n-1)||f||_p$$

for each $p>0$ and $R\ge 1$. This completes the proof of Theorem 1.

Proof of Theorem 2. Since the polynomial $f(z)$ does not vanish in $|z|<1$, it follows from Lemma 2 that

$$|f(Rz)-f(z)| \le |R^n f(z/R)-f(z)| \quad \text{for } |z|=1 \quad \text{and } R\ge 1. \tag{26}$$

By Lemma 5, we have for each $p>0$, $R\ge 1$ and α real,

$$\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \,d\theta \le (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \,d\theta \tag{27}$$

where

$$A(\theta)=f(Re^{i\theta})-f(e^{i\theta}) \text{ and } B(\theta)=R^n f(e^{i\theta}/R)-f(e^{i\theta}).$$

Integrating both sides of (27) with respect to α from 0 to 2π, we get for each $p>0$, $R\ge 1$ and α real,

$$\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \,d\theta\,d\alpha \le 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \,d\theta. \tag{28}$$

Now for every real α and $t\ge 1$, we have $|t+e^{i\theta}| \ge |1+e^{i\alpha}|$, which implies

$$\int_0^{2\pi} |t+e^{i\alpha}|^p \,d\alpha \ge \int_0^{2\pi} |1+e^{i\alpha}|^p d\alpha, \quad p>0.$$

If $A(\theta)\ne 0$, we take $t=|B(\theta)|/|A(\theta|$, then by (26), $t\ge 1$ and we get

$$\begin{aligned}
\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \,d\alpha &= |A(\theta)|^p \int_0^{2\pi} \left|1+\frac{B(\theta)}{A(\theta)}e^{i\alpha}\right|^p d\alpha \\
&= |A(\theta)|^p \int_0^{2\pi} \left|\frac{B(\theta)}{A(\theta)}+e^{i\alpha}\right|^p d\alpha \\
&= |A(\theta)|^p \int_0^{2\pi} \left|\left|\frac{B(\theta)}{A(\theta)}\right|+e^{i\alpha}\right|^p d\alpha \\
&\ge |A(\theta)|^p \int_0^{2\pi} |1+e^{i\alpha}|^p \,d\alpha \\
&= |f(Re^{i\theta})-f(e^{i\theta})|^p \int_0^{2\pi} |1+e^{i\alpha}|^p \,d\alpha.
\end{aligned}$$

For $A(\theta)=0$, this inequality is trivially true. Using this in (28), we conclude that for each $p>0$, $R\ge 1$ and α real,

$$\int_0^{2\pi} |1+e^{i\alpha}|^p \, d\alpha \int_0^{2\pi} |f(Re^{i\theta})-f(e^{i\theta})|^p \, d\theta$$
$$\leq 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\alpha,$$

which immediately leads to (9) and this completes the proof of Theorem 2.

Proof of Theorem 3. Since $f(z)$ is a self-inversive polynomial, we have $f(z)=Q(z)$ where $Q(z)=z^n\overline{f(1/\bar{z})}$. Therefore, for each $R\geq 1$,

$$|f(Rz)-f(z)|=|Q(Rz)-Q(z)| \quad \text{for all } z\in\mathbb{C},$$

so that

$$|B(\theta)/A(\theta)|=\left|\frac{R^nf(e^{i\theta}/R)-f(e^{i\theta})}{f(Re^{i\theta})-f(e^{i\theta})}\right|=1.$$

Using this in (28) and proceeding similarly as in the proof of Theorem 2, we get (11) and this proves Theorem 3.

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Denting and strongly extreme points in the unit ball of spaces of operators

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Abstract. For $1 \leq p \leq \infty$ we show that there are no denting points in the unit ball of $\mathcal{L}(\ell^p)$. This extends a result recently proved by Grząślewicz and Scherwentke when $p = 2$ [GS1]. We also show that for any Banach space X and for any measure space $(\Omega, \mathcal{A}, \mu)$, the unit ball of $\mathcal{L}(L^1(\mu), X)$ has denting points iff $L^1(\mu)$ is finite dimensional and the unit ball of X has a denting point. We also exhibit other classes of Banach spaces X and Y for which the unit ball of $\mathcal{L}(X, Y)$ has no denting points. When X^* has the extreme point intersection property, we show that all 'nice' operators in the unit ball of $\mathcal{L}(X, Y)$ are strongly extreme points.

Keywords. Denting point; strongly extreme point; M-ideal.

1. Introduction

Let X be a Banach space and let X_1 denote its closed unit ball. In this paper we consider several aspects of the extremal structure of the unit ball of the space of operators $\mathcal{L}(X, Y)$. A point $x_0 \in X_1$ is said to be a denting point if for all $\epsilon > 0$, $x_0 \notin \overline{CO}(X_1 \backslash B(x_0, \epsilon))$ ($B(x_0, \epsilon)$ denotes the open ball and $\overline{CO}$, the closed convex hull) (see [DU]). Any denting point is an extreme point. For an infinite dimensional Hilbert space H, Grząślewicz and Scherwentke have showed recently that there are no denting points in the unit ball of $\mathcal{L}(H)$, the space of bounded linear operators (see [GS1]). Their proof makes use of the description of the extreme points of $\mathcal{L}(H)_1$ as isometries and co-isometries and shows that they are not denting points. However for the case of $1 < p < \infty, p \neq 2$, there is no complete description of extreme points of $\mathcal{L}(\ell^p)_1$ known (see [G] for more information). In this paper we take the equivalent definition of a denting point given by the result of [LLT] as an extreme point and a point of weak-norm continuity for the identity mapping on the unit ball. Most of our arguments involve ideas from M-structure theory for which we refer to [HWW].

In the first section we show that there are no denting points in the unit ball of $\mathcal{L}(\ell^p, Y)$ whenever there is a non-compact operator and the space of compact operators $\mathcal{K}(\ell^p, Y)$ is a M-ideal in $\mathcal{L}(\ell^p, Y)$. Since this is the case when $Y = \ell^p$ (see [HWW]) we have an extension of the result from [GS1]. For measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ (μ_1 and μ_2 are positive measures) the same authors have proved in [GS2] that there are no denting points in $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ when $L^1(\mu_1)$ is infinite dimensional. We generalize this by showing that $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity iff $L^1(\mu)$ is finite dimensional and X_1 has a point of weak-norm continuity. Our ideas also work for other operator ideals in $\mathcal{L}(L^1(\mu), X)$. By taking advantage of the description of operators defined on $C(K)$ spaces (see [DU] Chap. VI), in some special cases we could describe

points of weak-norm continuity in $\mathcal{AS}(C(K),X)_1$ (the ideal of absolutely summing operators with the absolutely summing norm).

For continuous function spaces, we show that for any infinite, totally disconnected, compact Hausdorff space K and for any Banach space X there are no points of weak-norm continuity in $\mathcal{L}(X,C(K))_1$. Since ℓ^∞ can be identified as $C(\beta(N))$, this covers the case $p=\infty$. We also show that if Y is a Banach space with infinite dimensional centralizer then no 'nice' operator (defined in § 1) can be a point of weak-norm continuity for the identity map on $\mathcal{L}(X,Y)_1$.

We next consider a weaker extremal form, namely, strongly extreme point. A point $x_0 \in X_1$ is said to be a strongly extreme point, if for every pair of sequences $\{x_n\}$, $\{y_n\} \subset X_1$ such that $(x_n+y_n)/2 \to x_0$, implies $\|x_n - y_n\| \to 0$. Any denting point is clearly a strongly extreme point. It is well known that for any compact set K, and for any measure space $(\Omega, \mathcal{A}, \mu)$, every extreme point in $C(K)_1$ and $L^1(\mu)_1$ is a strongly extreme point. Extending this aspect to operator spaces the authors of [GS2] show that all the extreme points in $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ are strongly extreme. Isolating a property called the E. P. I .P that is common to both $C(K)$ and $L^1(\mu)$ we show that if X^* has the E. P. I. P then any 'nice' operator in $\mathcal{L}(X,Y)_1$ is a strongly extreme point, thus extending Theorem 1 and Corollary 4 of [GS2].

In the concluding part of the paper we briefly consider the stability aspect of another geometric property shared by the $C(K)$ and $L^1(\mu)$ spaces, namely every extreme point of the unit ball is also an extreme point of the unit ball of the bidual. These were called weak* extreme points in [KR] and it is known that any denting or strongly extreme point of the unit ball is a weak* extreme point. Having noted the non-existence of denting points and the availability of strongly extreme points, the permanence of extreme points is a natural question to consider. Our result considered in the more general setting of vector-valued continuous functions, states that, if K is a dispersed space then $\partial_e C(K,X)_1 \subset \partial_e C(K,X)_1^{**}$ whenever X has the same property.

All the Banach spaces considered here are infinite dimensional.

2. Denting points

In this section, using ideas from M-structure theory and the structure of basic sequences in ℓ^p spaces, we first show that there are no denting points in $\mathcal{L}(\ell^p, Y)_1$ for $1<p<\infty$ in the non-trivial situation. For $p=1$ we get a better result by showing that there are no points of weak-norm continuity in the unit ball of $\mathcal{L}(\ell^1, Y)$ for any Y.

We refer the reader to [Ans] Proposition 2.5 for a characterization of Banach spaces Y for which $\mathcal{K}(\ell^p, Y) = \mathcal{L}(\ell^p, Y)$ (see Proposition 2.c.3 in [LT] when $Y = \ell^r$). In what follows we assume that Y is such that there is a non-compact operator from ℓ^p and $\mathcal{K}(\ell^p, Y) \subset \mathcal{L}(\ell^p, Y)$ is a M-ideal. Examples of such Y include, ℓ^q for $p \le q$, $L^p([0,1])$, and the Schatten class c_p for $2 \le p$. We refer the reader to [KW] for more examples and characterizations of such Y. In particular we recall from Corollary 6.4 of [KW] that this property is hereditary for Y.

Theorem 2.1. *Let $1<p<\infty$. Suppose Y is such that $\mathcal{K}(\ell^p, Y)$ is a proper M-ideal in $\mathcal{L}(\ell^p, Y)$. There are no denting points in the unit ball of $\mathcal{L}(\ell^p, Y)$.*

Proof. Let $T \in \mathcal{L}(\ell^p, Y)$, $\|T\| = 1$ be a non-compact operator. We shall show that T is not a point of weak-norm continuity for the identity mapping on the unit ball. Once this is

established it would follow that any denting point must be a compact operator. However since $\mathcal{K}(\ell^p, Y)$ is a proper M-ideal in $\mathcal{L}(\ell^p, Y)$, applying Proposition 4.2 and Theorem 4.4 of Chapter 2 in [HWW] we see that there are no denting points in $\mathcal{K}(\ell^p, Y)_1$. This completes the proof.

Now let $T \in \mathcal{L}(\ell^p, Y)_1$ be a non-compact operator.

Case i. Let $\{e_n\}$ be the canonical basis of ℓ^p. Suppose $T(e_n) \not\to 0$ in the norm. It is easy to see that $T \circ (I - (e_n \otimes e_n)) \to T$ weakly. Also

$$\begin{aligned} \|I - (e_n \otimes e_n)\| &\leq 1 \text{ and} \\ \|T \circ (e_n \otimes e_n)\| &= \sup_{\|x\| \leq 1} \|e_n(x) T(e_n)\| \\ &= \|T(e_n)\|. \end{aligned}$$

We thus get the required contradiction.

Case ii. The general case follows from a similar argument. Since T is non-compact on a reflexive domain, assume without loss of generality that there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1, x_n \xrightarrow{w} 0$ but $T(x_n) \not\to 0$ in the norm topology. Applying Proposition 1.a.12 in [LT] we may assume that $\{x_n\}$ is equivalent to a block basis of the canonical basis $\{e_n\}$. Also by Proposition 2.a.1 in [LT], for any normalized block basis, its closed span is isometric to ℓ^p. Hence the conclusion follows from arguments similar to the ones given during the proof in Case i).

Remark 2.1. We note that in the situation $p = 2$, $Y = \ell^2$ where it is well-known that there are no extreme points in the unit ball of $\mathcal{K}(\ell^2)$, the second half of the proof gives the result, and the M-ideal argument is not needed. In the general case, even when $Y = \ell^p$, this does not immediately lead to a contradiction since there are plenty of compact extremal operators(see [H]).

Remark 2.2. It is apparent from the arguments given above that an isometry or co-isometry is not a point of weak-norm continuity in the unit ball. In the case of a complex Hilbert space H, since any operator is an average of an isometry and co-isometry, it is easy to see that there are no points of weak-norm continuity in $\mathcal{L}(H)_1$. Thus there are no points of weak*-norm continuity either. Since $\mathcal{L}(H)$ is the bidual of $\mathcal{K}(H)$, one can conclude that there are no points of weak-norm continuity in $\mathcal{K}(H)_1$ (see [HL]). For other finite p, we do not know if there can be points of weak-norm continuity in the unit ball of the space of operators.

Remark 2.3. If Y has the R. N. P and $\mathcal{K}(\ell^p, Y) = \mathcal{L}(\ell^p, Y)$, it follows from a Corollary in [DM] that $\mathcal{L}(\ell^p, Y)$ has the R. N. P and hence has plenty of denting points in the unit ball (see [DU]). We do not know an example of a space Y for which $\mathcal{K}(\ell^p, Y) = \mathcal{L}(\ell^p, Y)$ fails to have denting points in the unit ball.

In the following proposition we exhibit another class of Banach spaces where there are denting points in the unit ball of $\mathcal{L}(\ell^p, X)$. If X has the Schur property (i.e., weak and norm sequential convergences coincide) then clearly, $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$. If further X is infinite dimensional, then it contains an isomorphic copy of ℓ^1. In the following proposition we assume that X contains 'better' copies of ℓ^1.

PROPOSITION 2.1

Let X be a Banach space having the Schur property and an isometric copy of ℓ^1. *Suppose there exists a projection* $P : X^* \to X^*$ *of norm one such that* $\mathrm{Ker}(P) = \ell^{1\perp}$ *and* $P(X^*)_1$ *is weak* dense in* X_1^*. *Then there are denting points in* $\mathcal{L}(\ell^p, X)_1$.

Proof. Consider $\mathcal{L}(\ell^p, \ell^1) \subset \mathcal{L}(\ell^p, X)$. Since the latter space being a separable dual space has the R. N. P., it has denting points in the unit ball. We shall show that any denting point of this space is also a denting point of $\mathcal{L}(\ell^p, X)_1$. This is achieved by exhibiting a projection, $Q : \mathcal{L}(\ell^p, X)^* \to \mathcal{L}(\ell^p, X)^*$ of norm one such that $\mathrm{Ker}(Q) = \mathcal{L}(\ell^p, \ell^1)^{\perp}$ and $Q(\mathcal{L}(\ell^p, X)^*)_1$ is weak* dense in $\mathcal{L}(\ell^p, X)_1^*$. We then appeal to Proposition 2 and its proof in [R3] to conclude that denting points get preserved.

Since $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$, we identify $\mathcal{L}(\ell^p, X)^*$ with the space of integral operators from ℓ^q to X^* (see [DU] p. 232). Now Q is defined by composing such an operator with P. Using the properties of P it is fairly routine to verify that Q has the desired properties. Hence there are denting points in $\mathcal{L}(\ell^p, X)_1$.

Using arguments similar to the ones given above and results from [DM], the following corollary is easy to prove.

COROLLARY 2.1

Let X be a Banach space having the R. N. P. Suppose $\mathcal{L}(\ell^p, X^{**}) = \mathcal{K}(\ell^p, X^{**})$. *Then there are denting points in the unit ball of* $\mathcal{L}(\ell^p, X^{**})$.

Our next result deals with the question of points of weak-norm continuity in the unit ball of ℓ^∞-sums of Banach spaces.

PROPOSITION 2.2

If $\{X_i\}_{i\in I}$ *is any infinite family of non-trivial Banach spaces then there is no point of weak-norm continuity in the unit ball of the space* $X = \oplus_\infty X_i$.

Proof. Write $X = Y \oplus_\infty Z$ where Y and Z consist of ℓ^∞-sum of infinitely many X_i's. Suppose $x \in X_1$ is a point of weak-norm continuity. Let $x = y + z$, $y \in Y$, $z \in Z$.

If $\|y\| < 1$ then since Y is infinite dimensional, we can get a net $\{y_\alpha\}$ with $\|y_\alpha\| = 1$ and $y_\alpha \to y$ in the weak topology. Now $\|y_\alpha + z\| = \max\{\|y_\alpha\|, \|z\|\} = 1$ and $y_\alpha + z \to x$ weakly but not in the norm. Therefore $\|y\| = 1$. Similarly $\|z\| = 1$. It is also easy to see that each of y and z are points of weak norm continuity in Y_1 and Z_1 respectively.

Thus there is no loss of generality in assuming that I is countable. The same argument also shows that x can have at most finitely many components zero and for any infinite set $A \subset N$, $\sup_{a\in A} \|x(a)\| = 1$. Thus $\|x(n)\| \not\to 0$.

We may assume without loss of generality that there exists $0 < \delta < 1$ such that $\|x(n)\| > \delta\ \forall n$. Now for $e_n \in \ell^\infty$, let $e_n x \in \oplus_{c_0} X_n$ (the c_0-direct sum) denote the vector with x in the n-th place and zeros elsewhere. Clearly $e_n x \to 0$ weakly. Note that $\|x - e_n x\| = 1$ and $x - e_n x \to x$ weakly but not in the norm. Therefore there are no points of weak-norm continuity in the unit ball.

Remark 2.4. The corresponding question for ℓ^p-direct sums for $1 \le p < \infty$ has positive answer (see [HL]).

We recall that for any discrete set I, the space $\mathcal{L}(\ell^1(I), X)$ can be identified with $\oplus_\infty X_i$ where $X_i = X$ for all $i \in I$.

Theorem 2.2. *Let X be a Banach space and $(\Omega, \mathcal{A}, \mu)$ (μ-positive) be a measure space. $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity if and only if $L^1(\mu)$ is finite dimensional and X_1 has a point of weak-norm continuity.*

Proof. Suppose $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity.

For any $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$, the projection $P : L^1(\mu) \to L^1(\mu)$ defined by $Pf = f\chi_A$ has the property that $\|f\| = \|P(f)\| + \|f - P(f)\|$ for all $f \in L^1(\mu)$. For such a projection P (a so called L-projection) Å. Lima observed in [L1] that $Q : \mathcal{L}(L^1(\mu), X) \to \mathcal{L}(L^1(\mu), X)$ defined by $Q(T) = T \circ P$ is a projection and satisfies $\|T\| = \max\{\|Q(T)\|, \|T - Q(T)\|\}$ for all $T \in \mathcal{L}(L^1(\mu), X)$.

Thus if $L^1(\mu)$ is infinite dimensional we can choose a sequence $\{A_n\}$ of pairwise disjoint sets with $0 < \mu(A_n) < \infty$ such that

$$\mathcal{L}(L^1(\mu), X) = \bigoplus_{\infty} \mathcal{L}(L^1(\mu_n), X) \bigoplus_{\infty} M$$

where $\mu_n = \mu|A_n$ and M is a closed (possibly trivial) subspace of $\mathcal{L}(L^1(\mu), X)$. In view of the above proposition we obtain a contradiction.

Clearly if $L^1(\mu)$ is of dimension n, then $\mathcal{L}(L^1(\mu), X) = \oplus_{\infty}^n X$ (n-many copies of X) and the conclusion follows from the arguments given during the proof of the proposition.

From the definition of a denting point we chose, and from arguments similar to the ones indicated above the following corollary is immediate.

COROLLARY 2.2

$\mathcal{L}(L^1(\mu), X)_1$ has a denting point if and only if $L^1(\mu)$ is finite dimensional and X_1 has a denting point.

Remark 2.5. Note that the same argument shows that for any closed subspace $\mathcal{H} \subset \mathcal{L}(L^1(\mu), X)$ that is closed under composition by operators from $\mathcal{L}(L^1(\mu))$, there is no point of weak-norm continuity in the unit ball of $\mathcal{H}$. Examples of such $\mathcal{H}$ include the spaces of compact operators, weakly compact operators.

Similar idea is again used in the following proposition which generalizes also Theorem 3 of [GS2]. Recall that a compact set is totally disconnected, if it has a base consisting of clopen sets.

PROPOSITION 2.3

Let K be any infinite, compact, totally disconnected space. For any Banach space X, there are no points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$.

Proof. For any clopen set $A \subset K$, the projection $R : C(K) \to C(K)$ defined by $R(f) = f\chi_A$ has the property

$$\|f\| = \max\{\|R(f)\|, \|f - R(f)\|\}.$$

For such projections R it again follows from [L1], that $S : \mathcal{L}(X, C(K)) \to \mathcal{L}(X, C(K))$ defined by $S(T) = R \circ T$ is a projection and satisfies

$$\|T\| = \max\{\|S(T)\|, \|T - S(T)\|\}.$$

Thus since K is infinite we can find an infinite maximal family $\{A_i\}_{i\in I}$ of pairwise disjoint clopen sets. Hence $\mathcal{L}(X, C(K))$ is a ℓ^∞-sum of infinitely many non-trivial spaces. Thus there is no point of weak-norm continuity in $\mathcal{L}(X, C(K))_1$.

The main difficulty in dealing with the question of points of weak-norm continuity is that in general one does not have a description of the weak topology of $\mathcal{L}(X)$. We do not know if the identity operator can be a denting point of $\mathcal{L}(X)_1$. In the above arguments we took advantage of weak convergence of sequences in $\mathcal{K}(X)$. Thus for a general X a more reasonable space to consider is span$\{\mathcal{K}(X), I\}$.

In contrast with the situation for ℓ^∞-sums, for ℓ^1-sums of Banach spaces, we have a positive result. It is clearly enough to consider sum of two spaces. The following Lemma is easy to prove.

Lemma. Let X be a Banach space. Suppose M and N are two closed subspaces such that $X = M \oplus N$, is an ℓ^1 direct sum. x_0 is a denting point of X_1 if and only if $x_0 \in M$ or N and is a denting point of the corresponding unit ball.

Turning back to the question of denting points of span$\{\mathcal{K}(X), I\}_1$, suppose $\|T + I\| = \|T\| + 1$ for all $T \in \mathcal{K}(X)$. Then it follows from the above Lemma that I is a denting point of span$\{\mathcal{K}(X), I\}_1$. That this hypothesis is satisfied when $X = C[0, 1]$ is a well known result of Daugavet (see [A] for more information). This also shows that the preceding technique of working with finite rank or compact operators does not work here. In spite of this, this author has recently proved that for any infinite compact space Ω and for any Banach space X, there are no denting points in $\mathcal{L}(X, C(\Omega))_1$ (see [R2]). In the case of a non-atomic measure it was shown in [R3] that there are no points of weak-norm continuity in $L^1(\mu, X)_1$.

Before exhibiting another class of Banach spaces for which there are no denting points in $\mathcal{L}(X, Y)_1$, we need some notation and terminology.

An operator $T \in \mathcal{L}(X, Y)_1$ is said to be a 'nice' operator if $T^*(\partial_e Y_1^*) \subset \partial_e X_1^*$. Any such operator is clearly an extreme point of $\mathcal{L}(X, Y)_1$ (see [S]).

For any X, its centralizer $Z(X)$ is the set of all $T \in \mathcal{L}(X)$ for which there is a bounded function α and a $S \in \mathcal{L}(X)$ such that

$$\left.\begin{array}{l} T^*(p) = \alpha(p)p \\ S^*(p) = \overline{\alpha}(p)p \end{array}\right\} \quad \text{for all } p \in \partial_e X_1^*.$$

In our concluding result of this section we again consider extra conditions on the range space to conclude that certain operators cannot occur as points of weak-norm continuity. Since $Z(C(K))$ is isometric to $C(K)$ (see Ch. 3 in [B]), the next theorem generalizes Theorem 2 of [GS2].

Theorem 2.3. *Let Y be a Banach space such that $Z(Y)$ is infinite dimensional. Then any $T \in \mathcal{L}(X, Y)_1$ such that T^* maps extreme points of Y_1^* to unit vectors is not a point of weak-norm continuity. In particular I is not a denting point of $\mathcal{L}(Y)_1$.*

Proof. It is easy to see that if K is any infinite compact Hausdorff space, there exists a sequence $\{f_n\} \in C(K)_1, 0 \leq f_n \leq 1, f_n \xrightarrow{w} 0$ and $\|f_n\| = 1, \|1 - f_n\| \leq 1$ for all n.

Using Theorem 4.14 of [B], we represent Y as a maximal function module over a compact space K_Y. By our hypothesis on $Z(Y)$, K_Y is an infinite set. Therefore using the isometric correspondence between $C(K_Y)$ and $Z(Y)$, we may choose a sequence $T_n \in Z(Y)$

that corresponds to the f_n's mentioned above such that $\|T_n\|=1$ and $T_n \xrightarrow{w} 0$ and $\|I-T_n\| \leq 1$ for all n.

Now let $T \in \mathcal{L}(X,Y)_1$ be such that T^* maps extreme points of Y_1^* to unit vectors. Clearly $(I-T_n)\circ T \to T$ weakly and $\|(I-T_n)\circ T\| \leq 1$. Also

$$\|T_n \circ T\| = \|T^* \circ T_n^*\| = \sup_{p\in\partial_e Y_1^*} \|T^*(T_n^*(p)\|$$
$$= \sup_{p\in\partial_e Y_1^*} \|\alpha_n(p)T^*(p)\|$$

(since $T_n \in Z(Y)$ we have that $T_n^*(p) = \alpha_n(p)p$)

$$= \sup_{p\in\partial_e Y_1^*} |\alpha_n(p)|$$
$$= \|\alpha_n\| = \|T_n\| = 1.$$

Therefore T is not a point of weak-norm continuity.

COROLLARY 2.3

Let X and Y be such that extreme points of $\mathcal{L}(X,Y)_1$ are 'nice' operators. Suppose that $Z(Y)$ is infinite dimensional. Then there are no denting points in $\mathcal{L}(X,Y)_1$.

Remark 2.6. It is worth noting that in the case of $\ell^p, Z(\ell^p)$ is trivial (see [B] Corollary 4.23).

Remark 2.7. We do not know an answer to the denting point (point of weak-norm continuity) question for the space $\mathcal{L}(\ell^\infty, X)_1$ for a general X. When X is an infinite dimensional space with the Schur property, then since X has no copy of ℓ^∞, it follows from Corollary 3 on p. 149 (see also Theorem 15 on p. 159) of [DU] that every operator here is weakly compact and hence compact. When $X=\ell^1$ we first note that $\mathcal{K}(c_0,X)=\ell^1\otimes_\epsilon \ell^1 = \mathcal{L}(c_0,X)$, being a separable dual space has the R. N. P. Also $\mathcal{K}(\ell^\infty,X)=\ell^{\infty *}\otimes_\epsilon \ell^1 = \mathcal{L}(\ell^\infty,X)$. Using the canonical embedding of ℓ^1 in its bidual and arguments similar to the ones given during the proof of Proposition 1 we see that $\mathcal{L}(\ell^\infty,\ell^1)_1$ has points of weak-norm continuity. Also for any infinite compact set K and for any X with the R. N. P, it follows from the results in Chap. VI of [DU] that $\mathcal{AS}(C(K),X)$ can be identified as a subspace of rcabv(X) (space of X-valued countably additive regular Borel measures of finite variation). A complete description of points of weak-norm continuity of $\mathcal{AS}(C(K),X)_1$ can be deduced from Theorem 3 of [R3].

3. Strongly extreme points

In this section we consider the existence of strongly extreme points in the unit ball of the space of operators $\mathcal{L}(X,Y)$ and the permanence of extreme points.

It is known that for any compact set K, every extreme point of $C(K)_1$ is a strongly extreme point and a similar result is true of $L^1(\mu)_1$. The corresponding operator version ie., all extreme points in $\mathcal{L}(L^1(\mu_1),L^1(\mu_2))_1$ are strongly extreme has been recently proved in [GS2]. The authors of [GS2] also exhibit certain class of operators as strongly extreme points in $\mathcal{L}(C(K_1),C(K_2))_1$ for compact sets K_1,K_2.

We first isolate a property that is common to $C(K)$ and $L^1(\mu)$ spaces and use it to obtain a general version of the results in [GS2].

DEFINITION [L1]

A Banach space X is said to have the extreme point intersection property (E. P. I. P. for short) if for all $x \in \partial_e X_1$ and for all $x^* \in \partial_e X_1^*$, $|x^*(x)| = 1$. It is easy to see that both $C(K)$ and $L^1(\mu)$ have this property. Any Banach space whose dual is isometric to $L^1(\mu)$ also has this property.

Suppose $x_0 \in X_1$ is such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$ then we note that x_0 is a strongly extreme point of X_1. To see this, if $(x_n + y_n)/2 \to x_0$ for two sequences $\{x_n\}$, $\{y_n\} \subset X_1$. Then for $\epsilon > 0 \, \exists N$ such that $\forall n \geq N$, $\forall x^* \in \partial_e X_1^*$, $|(x^*(x_n) + x^*(y_n))/2 - x^*(x_0)| < \epsilon$. Since $|x^*(x_0)| = 1$, $|x^*(x_n)| \leq 1$ and $|x^*(y_n)| \leq 1$ we get that $\|x_n - y_n\| = \sup_{x^* \in \partial_e X_1^*} |x^*(x_n) - y^*(x_n)| \leq 2\epsilon$ for all $n \geq N$. Hence the claim.

Theorem 3.1. *Suppose X^* has the E.P.I.P. For any Banach space Y, any 'nice' operator $T \in \mathcal{L}(X, Y)_1$ is a strongly extreme point.*

Proof. Let $T \in \mathcal{L}(X, Y)_1$ be a 'nice' operator. In view of our observation above, we shall show that $|\wedge(T)| = 1$ for all $\wedge \in \partial_e \mathcal{L}(X, Y)_1^*$. Since $T^*(\partial_e Y_1^*) \subset \partial_e X_1^*$ and since X^* has the E. P. I. P., the conclusion now follows from the arguments given during the proof of Theorem 1 in [R1].

Remark 3.1. Since every extreme point of $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ is a nice operator (see [S] Theorem 2.2) we get that every extreme point is strongly extreme (see [GS2] Corollary 4).

Another extremal property enjoyed by both $C(K)$ and $L^1(\mu)$ spaces is that any extreme point of the unit ball is also an extreme point of the unit ball of the second dual of the space (under the canonical embedding). We do not know if this property holds for spaces of operators as well. We however have the following stability results.

PROPOSITION 3.1

Let $\{X_i\}_{i \in I}$ be a family of Banach spaces. If for all i, every extreme point of $(X_i)_1$ is an extreme point of the unit ball of the bidual then the same is true of their ℓ^1-direct sum $X = \oplus_1 X_i$.

Proof. Let $x_0 \in X_1$ be an extreme point. Clearly there exists a $i_0 \in I$ such that $x_0(i) = 0$ for all $i \neq i_0$.

It is easy to see using the hypothesis that x_0 is an extreme point of $\oplus_1 X_i^{**}$. It is well-known that $X^{**} = \oplus_1 X_i^{**} \oplus_1 (\oplus_{c_0} X_i^*)^{\perp}$. Therefore x_0 is an extreme point of X_1^{**}.

We next consider this property for the space of vector-valued continuous functions $C(K, X)$ equipped with the supremum norm. We deal with the cases K dispersed (or scattered) and K containing a perfect set separately. The well-known identification of the space of compact operators $\mathcal{K}(X, C(K))$ with $C(K, X^*)$ thus gives corresponding result for the space of compact operators.

The situation when K is dispersed is very similar to Proposition 1.

PROPOSITION 3.2

Let K be a dispersed compact set. Let X be such that $\partial_e X_1 \subset \partial_e X_1^{**}$. Then $\partial_e C(K, X)_1 \subset \partial_e C(K, X)_1^{**}$.

Proof. Let I denote the set of isolated points of K. It is well-known that $C(K,X)^*$ can be identified with $\oplus_1 X_i^*$ where $X_i^* = X^*$ for all i. Thus the bidual has the identification $\oplus_\infty X_i^{**}$.

If $f \in \partial_e C(K,X)_1$ then for any $i \in I$, since i is an isolated point, $f(i) \in \partial_e X_1 \subset \partial_e X_1^{**}$. Therefore $f \in \partial_e C(K,X)_1^{**}$.

We need stronger assumptions on X and $C(K,X)$ to deal with the case when K contains a perfect set. We have proved in [R1] that if X has the E. P. I. P. and for every $f \in \partial_e C(K,X)_1$, $f(K) \subset \partial_e X_1$ then $C(K,X)$ has the E. P. I. P. and thus from our observation above every element of $\partial_e C(K,X)_1$ is a weak* extreme point.

Our concluding remark deals with the question of permanence of extreme points in the unit ball of projective tensor products.

Remark 3.2. Let μ be a finite measure and X be such that $\partial_e X_1 \subset \partial_e X_1^{**}$. Since any extreme point of $L^1(\mu,X)_1$ is of the form $\chi_A x$ where $x \in \partial_e X_1$ and $A \in \mathcal{A}$ is a μ-atom (see [Su]), it is easy to see that $L^1(\mu,X) = X \oplus_1 M$ for a closed subspace $M \subset L^1(\mu,X)$ (X is identified as functions that are constant on the atom A). Therefore $\partial_e L^1(\mu,X)_1 \subset \partial_e L^1(\mu,X)_1^{**}$.

In the case of general projective tensor product $X \otimes_\pi Y$, it is known that any denting point of $(X \otimes_\pi Y)_1$ is of the form $x \otimes y$ where $x \in X_1$ and $y \in Y_1$ are denting points (see [RS]). Since $(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*)$ (see [DU] p. 230), it follows from Theorem 3.7 of [L2] that if $x \in X_1$ is a denting point and $y \in Y$ is a weak* extreme point then $x \otimes y$ is a weak* extreme point of $(X \otimes_\pi Y)_1$.

It is known (see [La]) that on the surface of the unit ball of $K(\ell^2)^* = \ell^2 \otimes_\pi \ell^2$ the weak and norm topologies coincide and thus any unit vector is a point of weak-norm continuity, whereas denting points in the unit ball are of the form $x \otimes y$ for $\|x\| = \|y\| = 1$. In view of these remarks it is natural to ask the following question.

Question. If $x \in X_1$ is a denting point and $y \in Y_1$ is a point of weak-norm continuity then will $x \otimes y$ always be a point of weak-norm continuity in $(X \otimes_\pi Y)_1$?

This is indeed the case when of X or Y is a $L^1(\mu)$ space (see [R3]). More generally the following proposition gives another instance when the above question has affirmative answer. It can be proved using arguments similar to the ones given during the proof of Corollary 3 in [R3].

PROPOSITION 3.3

Suppose X is such that the answer to the above question is affirmative for $X \otimes_\pi X$. For any compact set K, let $F \in M(K,X)_1$ be a denting point and $x \in X_1$ be a point of weak-norm continuity. Then $F \otimes x$ is a point of weak-norm continuity of $(M(K,X) \otimes_\pi X)_1$. If either K is dispersed or X also has the R. N. P., then the same conclusion holds, when F is a point of weak norm continuity and x is a denting point.

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Note added in proof

We now use the ideas contained in § 1 of this paper to completely answer the question on points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$. In what follows we shall use the identification of $\mathcal{L}(X, C(K))$ as $W^*C(K, X^*)$, the space of functions on K that are continuous when X^* has the weak* topology, equipped with the supremum norm.

Theorem. *Let K be an infinite compact Hausdorff space and let X be any Banach space. Let $f \in W^*C(K,X^*)$ be a unit vector. There exists a sequence $\{f_n\}_{n\geq 1} \subset W^*C(K,X^*)_1$ such that $f_n \to f$ weakly but not in the norm. Hence there are no points of weak-norm continuity in $\mathcal{L}(X,C(K))_1$.*

Proof. Let $f \in W^*C(K,X^*)$, $\|f\| = 1$. It follows from Theorem 3 of [DHS] (see also [R2]) that the result is true if f is continuous w.r.t the norm topology on X^*. Note that if $f(K)$ is a norm compact subset of X^*, then since weak* and norm topologies coincide on $f(K)$, we get that f is continuous w.r.t the norm topology. Thus we assume w.l.o.g that $f(K)$ is not norm compact. Therefore there exists a sequence $\{t_n\}_{n\geq 1} \subset K$ of distinct terms such that $\{f(t_n)\}_{n\geq 1}$ has no convergent subsequence ($f(K)$ being weak* compact is norm closed).

We choose as before sequences of pairwise disjoint open sets $\{U_n\}$ and $\{g_n\} \subset C(K)_1^+$ such that $g_n(t_n) = 1$ and $g_n = 0$ on $K - U_n$ for all n. Using the 'dominated convergence' and the 'Riesz representation' theorems it is easy to see that $(1 - g_n) \to 0$ in the weak topology of $C(K)$. Since the map $g \to gf$ is a bounded linear contraction from $C(K)$ into $W^*C(K,X^*)$, it preserves weak convergence. Thus it follows that $f_n = (1 - g_n)f \to f$ in the weak topology. Since $\mathrm{Sup}\|f(t_n)\| \leq \mathrm{Sup}\|g_n f\|$ we get that $f_n \not\to f$ in the norm.

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A note on the non-commutative neutrix product of distributions

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Abstract. The distribution $F(x_+,-r)\ln x_+$ and $F(x_-,-s)$ corresponding to the functions $x_+^{-r}\ln x_+$ and x_-^{-s} respectively are defined by the equations

$$\langle F(x_+,-r)\ln x_+,\phi(x)\rangle = \int_0^\infty x^{-r}\ln x\left[\phi(x)-\sum_{i=0}^{r-2}\frac{\phi^{(i)}(0)}{i!}x^i-\frac{\phi^{(r-1)}(0)}{(r-1)!}H(1-x)x^{r-1}\right]\mathrm{d}x \tag{1}$$

and

$$\langle F(x_-,-s),\phi(x)\rangle = \int_0^\infty x^{-s}\left[\phi(-x)-\sum_{i=0}^{s-2}\frac{\phi^{(i)}(0)}{i!}(-x)^i-\frac{\phi^{(s-1)}(0)}{(s-1)!}H(1-x)x^{s-1}\right]\mathrm{d}x \tag{2}$$

where $H(x)$ denotes the Heaviside function. In this paper, using the concept of the neutrix limit due to J G van der Corput [1], we evaluate the non-commutative neutrix product of distributions $F(x_+,-r)\ln x_+$ and $F(x_-,-s)$. The formulae for the neutrix products $F(x_+,-r)\ln x_+\circ x_-^{-s}, x_+^{-r}\ln x_+\circ x_-^{-s}$ and $x_-^{-s}\circ F(x_+,-r)\ln x_+$ are also given for $r,s=1,2,\ldots$

Keywords. Distribution; delta function; neutrix; neutrix limit; neutrix product.

1. Introduction

The technique of neglecting appropriately defined infinite quantities was devised by J Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Hadamard's method can be regarded as a particular application of the neutrix calculus developed by J P van der Corput. This is a very general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in the context of distributions, by B Fisher in connection with the problem of distributional multiplication, see [3] or [6].

To define the neutrix product of distributions, we shall first of all let ρ be a fixed infinitely differentiable function having the properties:

(i) $\rho(x)=0 \quad$ for $|x|\geq 1$,
(ii) $\rho(x)\geq 0$,
(iii) $\rho(x)=\rho(-x)$,
(iv) $\int_{-1}^{1}\rho(x)\,\mathrm{d}x=1$.

We now define the function $\delta_n(x)=n\rho(nx)$ for $n=1,2,\ldots$ It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac

delta-function $\delta(x)$. Now let f be an arbitrary distribution and define the function f_n by

$$f_n(x) = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)\,\mathrm{d}t.$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f. Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ be the space of distributions. Now let f and g be arbitrary distributions and let $g_n = g * \delta_n$. We will say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a,b) $(-\infty \leq a < b \leq \infty)$ if

$$N\!-\!\lim_{n\to\infty}\langle fg_n, \phi\rangle = \langle h, \phi\rangle \tag{3}$$

for all $\phi \in \mathcal{D}$, where N is the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity, see [3] or [4].

In [2], the product of two distributions f and g was defined to be the limit of the sequence $\{fg_n\}$ in the usual sense, provided this limit exists. It is obvious that if the product fg exists by this definition, then it will exist by the new definition and will define the same distribution. In the following two theorems, which were proved in [4] and [7] respectively, the distributions x_+^{-r} and x_-^{-s} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)},$$

which is distinct from the definition given by Gel'fand and Shilov [5]. Further the distribution $x_+^{-r}\ln x_+$ is defined by (see [8])

$$x_+^{-r}\ln x_+ = F(x_+, -r)\ln x_+ + \frac{(-1)^r}{(r-1)!}\psi_1(r-1)\delta^{(r-1)}(x) \tag{4}$$

for $r = 1, 2, \ldots$, where

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r i^{-1}, & r \geq 1, \end{cases} \quad \text{and} \quad \psi_1(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r \frac{\psi(i)}{i}, & r \geq 1. \end{cases}$$

Theorem 1. *The neutrix products $x_+^{-s} \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ x_+^{-s}$ exist and are defined by*

$$x_+^{-s} \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{2(r+s)!}\delta^{(r+s)}(x) \tag{5}$$

$$\delta^{(r)}(x) \circ x_+^{-s} = 0 \tag{6}$$

for $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

Theorem 2. *The neutrix products $F(x_+, -r)\,\ln x_+ \circ \delta^{(s)}(x)$ and $\delta^{(s)}(x) \circ F(x_+, -r)\,\ln x_+$ exist and*

$$F(x_+,-r)\ln x_+ \circ \delta^{(s)}(x) = \frac{(-1)^s s!}{(r+s)!}\left[c_0(\rho)+\frac{1}{2}\psi(s)\right]\delta^{(r+s)}(x)$$
$$= x_+^{-r}\ln x_+ \circ \delta^{(s)}(x) \tag{7}$$
$$\delta^{(s)}(x)\circ F(x_+,-r)\ln x_+ = \delta^{(s)}(x)\circ x_+^{-r}\ln x_+ = 0 \tag{8}$$

for $r=1,2,\ldots$ *and* $s=0,1,2,\ldots$ *where* $c_0(\rho)=\int_0^1 \ln t\rho(t)\,dt$.

We now prove the following theorem.

Theorem 3. *The products* $F(x_+,-r)\ln x_+ \circ x_-^{-s}, x_+^{-r}\ln x_+ \circ x_-^{-s}, x_-^{-s}\circ F(x_+,-r)\ln x_+$ *and* $x_-^{-s}\circ x_+^{-r}\ln x_+$ *exist and*

$$F(x_+,-r)\ln x_+ \circ x_-^{-s} = \frac{(-1)^r}{(r+s-1)!}\kappa_s(\rho)\delta^{(r+s-1)}(x) \tag{9}$$
$$= x_+^{-r}\ln x_+ \circ x_-^{-s} \tag{10}$$
$$x_-^{-s}\circ F(x_+,-r)\ln x_+ = \frac{(-1)^r}{(r+s-1)!}\left[\Lambda_r(\rho)-\frac{1}{2}\psi_1(r-1)\right]\delta^{(r+s-1)}(x) \tag{11}$$
$$x_-^{-s}\circ x_+^{-r}\ln x_+ = \frac{(-1)^r}{(r+s-1)!}\Lambda_r(\rho)\delta^{(r+s-1)}(x) \tag{12}$$

for $r,s=1,2,\ldots,$ *where*

$$\kappa_s(\rho) = c_1(\rho)+\psi(s-1)c_0(\rho)+\psi_1(s)-\frac{1}{2}[\psi(s)]^2-\frac{1}{2}\xi(2)-\frac{1}{2}\sum_{i=1}^{s} i^{-2}$$

$$\Lambda_r(\rho) = \frac{1}{2}\left[c_1(\rho)+[\psi(r)+2\psi(r-1)]c_0(\rho)+2\psi_1(r)-\frac{1}{2}[\psi(r)]^2-\sum_{i=1}^{r} i^{-2}\right]$$

and

$$c_1(\rho) = \int_0^1 \ln^2 t\rho t\,dt, \quad \xi(2)=\sum_{i=1}^{\infty} i^{-2}.$$

Proof. We begin the proof by stating that $x_+^{-r}\ln x_+ \circ x_-^{-s} = F(x_+,-r)\ln x_+ \circ x_-^{-s}$ is an immediate consequence of eqs (4) and (6).

Now put $(x_-^{-s})_n = x_-^{-s} * \delta_n(x)$ so that

$$(x_-^{-s})_n = -\frac{1}{(s-1)!}\int_x^{1/n} \ln(t-x)\delta_n^{(s)}(t)\,dt$$

on the interval $[0,1/n]$, the intersection of the supports of $F(x_+,-r)\ln x_+$ and $(x_-^{-s})_n$. Then on using eq. (1) we have

$$\langle F(x_+,-r)\ln x_+, (x_-^{-s})_n x^k\rangle = \int_0^{1/n} x^{-r}\ln x\left[\Psi_k(x)-\sum_{i=0}^{r-1}\frac{x^i}{i!}\Psi_k^{(i)}(0)\right]dx \tag{13}$$

where $\Psi_k(x) = (x_-^{-s})_n x^k$.

Now

$$\Psi_k^{(i)}(x) = \begin{cases} \sum_{j=0}^{i}\binom{i}{j}\frac{k!}{(k-j)!}(x_-^{-s})_n^{(i-j)}x^{k-j}, & 0\le i<k,\\ \sum_{j=0}^{k}\binom{i}{j}\frac{k!}{(k-j)!}(x_-^{-s})_n^{(i-j)}x^{k-j}, & i\ge k.\end{cases}$$

Thus

$$\Psi_k^{(i)}(0) = 0 \tag{14}$$

for $i = 0, 1, 2, \dots, k-1$ and

$$\begin{aligned}\Psi_k^{(i)}(0) &= \frac{i!}{(i-k)!}(x_-^{-s})_n^{(i-k)}\rfloor_{x=0} \\ &= -\frac{i!n^{s+i-k}}{(i-k)!(s-1)!}\int_0^1 [\ln u - \ln n]\rho^{(s+i-k)}(u)\,du \end{aligned}\tag{15}$$

for $i = k, k+1, \dots,$ where the substitution $nt = u$ has been made. On making the substitution $nx = v$, it follows from eqs. (13), (14) and (15) that

$$\begin{aligned}&\langle F(x_+, -r)\ln x_+, (x_-^{-s})_n x^k\rangle \\ &= n^{r-1}\int_0^1 v^{-r}[\ln v - \ln n]\left[\Psi(v/n) - \sum_{i=k}^{r-1}\frac{v^i}{i!n^i}\Psi_k^{(i)}(0)\right]dv \\ &= -\frac{n^{r+s-k-1}}{(s-1)!}\int_0^1\left[v^{k-r}[\ln v - \ln n]\int_v^1 [\ln(u-v) - \ln n]\rho^{(s)}(u)\,du\right. \\ &\quad \left. - \sum_{i=k}^{r-1}\frac{v^{i-r}[\ln v - \ln n]}{(i-k)!}\int_0^1 [\ln u - \ln n]\rho^{(s+i-k)}(u)\,du\right]dv\end{aligned}$$

and so

$$\underset{n\to\infty}{N\text{-}\lim}\langle F(x_+, -r)\ln x_+, (x_-^{-s})_n x^k\rangle = 0 \tag{16}$$

for $k = 0, 1, 2, \dots, r+s-2$, the sum in the integral being empty if $k > r-1$.

When $k = r+s-1$ we have

$$\begin{aligned}&\langle F(x_+, -r)\ln x_+, (x_-^{-s})_n x^{r+s-1}\rangle \\ &= -\frac{1}{(s-1)!}\int_0^1 v^{s-1}[\ln v - \ln n]\int_v^1 [\ln(u-v) - \ln n]\rho^{(s)}(u)\,du\,dv.\end{aligned}$$

It follows that

$$\begin{aligned}&\underset{n\to\infty}{N\text{-}\lim}\langle F(x_+, -r)\ln x_+, (x_-^{-s})_n x^{r+s-1}\rangle \\ &= -\frac{1}{(s-1)!}\int_0^1 v^{s-1}\ln v\int_v^1 \ln(u-v)\rho^{(s)}(u)\,du\,dv \\ &= -\frac{1}{(s-1)!}\int_0^1 \rho^{(s)}(u)\int_0^u v^{s-1}\ln v\ln(u-v)\,dv\,du\end{aligned}$$

and on making the substitution $v = uy$

$$\begin{aligned}&\int_0^1 v^{s-1}\ln v\int_v^1 \ln(u-v)\rho^{(s)}(u)\,du\,dv \\ &= \int_0^1 \rho^{(s)}(u)\int_0^u v^{s-1}\ln v\ln(u-v)\,dv\,du = \int_0^1 u^s\ln^2 u\rho^{(s)}(u)\int_0^1 y^{s-1}\,dy\,du \\ &\quad + \int_0^1 u^s\ln u\rho^{(s)}(u)\int_0^1 y^{s-1}\ln(1-y)\,dy\,du + \int_0^1 u^s\ln u\rho^{(s)}(u)\end{aligned}$$

$$\times \int_0^1 y^{s-1} \ln y \, dy \, du + \int_0^1 u^s \rho^{(s)}(u) \int_0^1 y^{s-1} \ln y \ln(1-y) \, dy \, du.$$

It can be easily seen that

$$\int_0^1 y^{s-1} \ln y \, dy = \frac{1}{s^2} \tag{17}$$

$$\int_0^1 y^{s-1} \ln(1-y) \, dy = -\frac{\psi(s)}{s} \tag{18}$$

$$\int_0^1 y^{s-1} \ln y \ln(1-y) \, dy = \frac{\psi(s)}{s^2} - \frac{1}{s}\left[\xi(2) - \sum_{i=1}^{s} \frac{1}{i^2}\right] \tag{19}$$

$$\int_0^1 u^s \rho^{(s)}(u) \, du = \frac{1}{2}(-1)^s s! \tag{20}$$

$$\int_0^1 u^s \ln u \rho^{(s)}(u) \, du = (-1)^s s!\left[c_0(\rho) + \frac{1}{2}\psi(s)\right] \tag{21}$$

$$\int_0^1 u^s \ln^2 u \rho^{(s)}(u) \, du = (-1)^s s!\left[c_1(\rho) + 2\psi(s)c_0(\rho) + \sum_{i=1}^{s} \frac{\psi(i-1)}{i}\right] \tag{22}$$

Thus

$$N\!-\!\lim_{n\to\infty} \langle F(x_+, -r) \ln x_+, (x_-^{-s})_n x^{r+s-1} \rangle = -(-1)^s \kappa_s(\rho). \tag{23}$$

Further, when $k = r + s$ we have

$$\begin{aligned} &\langle F(x_+, -r) \ln x_+, (x_-^{-s})_n x^{r+s} \rangle \\ &= -\frac{n^{-1}}{(s-1)!} \int_0^1 v^s [\ln v - \ln n] \int_v^1 [\ln(u-v) - \ln n] \rho^{(s)}(u) \, du \, dv \\ &= 0(n^{-1} \ln^2 n). \end{aligned}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x) = \sum_{k=0}^{r+s-1} \frac{x^k}{k!} \phi^{(k)}(0) + \frac{x^{r+s}}{(r+s)!} \phi^{(r+s)}(\xi x).$$

It follows that

$$\begin{aligned} \langle F(x_+, -r) \ln x_+ (x_-^{-s})_n, \phi(x) \rangle = &\sum_{k=0}^{r+s-1} \frac{\phi^{(k)}(0)}{k!} \langle F(x_+, -r) \ln x_+, (x_-^{-s})_n x^k \rangle \\ &+ \frac{1}{(r+s)!} \langle F(x_+, -r) \ln x_+, (x_-^{-s})_n \phi^{(r+s)}(\xi x) x^{r+s} \rangle \end{aligned}$$

and so

$$\begin{aligned} N\!-\!\lim_{n\to\infty} \langle F(x_+, -r) \ln x_+ (x_-^{-s})_n, \phi(x) \rangle &= -\frac{(-1)^s}{(r+s-1)!} \kappa_s(\rho) \phi^{(r+s-1)}(0) \\ &= \frac{(-1)^r}{(r+s-1)!} \kappa_s(\rho) \langle \delta^{(r+s-1)}(x), \phi(x) \rangle \end{aligned}$$

on using eqs (16) and (23). Equation (9) follows.

Next, we consider the neutrix product of $F(x_-,-s)$ and $x_+^{-r}\ln x_+$.

Similarly we have on the interval $[0,1/n]$, the intersection of the supports of $F(x_-,-s)$ and $x_+^{-r}\ln x_+$ that

$$\langle F(x_-,-s),(x_+^{-r}\ln x_+)_n x^k\rangle=\int_0^{1/n}x^{-s}\left[\Phi_k(-x)-\sum_{i=0}^{s-1}\frac{(-x)^i}{i!}\Phi_k^{(i)}(0)\right]\mathrm{d}x$$

where $\Phi_k(x)=(x_+^{-r}\ln x_+)_n x^k$ and $\Phi_k(-x)=(-1)^k(x_-^{-r}\ln x_-)_n x^k$ and

$$\Phi_k^{(i)}(x)=\begin{cases}\sum_{j=0}^{i}\binom{i}{j}\frac{k!}{(k-j)!}(x_+^{-r}\ln x_+)_n^{(i-j)}x^{k-j}, & 0\le i<k,\\ \sum_{j=0}^{k}\binom{i}{j}\frac{k!}{(k-j)!}(x_+^{-r}\ln x_+)_n^{(i-j)}x^{k-j}, & i\ge k.\end{cases}$$

It follows that

$$\Phi_k^{(i)}(0)=0 \tag{24}$$

for $i=0,1,2,\dots,k-1$ and

$$\begin{aligned}\Phi_k^{(i)}(0)&=\frac{i!}{(i-k)!}(x_+^{-r}\ln x_+)_n^{(i-k)}\rfloor_{x=0}\\ &=\frac{(-1)^{r-1}i!}{2(i-k)!(r-1)!}(\ln^2 x_+)_n^{(r+i-k)}\rfloor_{x=0}+\frac{(-1)^{r-1}i!\psi(r-1)}{(i-k)!(r-1)!}(\ln x_+)_n^{(r+i-k)}\rfloor_{x=0}\end{aligned}$$

for $i=k,k+1,\dots$, where

$$x_+^{-r}\ln x_+=\frac{(-1)^{r-1}}{2(r-1)!}(\ln^2 x_+)^{(r)}+\frac{(-1)^{r-1}\psi(r-1)}{(r-1)!}(\ln x_+)^{(r)}.$$

It follows from the definition of $(x_+^{-r}\ln x_+)_n$ that

$$\begin{aligned}(x_+^{-r}\ln x_+)_n^{(i-k)}\rfloor_{x=0}=&\frac{(-1)^{r-1}}{2(r-1)!}\int_{-1/n}^{0}\ln^2(-t)\delta_n^{(r+i-k)}(t)\,\mathrm{d}t\\ &+\frac{(-1)^{r-1}\psi(r-1)}{(r-1)!}\int_{-1/n}^{0}\ln(-t)\delta_n^{(r+i-k)}(t)\,\mathrm{d}t\end{aligned}$$

and on making the substitution $-nt=u$

$$\begin{aligned}\Phi_k^{(i)}(0)&=\frac{i!}{(i-k)!}(x_+^{-r}\ln x_+)_n^{(i-k)}\rfloor_{x=0}\\ &=\frac{(-1)^{r-1}i!n^{r+i-k}}{2(i-k)!(r-1)!}\int_0^1[\ln^2 u-2\ln n\ln u+\ln^2 n]\rho^{(r+i-k)}(u)\mathrm{d}u\\ &\quad+\frac{(-1)^{r-1}i!\psi(r-1)n^{r+i-k}}{(i-k)!(r-1)!}\int_0^1[\ln u-\ln n]\rho^{(r+i-k)}(u)\,\mathrm{d}u\end{aligned} \tag{25}$$

for $i=k,k+1,\dots$

On making the substitution $nx=v$ it follows from above that

$$\langle F(x_-,-s),(x_+^{-r}\ln x_+)_n x^k\rangle=n^{s-1}\int_0^1 v^{-s}\left[\Phi(-v/n)-\sum_{i=k}^{s-1}\frac{v^i}{i!n^i}\Phi_k^{(i)}(0)\right]\mathrm{d}v$$

$$= \frac{(-1)^k n^{r+s-k-1}}{(r-1)!} \int_0^1 \left[v^{k-s} \int_v^1 \left[\frac{1}{2} [\ln(u-v) - \ln n]^2 \right.\right.$$
$$\left. + \psi(r-1)[\ln(u-v) - \ln n] \right] \rho^{(r)}(u)\, du - \sum_{i=k}^{s-1} \frac{v^{i-s}}{(i-k)!}$$
$$\left. \times \int_0^1 \left[\frac{1}{2} [\ln^2 u - 2 \ln n \ln u + \ln^2 n] + \psi(r-1)[\ln u - \ln n] \right] \rho^{(r+i-k)}\, du \right] dv.$$

We thus have that

$$\underset{n\to\infty}{N\text{—}\lim} \langle F(x_-, -s), (x_+^{-r} \ln x_+)_n x^k \rangle = 0 \tag{26}$$

for $k = 0, 1, 2, \ldots, r+s-2$ and the sum in the integral again being empty if $k > s-1$. When $k = r+s-1$ we have

$$\langle F(x_-, -s), (x_+^{-r} \ln x_+)_n x^{r+s-1} \rangle = \frac{(-1)^{r+s-1}}{(r-1)!} \int_0^1 v^{r-1} \int_v^1$$
$$\times \left[\frac{1}{2} [\ln(u-v) - \ln n]^2 + \psi(r-1)[\ln(u-v) - \ln n] \right] \rho^{(r)}(u)\, du\, dv$$

and on making the substitution $v = uy$

$$\underset{n\to\infty}{N\text{—}\lim} \langle F(x_-, -s), (x_+^{-r} \ln x_+)_n x^{r+s-1} \rangle$$
$$= \frac{(-1)^{r+s-1}}{(r-1)!} \int_0^1 \rho^{(r)}(u) \int_0^u v^{r-1} \left[\frac{1}{2} \ln^2(u-v) + \psi(r-1) \ln(u-v) \right] dv\, du$$
$$= \frac{(-1)^{r+s-1}}{(r-1)!} \int_0^1 u^r \rho^{(r)}(u) \int_0^1 y^{r-1} \left[\frac{1}{2} [\ln^2 u + \ln u \ln(1-y) + \ln^2(1-y)] \right.$$
$$\left. + \psi(r-1)[\ln u + \ln(1-y)] \right] dy\, du.$$

It is easily seen that

$$\int_0^1 y^{r-1} \ln^2(1-y)\, dy = \frac{2}{r} \psi_1(r). \tag{27}$$

It now follows from eqs (17), (18), (19), (20), (21) and (27) that

$$\underset{n\to\infty}{N\text{—}\lim} \langle F(x_-, -s), (x_+^{-r} \ln x_+)_n x^{r+s-1} \rangle = -(-1)^s \Lambda_r(\rho). \tag{28}$$

Similarly, when $k = r+s$ we have

$$\underset{n\to\infty}{N\text{—}\lim} \langle F(x_-, -s), (x_+^{-r} \ln x_+)_n x^{r+s} \rangle$$
$$= \frac{(-1)^{r+s} n^{-1}}{(r-1)!} \int_0^1 v^r \int_0^1 \left[\frac{1}{2} [\ln(u-v) - \ln n]^2 \right.$$
$$\left. + \psi(r-1)[\ln(u-v) - \ln n] \right] \rho^{(r)}(u)\, du\, dv$$
$$= 0(n^{-1} \ln^2 n).$$

It follows that

$$\langle F(x_-,-s)(x_+^{-r}\ln x_+)_n,\phi(x)\rangle-\sum_{k=0}^{r+s-1}\frac{\phi^{(k)}(0)}{k!}\langle F(x_-,-s),(x_+^{-r}\ln x_+)_n x^k\rangle$$
$$=0(n^{-1}\ln^2 n).$$

It follows from eqs (26) and (28) that

$$N-\lim_{n\to\infty}\langle F(x_-,-s)(x_+^{-r}\ln x_+)_n,\phi(x)\rangle=F(x_-,-s)\circ x_+^{-r}\ln x_+$$
$$=-\frac{(-1)^s\Lambda_r(\rho)}{(r+s-1)!}\phi^{(r+s-1)}(0)=\frac{(-1)^r\Lambda_r(\rho)}{(r+s-1)!}\langle\delta^{(r+s-1)}(x),\phi(x)\rangle.$$

It was proved in [8] that

$$x_+^{-r}=F(x_+,-r)+\frac{(-1)^r\psi(r-1)}{(r-1)!}\delta^{(r-1)}(x) \tag{29}$$

$$x_-^{-r}=F(x_-,-r)-\frac{\psi(r-1)}{(r-1)!}\delta^{(r-1)}(x). \tag{30}$$

Equation (12) follows from eqs (8) and (30). Equation (11) follows from eqs (4) and (5). The neutrix products $F(x_+,-r)\ln x_+\circ F(x_-,-s)$ and $F(x_+,-s)\circ F(x_-,-r)\ln x_+$ can be easily obtained from eqs (7), (8), (11), (29) and (30).

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Thermoelasticity with thermal relaxation: An alternative formulation

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Abstract. The theory of thermoelasticity with thermal relaxation for homogeneous materials is formulated upon the basis of the law of balance of energy and the law of balance of entropy, proposed by Green and Naghdi [5]. The non-linear theory is formulated first; then the linearized theory is deduced. The uniqueness of solution of a typical initial, mixed boundary value problem is established.

Keywords. Generalized thermoelasticity; thermal relaxation; uniqueness of solution.

1. Introduction

In the conventional approach to thermo-mechanical theories of continua, the constitutive equations are formulated upon the basis of the law of balance of energy and an entropy production inequality. In 1991, Green and Naghdi [5] suggested an alternative procedure that is significantly different from the conventional one. In this new procedure, the constitutive equations are formulated upon the basis of a reduced energy equation which is a blend of the equation of balance of energy and an equation of balance of entropy. A novel feature of this procedure is that an entropy production inequality is not employed in the process of obtaining the constitutive equations; the inequality is utilized to impose additional restrictions, if any, on the constitutive variables *only after* the constitutive equations have been derived. By adopting this new procedure, Green and Naghdi [6, 7] have formulated three models of the thermoelasticity theory for a homogeneous and isotropic material. The first of these models is identical with the conventional parabolic model [1], and the second and third models are hyperbolic models that were not formulated earlier.

In this article, we formulate the thermoelasticity theory with thermal relaxation by adopting the new procedure suggested in [5] and by including the entropy-flux as an independent constitutive variable. This formulation may be regarded as an alternative to the formulations of the theory presented by Lord and Shulman [9] and Dhaliwal and Sherief [4] by employing the conventional procedure. The idea of using the entropy-flux as an independent constitutive variable is analogous to the idea of taking the heat-flux as a constitutive variable [8].

In § 2, we derive the reduced energy equation, in the material description, by starting with the equation of balance of energy and the equation of balance of entropy postulated in [5]. In § 3, we employ this reduced energy equation to construct the constitutive equations of the non-linear theory. Thereafter, in § 4, we obtain the energy equation in a form that is appropriate for our formulation and a dissipation inequality that arises from the second law of thermodynamics. The remaining sections are concerned with the

linearized version of the theory. In § 5, we linearize the constitutive equations and write down the complete set of field equations of the linearized theory for a homogeneous and anisotropic material. We find that the dissipation inequality imposes a restriction only on one material constant, χ, which appears in the heat conduction law of the theory and which dictates the damping of thermal signals, the restriction being $\chi \geq 0$, and that the linearized theory admits wave-like thermal signals propagating at a finite speed. In § 6, we deduce the field equations for an isotropic material. In § 7, we analyse the field equations obtained in § 5 and find that: (i) when χ is finite and positive, our theory becomes coincident with the theory presented in [4, 9], and (ii) when $\chi = 0$, our theory serves as the anisotropic counterpart of the thermoelasticity theory without energy dissipation formulated in [7]. In the case when $\chi \to \infty$, we recover the conventional coupled thermoelasticity theory [1]. Thus, the linearized thermoelasticity theory presented here may be regarded as a unified thermoelasticity theory from which some known thermoelasticity theories emerge as particular cases.

In § 8 we obtain a global energy equation for the linearized anisotropic theory; in § 9, we use it to prove the uniqueness of solution of a typical initial, mixed boundary value problem. The energy equation and the uniqueness of solution hold for all the three models of thermoelasticity mentioned in the preceding paragraph.

2. Laws of balance of energy and entropy

We start with the equations representing the law of balance of energy (first law of thermodynamics) and the law of balance of entropy postulated in [5]. In the local form, these equations are as given below in the material description:

$$\rho_0(\dot{\varepsilon} - r) = \mathbf{S} \cdot \dot{\mathbf{E}} - \operatorname{div} \mathbf{q}, \tag{2.1}$$

$$\rho_0 \dot{\eta} = \rho_0(s + \xi) - \operatorname{div} \mathbf{p}. \tag{2.2}$$

Here, ρ_0 (positive constant) is the mass density of the material in the referential (undeformed) state; ε, η, r, s and ξ are, in this order, the internal energy, the entropy density, the external rate of supply of heat, the external rate of supply of entropy and the internal rate of production of entropy, each measured per unit mass; $\mathbf{q}$ and $\mathbf{p}$ are the heat-flux and the entropy-flux vectors referred to the unit area in the referential state; and $\mathbf{E}$ and $\mathbf{S}$ are the Green strain tensor and the (second) Piola–Kirchhoff stress tensor, respectively. A superposed dot denotes material differentiation with respect to time t and div denotes the divergence operator with respect to the position $\mathbf{X}$ in the referential state. Also, the direct vector/tensor notation [2] is employed. The energy balance eq. (2.1) is well-known and is available in the textbook literature; for example, see [10]. The entropy balance equation (2.2) is the counterpart, in the material description, of eq. (7.20a) of [5]. Our notation is slightly different from that used in [5, 10].

As in [5], we suppose that $(\mathbf{p}, s)$ are related to $(\mathbf{q}, r)$ through the equations

$$(\mathbf{p}, s) = \frac{1}{\vartheta}(\mathbf{q}, r), \tag{2.3}$$

where ϑ is a scalar function of an empirical temperature T and other constitutive variables such that

$$\vartheta > 0, \quad \frac{\partial \vartheta}{\partial T} > 0. \tag{2.4}$$

Further, we consider the free energy function ψ defined by

$$\psi = \varepsilon - \eta\vartheta \tag{2.5}$$

for the constitutive development of the theory.

Substituting for $\mathbf{q}$ and r from eqs (2.3) and for ε from expression (2.5) in the energy balance equation (2.1) and eliminating $\dot{\eta}$ from the resulting equation with the aid of the entropy balance eq. (2.2), we arrive at the following reduced energy equation:

$$\mathbf{S} \cdot \dot{\mathbf{E}} - \mathbf{p} \cdot \nabla\vartheta - \rho_0(\dot{\psi} + \eta\dot{\vartheta}) - \rho_0\zeta = 0. \tag{2.6}$$

Here,

$$\zeta = \vartheta\xi \tag{2.7}$$

is the internal rate of production of heat per unit mass, and ∇ is the gradient operator with respect to $\mathbf{X}$. Equation (2.6) is the counterpart, in the material description, of eq. (9.9) of [5].

3. Constitutive equations

Guided by the form of the reduced energy eq. (2.6), we proceed with the hypothesis that the scalar functions ψ and ϑ depend, in general, on $\mathbf{E}, T, \nabla T \equiv \mathbf{g}$, and $\mathbf{p}$. Since the explicit dependence on $\mathbf{X}$ is not assumed, it is understood that we are concerned with a homogeneous material. To compensate the lack of evolution equations, we treat the entropy-flux rate $\dot{\mathbf{p}}$ as a dependent variable and assume that, like ψ and $\vartheta, \dot{\mathbf{p}}$ is also a function of the constitutive variables; that is,

$$\dot{\mathbf{p}} = \dot{\mathbf{p}}(\mathbf{E}, T, \mathbf{g}, \mathbf{p}). \tag{3.1}$$

But, $\dot{\mathbf{p}}$ cannot be an arbitrary function; it has to meet appropriate constitutive restrictions. We postulate that the following restrictions (expressed in the Cartesian component form) hold:

(i) $$\frac{\partial \dot{p}_i}{\partial p_j} = 0 \quad \text{for} \quad i \neq j, \tag{3.2}$$

(ii) The matrix $[\partial \dot{p}_i/\partial g_j]$ is negative definite, with

$$\left[\frac{\partial \dot{p}_i}{\partial g_j}\right]^{-1} = -\rho_0 \left[\frac{\partial^2 \psi}{\partial p_i \partial p_j}\right]. \tag{3.3}$$

Now, carrying out the material differentiations of ψ and ϑ in the reduced energy eq. (2.6), we obtain the identity:

$$\left[\mathbf{S} - \rho_0\left(\frac{\partial\psi}{\partial\mathbf{E}} + \eta\frac{\partial\vartheta}{\partial\mathbf{E}}\right)\right]\dot{\mathbf{E}} - \rho_0\left[\frac{\partial\psi}{\partial T} + \eta\frac{\partial\vartheta}{\partial T}\right]\dot{T} - \rho_0\left[\frac{\partial\psi}{\partial\mathbf{g}} + \eta\frac{\partial\vartheta}{\partial\mathbf{g}}\right]\dot{\mathbf{g}}$$
$$- \rho_0\left[\frac{\partial\psi}{\partial\mathbf{p}} + \eta\frac{\partial\vartheta}{\partial\mathbf{p}}\right]\dot{\mathbf{p}} - \mathbf{p} \cdot \nabla\vartheta - \rho_0\zeta = 0. \tag{3.4}$$

If we assume that $\mathbf{S}$ is independent of $\dot{\mathbf{E}}$ and η is independent of $\dot{T}$, then the identity (3.4) is satisfied for arbitrary $\dot{\mathbf{E}}$ and $\dot{T}$ provided the following relations hold:

$$\frac{\partial\vartheta}{\partial\mathbf{E}} = 0, \quad \frac{\partial\vartheta}{\partial\mathbf{g}} = \frac{\partial\vartheta}{\partial\mathbf{p}} = 0, \tag{3.5}$$

$$\mathbf{S} = \rho_0 \frac{\partial \psi}{\partial \mathbf{E}}, \tag{3.6}$$

$$\frac{\partial \psi}{\partial T} + \eta \frac{\partial \vartheta}{\partial T} = 0, \tag{3.7}$$

$$\frac{\partial \psi}{\partial \mathbf{g}} = 0, \tag{3.8}$$

$$\rho_0 \frac{\partial \psi}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \mathbf{p} \cdot \nabla \vartheta + \rho_0 \zeta = 0. \tag{3.9}$$

Relations (3.5) specify that ϑ is a function of T only. In view of the restrictions (2.4), it follows that ϑ may be regarded as an absolute temperature. Therefore, without loss of generality we may set $\vartheta \equiv T$, where $T > 0$ is now an absolute temperature. Then eqs (3.7) and (3.9) become

$$\eta = -\frac{\partial \psi}{\partial T} \tag{3.10}$$

$$\rho_0 \zeta = -\left\{ \rho_0 \frac{\partial \psi}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \mathbf{p} \cdot \mathbf{g} \right\}. \tag{3.11}$$

Evidently, (3.6) and (3.10) are the constitutive equations for $\mathbf{S}$ and η in the theory being considered here; we observe that $\mathbf{S}$ and η are derivable from the potential ψ which is independent of $\mathbf{g}$ as in the conventional thermoelasticity theory [1]. Equation (3.11) serves as the formula for ζ in the present development.

4. Energy equation and dissipation inequality

Now, we turn our attention back to the entropy balance eq. (2.2). Substituting for ξ, with the use of the relation (2.7) and the formula (3.11), in this equation, and bearing in mind that $\vartheta \equiv T$, we obtain the equation

$$\rho_0 \left[\frac{\partial \psi}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + T(\dot{\eta} - s) \right] + \operatorname{div}(T\mathbf{p}) = 0. \tag{4.1}$$

This serves as the energy equation for the theory being considered.

It is to be noted that, in the construction of the constitutive and other equations in § 3 and in the derivation of the energy eq. (4.1) above, we have not made use of the entropy production inequality. Now, we invoke this inequality to impose the necessary thermodynamic restriction on the constitutive variables.

In the Clausius–Duhem localized form, the entropy production inequality (also called the second law of thermodynamics) reads [10]

$$\rho_0 \dot{\eta} \geq -\operatorname{div} \mathbf{p} + \rho_0 s. \tag{4.2}$$

On using the entropy balance eq. (2.2), this inequality becomes

$$\rho_0 \xi \geq 0 \tag{4.3}$$

which, with the aid of the relation (2.7) and the formula (3.11), and the fact that $\vartheta \equiv T > 0$, takes the form

$$\rho_0 \frac{\partial \psi}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \mathbf{p} \cdot \mathbf{g} \leq 0. \tag{4.4}$$

This is the dissipation inequality to be satisfied in the present development.

The formulation of the entropy-flux dependent non-linear thermoelasticity theory for a homogeneous material, by following the Green and Naghdi approach [5], is now complete. To sum up: Under the thermodynamic restriction (4.4) and the restrictions on $\dot{\mathbf{p}}$ postulated in § 3, eqs (3.1), (3.6) and (3.10) are the constitutive equations and eq. (4.1) is the energy equation of the theory. These equations together with the equations of balance of forces and moments and appropriate kinematic relations form a complete system of governing equations of the theory.

5. Linearized theory

We now proceed to deduce the governing equations of the linearized version of the theory by making the usual assumptions and approximations.

We set $\theta = T - \theta_0$, where $\theta_0(> 0)$ is the reference temperature and use θ in place of T as the temperature field. We suppose that all the independent and dependent field variables vanish in the referential state.

Now, expanding ψ in Maclaurin series up to quadratic terms, and bearing in mind the equations (3.6), (3.8) and (3.10), we obtain the expression

$$\rho_0\psi = \frac{1}{2}\left\{\mathbf{E}\cdot\mathbf{C}[\mathbf{E}] + 2(\mathbf{M}\cdot\mathbf{E})\theta - \frac{c}{\theta_0}\theta^2 + \mathbf{p}\cdot\mathbf{N}\mathbf{p}\right\}. \tag{5.1}$$

Here, $\mathbf{C}$ is the elasticity tensor with its usual symmetries, $\mathbf{M}$ is the symmetric stress-temperature tensor and c is the specific heat, referred to the isothermal state. Further, $\mathbf{N}$ is a symmetrized constant tensor characteristic of the theory being formulated.

Next, expanding $\dot{\mathbf{p}}$, given by (3.1), in Maclaurin series up to linear terms, and bearing in mind the restrictions on $\dot{\mathbf{p}}$ postulated in § 3, we obtain

$$\dot{\mathbf{p}} = -\chi\mathbf{p} - \mathbf{H}(\nabla\theta). \tag{5.2}$$

Here, χ is a scalar constant and $\mathbf{H}$ is a positive definite constant tensor such that $\mathbf{H}^{-1} = \mathbf{N}$. It immediately follows that $\mathbf{N}$ is also positive definite. Further, since $\mathbf{N}$ is symmetrized, $\mathbf{H}$ is symmetric.

Using expressions (5.1) and (5.2), we find that

$$\rho_0\frac{\partial\psi}{\partial\mathbf{p}}\cdot\dot{\mathbf{p}} + \mathbf{p}\cdot\mathbf{g} = -\chi\mathbf{p}\cdot\mathbf{N}\mathbf{p}. \tag{5.3}$$

Consequently, and in view of the positive definiteness of $\mathbf{N}$, the dissipation inequality (4.4) becomes

$$\chi \geq 0. \tag{5.4}$$

With the aid of expression (5.3), the formula (3.11) yields (in quadratic approximation)

$$\rho_0\zeta = \chi\mathbf{p}\cdot\mathbf{N}\mathbf{p}. \tag{5.5}$$

This simple and elegant formula for ζ shows that $\zeta = 0$ if and only if $\chi = 0$. On the other hand, we note from eq. (5.2) that $\chi = 0$ if and only if $\dot{\mathbf{p}}$ is independent of $\mathbf{p}$. Since dissipation of energy occurs only when $\xi > 0$, or equivalently $\zeta > 0$, it follows that the linearized theory being considered here admits dissipation of energy if and only if $\chi > 0$ (or, equivalently, $\dot{\mathbf{p}}$ depends on $\mathbf{p}$). If $\chi = 0$, the theory does obey the second law of thermodynamics but does not admit dissipation of energy.

We now turn our attention to the constitutive equations. Equation (5.2) is indeed the constitutive equation for $\dot{\mathbf{p}}$ (in the linearized theory). With the aid of the expression (5.1), eqs (3.6) and (3.10) yield

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] + \mathbf{M}\theta, \tag{5.6}$$

$$\rho_0\eta = \frac{c}{\theta_0}\theta - \mathbf{M}\cdot\mathbf{E}. \tag{5.7}$$

Equations (5.2), (5.6) and (5.7) form the full set of constitutive equations for the linearized version of the theory being formulated here. We observe that equations (5.6) and (5.7) are identical with their counterparts in the conventional thermoelasticity theory (CTE) [1].

Next, we consider the balance laws and the kinematic relation (strain-displacement relation) applicable in the linearized theory. Substituting for ψ and η from eqs (5.1) and (5.7) in the energy eq. (4.1) and neglecting the non-linear terms in the field variables, we obtain the following linearized energy equation

$$\frac{c}{\theta_0}\dot{\theta} - \mathbf{M}\cdot\dot{\mathbf{E}} - \rho_0 s + \operatorname{div}\mathbf{p} = 0. \tag{5.8}$$

Also, the linearized equation of motion is

$$\operatorname{div}\mathbf{S} + \rho_0\mathbf{b} = \rho_0\ddot{\mathbf{u}} \tag{5.9}$$

and the linearized strain-displacement relation is

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T). \tag{5.10}$$

Here, **u** is the displacement vector and **b** is the external force per unit mass. Also, now, **S** is the Cauchy's stress tensor, and **E** is the Cauchy's strain tensor, and a superposed dot denotes partial differentiation with respect to t.

Eliminating **S**, **E**, **p** and η from eqs (5.2) and (5.6)–(5.10), we obtain the following system of equations:

$$\operatorname{div}\mathbf{C}[\nabla\mathbf{u}] + \mathbf{M}\nabla\theta + \rho_0\mathbf{b} = \rho_0\ddot{\mathbf{u}}, \tag{5.11}$$

$$\operatorname{div}(\mathbf{H}\nabla\theta) = \left(\chi + \frac{\partial}{\partial t}\right)\left[\frac{c}{\theta_0}\dot{\theta} - \mathbf{M}\cdot\nabla\dot{\mathbf{u}} - \rho_0 s\right]. \tag{5.12}$$

The system (5.11) and (5.12) contains four scalar equations that govern four field variables: the three components of the displacement vector **u** and the temperature θ, which are coupled together. This system forms a complete system of field equations for the entropy-flux dependent linearized thermoelasticity theory (or briefly EFDTE) formulated here; in this theory, eq. (5.11) is the equation of motion and eq. (5.12) is the equation of heat transport.

We observe that the equation of motion (5.11) is identical with its counterpart in CTE [1]. But the heat transport eq. (5.12) is different from its conventional counterpart. If we suppose that the specific heat c is positive (as usual), then we note that, like the equation of motion (5.11), the heat transport eq. (5.12) is of the hyperbolic-type (wave-type) for all admissible values of χ, and consequently it allows wave-like thermal signals propagating with finite speed. Thus, the linear/linearized theory of thermoelasticity formulated here is governed by a fully hyperbolic system of field equations; it is therefore a *hyperbolic*

thermoelasticity theory. (Traditionally, such theories are referred to as generalized thermoelasticity theories or thermoelasticity theories with second sound [3]).

6. Isotropy

For isotropic materials, all the tensors present in the governing equations obtained in the preceding section are isotropic. In this case, we have

$$\mathbf{C}[\mathbf{E}] = \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{6.1}$$

$$\mathbf{M} = -\gamma\mathbf{I}, \tag{6.2}$$

$$\mathbf{H} = h\mathbf{I}. \tag{6.3}$$

Here, λ and μ are the usual Lame' constants; $\gamma = (3\lambda + 2\mu)a$, where a is the coefficient of volume expansion; and $h > 0$ is a material constant characteristic of the theory being considered. Consequently, the constitutive eqs (5.2), (5.6) and (5.7) become

$$\dot{\mathbf{p}} = -\chi\mathbf{p} - h\nabla\theta, \tag{6.4}$$

$$\mathbf{S} = \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} - \gamma\theta\mathbf{I}, \tag{6.5}$$

$$\rho_0\eta = \frac{c}{\theta_0}\theta + \gamma(\operatorname{tr}\mathbf{E}). \tag{6.6}$$

Also, now, the field eqs (5.11) and (5.12) reduce to

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} - \gamma\nabla\theta + \rho_0\mathbf{b} = \rho_0\ddot{\mathbf{u}}, \tag{6.7}$$

$$h\nabla^2\theta = \left(\chi + \frac{\partial}{\partial t}\right)\left[\frac{c}{\theta_0}\dot{\theta} + \gamma \operatorname{div}\dot{\mathbf{u}} - \rho_0 s\right]. \tag{6.8}$$

We note that the constitutive eqs (6.5) and (6.6) and the equation of motion (6.7) are identical with their counterparts in CTE [1]. However, the heat transport eq. (6.8) is different from its conventional counterpart.

The hyperbolic nature of eq. (6.8) is more transparent than that of its anisotropic counterpart: eq. (5.12). Indeed, eq. (6.8) predicts the *finite speed*

$$c_T = \left(\frac{h\theta_0}{c}\right)^{1/2} \tag{6.9}$$

for thermal signals. We observe that, of the two material constants characteristic of the theory being considered, namely χ and h, only h contributes to the speed c_T. The constant χ, when it is not zero, dictates damping of thermal signals, as can be seen from eqs (5.12) and (6.8).

7. Comparison with earlier formulations

It has been noted that in the linearized theory, formulated in § 5, the entropy production inequality imposes only one restriction: $\chi \geq 0$. In this section, we consider three particular cases that are consistent with this restriction and show that each case corresponds to one already known model of thermoelasticity. First, we note that, in the linearized theory, the relations (2.3) read

$$(\mathbf{p}, s) = \frac{1}{\theta_0}(\mathbf{q}, r). \tag{7.1}$$

Case (i): Suppose χ is finite and positive. Then, with the use of relation $(7.1)_1$, the constitutive equation (5.2) may be rewritten as

$$\left(1+\tau\frac{\partial}{\partial t}\right)\mathbf{q} = -\mathbf{K}\nabla\theta, \tag{7.2}$$

where

$$\tau = \frac{1}{\chi} \quad \text{and} \quad \mathbf{K} = \frac{\theta_0}{\chi}\mathbf{H}. \tag{7.3}$$

We note that eq. (7.2) is the anisotropic version of the Cattaneo–Vernote-type heat conduction law for the heat flux vector $\mathbf{q}$ with τ representing the thermal relaxation time parameter and $\mathbf{K}$ representing the thermal conductivity tensor [3]. The physical meaning of τ is well-known: It represents the time-lag needed to establish the steady state of heat conduction in a material element when a temperature-gradient is suddenly imposed on that element [3]. Thus, the material constant χ and the tensor $\mathbf{H}$ that we introduced in § 5 do have definite physical meanings. Since $\mathbf{N} = \mathbf{H}^{-1}$, the tensor $\mathbf{N}$ also has a definite physical meaning; indeed,

$$\mathbf{N} = \frac{\theta_0}{\chi}\mathbf{G}, \tag{7.4}$$

where $\mathbf{G} = \mathbf{K}^{-1}$ is the thermal resistivity tensor.

Next, with the use of the relations (7.3) and (7.1), eq. (5.12) may be rewritten as

$$\operatorname{div}[\mathbf{K}(\nabla\theta)] = \left(1+\tau\frac{\partial}{\partial t}\right)[c\dot{\theta} - \theta_0\mathbf{M}\cdot\nabla\dot{\mathbf{u}} - \rho_0 r]. \tag{7.5}$$

This equation is identical (apart from the notation) with the heat transport equation obtained in [4] in the conventional formulation of the linear thermoelasticity theory with one relaxation time, or the extended thermoelasticity theory (ETE).

In the case of isotropic materials, eq. (7.2) reduces to the Cattaneo–Vernotte law

$$\left(1+\tau\frac{\partial}{\partial t}\right)\mathbf{q} = -k\nabla\theta, \tag{7.6}$$

and the heat transport eq. (7.5) reduces to

$$k\nabla^2\theta = \left(1+\tau\frac{\partial}{\partial t}\right)[c\dot{\theta} + \theta_0\gamma\operatorname{div}\dot{\mathbf{u}} - \rho_0 r], \tag{7.7}$$

where

$$k = \frac{h\theta_0}{\chi} \tag{7.8}$$

is the (scalar) thermal conductivity.

We note that eq. (7.7) is the heat transport equation of the Lord–Shulman theory [9].

Thus, when χ is finite and positive, the heat transport equation formulated here becomes identical with those obtained in the conventional formulations of ETE, as presented in [4, 9].

Since $\chi > 0$ in the present case, the formula (5.5) implies that $\zeta > 0$; consequently, there *does occur* dissipation of thermal energy. Thus, ETE is a thermoelasticity theory

with energy dissipation. Further, with the aid of expressions (7.1) and (7.4), formula (5.5) yields the following formulas for ζ in ETE in the anisotropic and isotropic cases respectively:

$$\rho_0\zeta = \frac{1}{\theta_0}\mathbf{q}\cdot\mathbf{G}\mathbf{q}, \tag{7.9}$$

$$\rho_0\zeta = \frac{q^2}{k\theta_0}. \tag{7.10}$$

Case (ii): Suppose χ is so large that $(1/\chi)$ may be ignored but $\mathbf{H}$ is such that $(\mathbf{H}/\chi)$ is finite and the relation $(7.3)_2$ remains valid. Then, the constitutive equation (5.2) reduces, with the aid of $(7.1)_1$, to the classical Fourier's law

$$\mathbf{q} = -\mathbf{K}(\nabla\theta). \tag{7.11}$$

Consequently, eq. (5.12) reduces to

$$\operatorname{div}[\mathbf{K}(\nabla\theta)] = c\dot{\theta} - \theta_0\mathbf{M}\cdot\nabla\dot{\mathbf{u}} - \rho_0 r, \tag{7.12}$$

which is the heat transport equation in CTE [1]. We note that the constitutive eq. (5.6) and therefore the equation of motion (5.11) are valid in CTE as well. Thus, CTE is recovered as a limiting case of our EFDTE.

When χ is large, formula (5.5) indicates that ζ is also large. Hence, CTE admits energy dissipation of high intensity. With the aid of relations $(7.1)_1$, (7.4) and (7.11), formula (5.5) yields the following formula for ζ in CTE:

$$\rho_0\zeta = -\mathbf{p}\cdot\nabla\theta. \tag{7.13}$$

This formula, which can be rewritten in different equivalent forms with the use of relations $(7.1)_1$ and (7.7), agrees with expression (2.34) of [7] obtained in an alternative way.

The isotropic versions of the governing equations of CTE can be recovered as in the preceding case. These equations agree with those of "Thermoelasticity, type I", formulated in [7].

Case (iii): Suppose $\chi = 0$. Then, formula (5.5) implies that $\zeta = 0$. Hence, in this case, there occurs *no* energy dissipation. As mentioned in § 5, this situation is not thermodynamically inadmissible, and our theory remains valid in this situation also. The theory is now (when $\chi = 0$) is referred to as "Thermoelasticity without energy dissipation" (or, briefly, TEWOED).

Since the constitutive eqs (5.6) and (5.7) and the equation of motion (5.11) are independent of χ, these equations remain valid in the present case also. But the constitutive eq. (5.2) and therefore the heat transport eq. (5.12) get modified; the modified equations are

$$\dot{\mathbf{p}} = -\mathbf{H}(\nabla\theta), \tag{7.14}$$

$$\operatorname{div}[\mathbf{K}^*(\nabla\theta)] = c\ddot{\theta} - \theta_0\mathbf{M}\cdot\nabla\ddot{\mathbf{u}} - \rho_0\dot{r}. \tag{7.15}$$

Here

$$\mathbf{K}^* = \theta_0\mathbf{H}. \tag{7.16}$$

We immediately find that $\mathbf{K}^*$ is the limiting value of $\chi\mathbf{K}$, as $\chi \to 0$.

Thus, (5.6), (5.7) and (7.14) are the constitutive equations and (5.11) and (7.15) are the field equations of TEWOED. We note that, like eq. (5.11), the heat transport eq. (7.15) is

of hyperbolic-type and consequently predicts the occurrence of wave-like thermal signals propagating with finite speed. Thus, TEWOED is also a hyperbolic/generalized thermo-elasticity theory.

We note that there is a significant difference between the hyperbolic heat transport eq. (7.5) of ETE and the hyperbolic heat transport eq. (7.15) of TEWOED. Whereas eq. (7.5) predicts damping of thermal signals, eq. (7.15) does not. Thus, unlike in ETE, heat waves are *undamped* in TEWOED.

For an isotropic material, eqs (6.5), (6.6) and (6.7) remain valid for $\chi = 0$. But, eqs (6.4) and (6.8) become

$$\dot{\mathbf{p}} = -\frac{\kappa^*}{\theta_0}\nabla\theta, \tag{7.17}$$

$$\kappa^*\nabla^2\theta = c\ddot{\theta} + \theta_0\gamma\,\mathrm{div}\,\ddot{\mathbf{u}} - \rho_0\dot{r}. \tag{7.18}$$

Here

$$\kappa^* = h\theta_0. \tag{7.19}$$

Thus, eqs (6.5), (6.6) and (7.17) are the constitutive equations and eqs (6.7) and (7.18) are the field equations of isotropic TEWOED. We notice that these equations are identical (but for the notation) with eqs $(2.38)_{1,2}$, (3.13), (3.15) and (3.16) of "Thermoelasticity, type II" formulated in [7].

From the heat transport eq. (7.18), it is evident that in the isotropic TEWOED the speed of thermal signals is

$$c_T = \left(\frac{\kappa^*}{c}\right)^{1/2}. \tag{7.20}$$

Thus, the material constant κ^* (which is a characteristic of the isotropic TEWOED) together with c determines the speed of thermal signals. The relations (7.9) and (7.19) provide a physical meaning for this constant: κ^* is the limiting value of χk, as $\chi \to 0$.

It may be mentioned that Green and Naghdi [7] have not considered the anisotropic case. The theory governed by our eqs (5.6), (5.7), (5.11), (7.14) and (7.15) serves as the anisotropic counterpart of Thermoelasticity, type II of [7].

8. Global energy equation

We now turn our attention back to the linearized anisotropic theory formulated in § 5 and derive an energy equation in global form in the context of this theory.

Substituting for ζ from the relation (2.7) and setting $\vartheta \equiv T = \theta_0 + \theta$ in the reduced energy equation (2.6), and eliminating $\theta\xi$ from the resulting equation and the entropy balance eq. (2.2), we obtain the equation

$$\rho_0\overline{(\psi + \eta\theta)}^{\,\cdot} = \mathbf{S}\cdot\dot{\mathbf{E}} - \mathrm{div}(\theta\mathbf{p}) + \rho_0 s\theta - \rho_0\theta_0\xi. \tag{8.1}$$

Substituting for ψ from eq. (5.1), for η from eq. (5.7) and for $\mathbf{E}$ from the strain-displacement relation (5.10) in eq. (8.1), and simplifying the resulting equation with the aid of standard tensor identities and the equation of motion (5.9), we arrive at the equation

$$\dot{\Lambda} = \mathrm{div}(\mathbf{S}\dot{\mathbf{u}} - \theta\mathbf{p}) + \rho_0(\mathbf{b}\cdot\dot{\mathbf{u}} + s\theta - \theta_0\xi) \tag{8.2}$$

where

$$\Lambda = \frac{1}{2}\left\{\rho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \mathbf{C}[\nabla \mathbf{u}] \cdot \nabla \mathbf{u} + \frac{c}{\theta_0}\theta^2 + \mathbf{p} \cdot \mathbf{N}\mathbf{p}\right\}. \tag{8.3}$$

Let B be a regular region in three-dimensional space, ∂B be the boundary surface enclosing B, and $\mathbf{n}$ be the unit outward normal to ∂B. Integrating both sides of eq. (8.2) over B and applying the divergence theorem to the resulting equation, we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \Lambda \mathrm{d}v = \int_{\partial B} (\mathbf{S}\dot{\mathbf{u}} - \theta \mathbf{p}) \cdot \mathbf{n}\, \mathrm{d}A + \int_B \rho_0(\mathbf{b} \cdot \dot{\mathbf{u}} + s\theta - \theta_0 \xi)\, \mathrm{d}v. \tag{8.4}$$

This is the desired energy equation in the global form. In the next section we employ this equation to establish the uniqueness of solutions.

9. Uniqueness of solution

In the context of the linearized theory formulated in § 5, eqs (5.11) and (5.12) form a complete system of field equations for the pair $(\mathbf{u}, \theta)$, as already noted. To this system, we now adjoin the *initial conditions*

$$\mathbf{u} = \mathbf{u}^0, \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}^0, \quad \theta = T_0, \quad \mathbf{p} = \mathbf{p}^0 \text{ in } B \text{ at } t = 0 \tag{9.1}$$

and the *boundary conditions*

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \partial B_1 \times [0, \infty), \tag{9.2a}$$

$$\mathbf{S}\mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \partial B_1^c \times [0, \infty), \tag{9.2b}$$

$$\theta = \hat{\theta} \quad \text{on } \partial B_2 \times [0, \infty), \tag{9.2c}$$

$$\mathbf{p} \cdot \mathbf{n} = \hat{p} \quad \text{on } \partial B_2^c \times [0, \infty). \tag{9.2d}$$

Here, ∂B_1 and ∂B_2 are arbitrary parts of ∂B, and ∂B_1^c and ∂B_2^c are their respective complements in ∂B. Also, $[0, \infty)$ is the interval over which t varies. Further, $\mathbf{u}^0$, $\dot{\mathbf{u}}^0$, T_0, $\mathbf{p}^0$, $\hat{\mathbf{u}}$, $\hat{\mathbf{t}}$, $\hat{\theta}$ and $\hat{p}$ are prescribed functions in their respective domains. Notice that the initial conditions (9.1) involve a condition on $\mathbf{p}$ as well. Further, in view of the relation $(7.1)_1$, it may be noted that prescribing $\mathbf{p} \cdot \mathbf{n}$ on $\partial B_2^c \times [0, \infty)$ is precisely the same as prescribing $\mathbf{q} \cdot \mathbf{n}$ on $\partial B_2^c \times [0, \infty)$.

With $\mathbf{b}$ and s (or r) as prescribed functions in $B \times [0, \infty)$, eqs (5.11) and (5.12) together with the initial conditions (9.1) and the boundary conditions (9.2a–d) govern an initial mixed boundary value problem in the context of the entropy-flux dependent linear thermoelasticity (EFDTE) formulated here. A pair $(\mathbf{u}, \theta)$ that satisfies eqs (5.11) and (5.12) in $B \times [0, \infty)$ and the conditions (9.1) and (9.2a–d) in the appropriate domains constitute a "solution" of this problem.

We now establish the following

Uniqueness Theorem. *In addition to the postulates already made, if we suppose that the elasticity tensor is positive definite, then there exists at most one solution for the problem described above.*

Proof. It is sufficient to show that for $\mathbf{b} \equiv \mathbf{0}$ and $s \equiv 0$ and homogeneous initial and boundary conditions, the only possible solution is the trivial solution.

For $\mathbf{b} \equiv \mathbf{0}$ and $s \equiv 0$ and homogeneous boundary conditions, the energy eq. (8.4) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \Lambda \, \mathrm{d}v = - \int_B \rho_0 \theta_0 \xi \, \mathrm{d}v. \tag{9.3}$$

By virtue of the entropy production inequality (4.3) and the fact that ρ_0 and θ_0 are positive, it follows from eq. (9.3) that Λ is a nonincreasing function of t. From the homogeneous initial conditions and expression (8.3) we find that $\Lambda = 0$ at $t = 0$. Consequently, we should have $\Lambda \leq 0$ for $t \geq 0$. On the other hand, since $\rho_0 > 0$, $c > 0$ and $\mathbf{N}$ is positive definite, it follows from expression (8.3) that $\Lambda \geq 0$ for $t \geq 0$, provided that the elasticity tensor $\mathbf{C}$ is positive definite. Hence if we take $\mathbf{C}$ to be positive definite, then $\Lambda \equiv 0$ in $B \times [0, \infty)$. This implies

$$\dot{\mathbf{u}} \equiv \mathbf{0}, \quad \nabla \mathbf{u} \equiv \mathbf{0}, \quad \theta \equiv \mathbf{0}, \quad \mathbf{p} \equiv \mathbf{0} \quad \text{in } B \times [0, \infty). \tag{9.4}$$

The homogeneous initial conditions and the first of the relations (9.4) imply that $\mathbf{u}$ is identically zero, in addition to θ, in $B \times [0, \infty)$. This completes the proof of the theorem.

It is interesting to note that the material constant χ does not appear in the energy eq. (8.4) and plays no role in the proof of the uniqueness of solution given above. As such, the energy eq. (8.4) and the uniqueness theorem proved above hold for all admissible values of χ (including $\chi = 0$ which corresponds to TEWOED).

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Statistical stationary states for a two-layer quasi-geostrophic system

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Abstract. Existence of a family of locally invariant probability measures for large scale flows in enclosed temperate sea is proved. This model is extremely important for understanding the meso-scale phenomena in oceans. The techniques used are those developed by Albeverio and his collaborators.

Keywords. Quasi-geostrophic flows; infinite dimensional dynamical system; Luoville's theorem; invariant measures; rigged Hilbert spaces; white noise; Koopmann's formulation; infinite dimensional analysis; Louville operator; self-adjointness.

1. Introduction

Dynamical systems with infinite degrees of freedom arise in all field theories in physics. At a formal level they have many similarities with dynamical systems with finitely many degrees of freedom. Often they admit a Hamiltonian description. It is tempting to attempt an investigation of their ergodic properties. Any such attempt requires the construction of an invariant probability measure on the phase space of the system. As the phase space is in general infinite dimensional a natural approach would be to exploit the specific topological structure which may be present on the phase space. If the phase space is a Hilbertian space (i.e. admits a structure of a Hilbert space) one could exploit Gaussian measures on this space. The only canonical Gaussian probability measures associated with such spaces are of the white noise type. But these measures are supported on generalized vectors, the Hilbert space itself being measurable but with measure zero. As the system is non-linear the existence of dynamics would have to be established, as the vector field concerned would involve products of distributions. These non-linear terms would have to be regularized and energy renormalized. It is this approach that is successfully exploited in [1, 2] to construct locally invariant measures for two dimensional Eulerian flows on a flat torus.

Periodic two dimensional Eulerian flows which have been investigated at great length in [2, 3, 5, 7], although of great interest in understanding two dimensional flows in laboratory experiments, are not of much use in understanding the motion of temperate seas. Flows with horizontal spatial scale of a few hundred kilometers and temporal scales of a few weeks are essentially two dimensional with little or no vertical variability (the depth of the sea is about 4 kms on an average). The motion to a large extent is controlled by the variable rotation of the Earth. This motion is called quasi-geostrophic motion. The variability of Coriolis force with latitude gives rise to a restoring mechanism, which allows for the existence of propagating Rossby waves and leads to the observed westward intensification of oceanic currents. These effects stand in sharp contrast to 2-D Eulerian flows. A clear account of quasi-geostrophy can be found in [11, 12]. Quasi-geostrophic flows are very close to 2-D Eulerian flows. Formal investigations into the construction of

invariant probability measures for such flows have been carried out in [9, 13, 17]. A rigorous investigation on the construction of locally invariant probability measures for such flows using the techniques of Albeverio and his group was carried out for single layer model in [10]. The analysis of the authors is strictly valid for an unstratified ocean, therefore of limited validity. Real oceans have a vertical density stratification. This stratification gives rise to baroclinic motion. The quasi-geostrophic motion in the sea is characterized by the coupling of the barotropic and baroclinic modes. In the sequel a simple two layer quasi-geostrophic model, which has been used with great effect by oceanographers, is studied. The existence of locally invariant probability measures is proved. The techniques used are essentially those of Albeverio and his collaborators.

2. Two layer quasi-geostrophic model

A two layer quasi-geostrophic (see [12]) model consists of a layer of an ideal fluid with density ρ_1 and thickness H_1 superposed on another layer with density ρ_2 and thickness H_2 with $\rho_1 < \rho_2$. The equations governing this model on the β-plane i.e. local tangent plane at a reference latitude λ_0, with the X-axis oriented eastwards and the Y-axis oriented northwards are

$$\left[\frac{\partial}{\partial t}+\frac{\partial\Psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi_1}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\Psi_1 - F_1(\Psi_1-\Psi_2)+\beta y]=0, \tag{2.1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial\Psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi_2}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\Psi_2 - F_2(\Psi_2-\Psi_1)+\beta y]=0. \tag{2.2}$$

Change of Coriolis frequency at latitude λ_0 and $f_0 = 2\alpha\sin\lambda_0$, with $\alpha = (2\pi/24)\,hr^{-1}$. $\Psi_1(x,y)$ and $\Psi_2(x,y)$ are the stream functions for the two dimensional velocity fields in the upper and lower layers respectively.

Introducing new stream functions $\Phi = \Psi_1 - \Psi_2$ the barotropic and $\Psi = (H_1H_2)^{1/2}$ $(H_1\Psi_1 + H_2\Psi_2)/H^2$ the baroclinic stream functions, where $H = H_1 + H_2$. Equations (2.1) and (2.2) take the form

$$\frac{\partial}{\partial t}(\nabla^2\Psi)+J(\Psi,\nabla^2\Psi)+J(\Phi,\nabla^2\Phi)+\beta\frac{\partial\Psi}{\partial x}=0, \tag{2.3}$$

$$\begin{aligned}&\frac{\partial}{\partial t}[(-\nabla^2\Phi+F\Phi)]+\kappa J(\Phi,\nabla^2\Phi)+J(\Psi,\nabla^2\Phi)\\&\quad+J(\Phi,\nabla^2\Psi)-FJ(\Psi,\Phi)+\beta\frac{\partial\Phi}{\partial x}=0,\end{aligned} \tag{2.4}$$

where $\kappa = (H_1-H_2)/(H_1H_2)^{1/2}, F = F_1 + F_2$ and

$$J(\Phi,\Psi)=\frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial y}-\frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial x}$$

is the Jacobian of Ψ and Φ.

This system admits two quadratic constants of motion.

$$H(\Psi,\Phi)=\int\!\!\int\{|\nabla\Psi|^2+|\nabla\Phi|^2+F\Phi^2\}\mathrm{d}x\,\mathrm{d}y, \tag{2.5}$$

$$S(\Psi,\Phi)=\frac{1}{2}\int\!\!\int\{(\nabla^2\Psi)^2+((-\nabla^2+F)\Phi)^2\}\mathrm{d}x\,\mathrm{d}y. \tag{2.6}$$

$S(\Psi,\Phi)$ is called the enstrophy, $H(\Psi,\Phi)$ is the energy of the system.

Let $\omega_1 = -\nabla^2\Psi$ be the vorticity of the baroclinic mode and $\omega_2 = (-\nabla^2 + F)\Phi$ be the modified vorticity of the barotropic mode. In terms of these modes enstrophy and energy take the form

$$S(\omega_1, \omega_2) = \frac{1}{2}\int\int\{\omega_1^2 + \omega_2^2\}\mathrm{d}x\,\mathrm{d}y, \tag{2.7}$$

$$H(\omega_1, \omega_2) = \frac{1}{2}\int\int\{(-\nabla^2)^{-1}(\omega_1)^2 + (-\nabla^2 + F)^{-1}(\omega_2)^2\}\mathrm{d}x\,\mathrm{d}y. \tag{2.8}$$

Energy and enstrophy being positive, attempts to construct Gibbs-like distributions on the space of vortices have been made in [9, 13, 14, 17]. There have been enthusiastic attempts at examining the ergodic properties of these systems in [4, 19]. As the space of vortices is infinite dimensional, such formal distribution functions do not exist. In this note we construct probability measures of the Gibbsian type for the two-layer quasi-geostrophic model following the techniques of Albeverio and his collaborators.

3. Mathematical formulation of the quasi-geostrophic model

In order to facilitate analysis, we consider an ocean basin occupying the region $\Omega = [0, \pi] \times [0, \pi] \subset R^2$. Consider the Hilbert space $L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$. The Friedrichs extension of the Dirichlet Laplacian $-\nabla^2$ on $C_0^\infty(\Omega) \subset L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$ is self-adjoint and has a Hilbert–Schmidt inverse. The eigenvalues and eigenfunctions can be computed explicitly. The eigenvalues of the operator are $K^2 = k_1^2 + k_2^2$, where $K = (k_1, k_2) \in Z^2$ and $k_1 \neq 0$ for $i = 1, 2$. Denote the eigenfunction corresponding to eigenvalue K^2 by ξ_K. It is assumed that $\omega_i, i = 1, 2$ are in $L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$. Expanding $\omega_i, i = 1, 2$ in terms of eigenfunctions ξ_K, we have $\omega_1 = \sum X_K\xi_K$ and $\omega_2 = \sum Y_K\xi_K$. Denote the sequences $\{X_K\}_{K\in Z}$ and $\{Y_K\}_{K\in Z}$ by X and Y. Using these expansions both equations (2.3) and (2.4) are reduced to a countably infinite system of equations. The explicit form of these equations are

$$\begin{aligned}\frac{\partial X_N}{\partial t} = {} & \frac{1}{2\pi}\sum_K K\cdot N\left\{\frac{X_K X_{N-K}}{(N-K)^2} + \frac{Y_K Y_{N-K}}{N-K)^2 + F}\right\} - \frac{1}{2\pi} \\ & \times\sum_K K\cdot N\left\{\frac{X_K X_{J(N-K)}}{(N-K)^2} + \frac{Y_K Y_{J(N-K)}}{N-K)^2 + F}\right\} = A_N(X, Y),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{\partial Y_N}{\partial t} = {} & \frac{\kappa}{2\pi}\sum_K E_K^N\{Y_K Y_{N-K} - Y_K Y_{J(N-K)}\} \\ & + \frac{1}{2\pi}\sum_K F_K^N\{Y_K X_{N-K} - Y_K X_{J(N-K)}\} = B_N(X, Y),\end{aligned} \tag{3.2}$$

where summation is taken over all $K \in Z^2, K \neq 0, J(N) = (n_1, -n_2), K.N = k_1 n_2 - k_2 n_1$,

$$E_K^N = K\cdot N/((N-K)^2 + F)$$

and

$$F_K^N = K\cdot N\{K^2 - (N-K)^2 - F\}/\{K^2((N-K)^2 + F)\}.$$

Energy and enstrophy take the form

$$H(X, Y) = \sum_K\left\{\frac{X_K^2}{K^2} + \frac{Y_K^2}{K^2 + F}\right\}, \tag{3.3}$$

and

$$S(X,Y) = \sum_K \{X_K^2 + Y_K^2\}, \tag{3.4}$$

the summation being taken over $K \in Z^2$ and $K \neq 0$.

For a fixed $K \in Z^2, K \neq 0, A_K(X,Y)$ and $B_K(X,Y)$ are independent of X_K and Y_K. Since enstrophy and energy are conserved it can be easily seen that

$$\sum_K \{X_K A_K(X,Y) + Y_K B_K(X,Y)\} = 0, \tag{3.5}$$

$$\sum_K \left\{ \frac{X_K}{K^2} A_K(X,Y) + \frac{Y_K}{K^2+F} B_K(X,Y) \right\} = 0. \tag{3.6}$$

Following Daletskii [6], consider the space $\{(X,Y)\}$ of real sequences in $\mathbb{R}^2$. For $m \in \mathbb{N}$, the subspace H_m defined by

$$H_m = \left\{ (X,Y) | \sum_K \{K_K^{2m} X_K^2 + (K^2+F)^m Y_K^2\} < \infty \right\}$$

is a Hilbert space. Set $H_\infty = \cap_{m=0}^{\infty} H_m$. Clearly H_∞ is a locally convex space with dual $H_{-\infty} = \cup_{m=0}^{\infty} H_{-m}$. Clearly $H_\infty \subset H_0 \subset H_{-\infty}$ is a Gelfand triplet (rigged Hilbert space).

For every $\gamma > 0$ the function $C : H_\infty \to \mathbb{R}$ defined by

$$C[X,Y] = \mathrm{e}^{(-\gamma/2)\sum_K \{X_K^2 + Y_K^2\}}$$

is a positive definite Frechet continuous functional on H_∞, normalized to 1 at 0. By Bochner–Minlos theorem (see [8]) there exists a Borel probability measure P_γ on $H_{-\infty}$, which has $C[X,Y]$ as its characteristic functional. The properties of P_γ are well known. Its two important properties are

1. H_0 is a measurable subset of $H_{-\infty}$ and has measure 0.
2. If $\gamma_1 \neq \gamma_2, P_\gamma \perp P_\gamma$.

The enstrophy allows us to construct a one parameter family of mutually singular measures, with support on $H_{-\infty}$. Clearly these measures are supported not on regular vortices but on generalized vortices i.e. distributional vortices. In the sequel these probability measures play the role analogous to the Lebsegue measure on $\mathbb{R}^n$. Further these measures are a pure Hilbert space construction.

If one were to regard these measures as the one sought, the baroclinic and modified barotropic vorticity components are statistically independent identically distributed normal random variable with mean zero and variance γ-indicating an equipartition of enstrophy and energy.

4. Renormalization of energy

PROPOSITION 4.1

Energy is P_γ almost surely infinite.

Proof. For each $n \in \mathbb{N}$, consider the cylinder function

$$H_n(X,Y) = \sum_{K^2<n} \left\{ \frac{X_K^2}{K^2} + \frac{Y_K^2}{K^2+F} \right\}. \tag{4.1}$$

H_n is measurable and H_n converges monotonically to H, hence H is measurable. For each $n, H_n > 0$ therefore

$$\lim_{n\to\infty} \int H_n \mathrm{d}P_\gamma = \int H \mathrm{d}P_\gamma.$$

But

$$\int H_n \mathrm{d}P_\gamma = \gamma \sum_{K^2<n} \left\{ \frac{1}{K^2} + \frac{1}{K^2+F} \right\}, \tag{4.2}$$

which diverges to infinity as $n \to \infty$. Hence H is almost surely infinite. [As $\{X_k^2/K^2 + Y_k^2/(K^2+F)\}_k$ are mutually independent random variables by the Kolmogorov zero-one law it follows that $\sum_K \{X_k^2/K^2 + Y_k^2/(K^2+F)\}$ either converges or diverges almost surely.] □

The support of P_γ indicates that the enstrophy is P_γ almost surely infinite. In a system occupying a finite region of space, clearly this is unphysical. For $n \in \mathbb{N}$, define

$$H_n^R(X,Y) = \sum_{K^2<n} \left\{ \frac{1}{K^2}(X_K^2 - \gamma) + \frac{1}{K^2+F}(Y_K^2 - \gamma) \right\}. \tag{4.3}$$

Then for each n

$$\int H_n^R(X,Y)\mathrm{d}P_\gamma = 0.$$

PROPOSITION 4.2

$\{H_n^R(X,Y)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(P_\gamma)$ and converges to a limit $H^R(X,Y)$ in $L^2(P_\gamma)$.

Proof. We prove that $\{H_n^R(X,Y)\}_{n\in\mathbb{N}}$ is Cauchy sequence in $L^2(P_\gamma)$. For $n, m \in \mathbb{N}$ and $n > m$,

$$\int |H_n^R(X,Y) - H_m^R(X,Y)|^2 \mathrm{d}P_\gamma = 2\gamma^2 \sum_{(m<K^2<n)} \left\{ \frac{1}{K^4} + \frac{1}{(K^2+F)^2} \right\}, \tag{4.4}$$

which tends to zero as $n, m \to \infty$. Hence the result. □

We now use the fact that $H^R(X,Y) \in L^2(P_\gamma)$ to construct the measure which corresponds to the formal distribution introduced by Rhines, Salmon and others.

Theorem 4.3. *For $\alpha > 0$, the formal object*

$$\mathrm{d}P_{\alpha,\gamma} = \mathrm{e}^{-\alpha H^R(X,Y)} \mathrm{d}P_\gamma$$

is a well-defined probability measure on $H_{-\infty}$, which is absolutely continuous with respect to P_γ and for $\gamma_1 \neq \gamma_2, P_{\alpha,\gamma}$ and $P_{\alpha,\gamma}$ are mutually singular. □

5. Regularization of $A_N(X,Y)$ and $B_N(X,Y)$

The right hand side of equations (3.1) and (3.2) involve products of distributions i.e. generalized vectors and are as such ill-defined. These can be given acceptable meaning

only by a process of regularization. In order to regularize A_N and B_N we resort to a high wave number cut-off and a subsequent limit process. Note that for $N, M \in Z^2$ and $N \neq M, X_N$ and X_M, Y_N and Y_M are statistically independent, normally distributed random variables. Also note that X_N and Y_M are independent. The random variable X_N is normally distributed with mean 0 and variance $\gamma N^2/(\alpha\gamma + N^2)$. Also Y_N is normally distributed with mean 0 and variance $\gamma(N^2+F)/[\alpha\gamma + (N^2+F)]$. For $m \in \mathbb{N}$ define A_N^m and B_N^m by

$$A_N^m(X,Y) = \frac{1}{2\pi} \sum_{K^2<m} K \cdot N \left\{ \frac{X_K X_{(N-K)}}{(N-K)^2} + \frac{Y_K Y_{(N-K)}}{(N-K)^2+F} - \frac{X_K X_{J(N-K)}}{(N-K)^2} - \frac{Y_K Y_{J(N-K)}}{(N-K)^2+F} \right\} \tag{5.1}$$

and

$$B_N^m(X,Y) = \frac{\kappa}{2\pi} \sum_{K^2<m} E_K^N \{ Y_K Y_{N-K} - Y_K Y_{J(N-K)} \} + \frac{1}{2\pi} \sum_{K^2<m} F_K^N \{ Y_K X_{N-K} - Y_K X_{J(N-K)} \}. \tag{5.2}$$

Theorem 5.1 *For each $m \in \mathbb{N}, A_N^m(X,Y)$ and $B_N^m(X,Y) \in L^2(P_{\alpha,\gamma})$ and converge to limit $A_N^R(X,Y)$ and $B_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$ respectively, as $m \to \infty$.*

Proof. $A_N^m(X,Y)$ and $B_N^m(X,Y)$ are cylinder functions and hence measurable. It is clear that $A_N^m(X,Y)$ and $B_N^m(X,Y) \in L^2(P_{\alpha,\gamma})$. We now prove that the sequence $\{A_N^m(X,Y)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(P_{\alpha,\gamma})$. It is enough to show that the sequence $\{D_N^m(X,Y)\}_{m\in\mathbb{N}}$ defined by

$$D_N^m(X,Y) = \sum_{K^2<m} K \cdot N \frac{X_K X_{N-K}}{(N-K)^2} \tag{5.3}$$

is a Cauchy sequence. Since KN is antisymmetric in K and N, equation (5.3) takes the form

$$D_N^m(X,Y) = \frac{1}{2} \sum_{K^2<m} K \cdot N \left\{ \frac{1}{(N-K)^2} - \frac{1}{K^2} \right\} X_N X_{N-K}. \tag{5.4}$$

Clearly $D_N^m(X,Y)$ is a cylinder function and hence measurable. Moreover

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} = \frac{1}{4} \sum_{K^2,L^2<m} (K\cdot N)(L\cdot N) \left\{ \frac{[K^2-(N-K)^2]}{K^2(N-K)^2} \right\} \times \left\{ \frac{[L^2-(N-L)^2]}{L^2(N-L)^2} \right\} \int X_K X_{N-K} X_L X_{N-L} \mathrm{d}P_{\alpha,\gamma}. \tag{5.5}$$

But

$$\int X_K X_{N-K} X_L X_{N-L} \mathrm{d}P_{\alpha,\gamma} = \left\{ \frac{\gamma K^2}{\alpha\gamma + K^2} \right\} \left\{ \frac{\gamma(N-K)^2}{\alpha\gamma + (N-K)^2} \right\} \{ \delta_{K,L} + \delta_{K,N-L} \}. \tag{5.6}$$

Note that $K^2 - (N-K)^2 = -N^2 + 2(N \cdot K)$. Hence

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq \frac{\gamma^2}{2} \sum_{K^2<m} \frac{[-N^2+2(N\cdot K)]^2 (K\cdot N)^2}{K^4(N-K)^4} \tag{5.7}$$

and

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq 72 N^6 \gamma^2 \sum_{K^2<m} \frac{1}{(N-K)^4}. \tag{5.8}$$

In the same way we get for $p, q \in \mathbb{N}$ and $p < q$

$$\int\int |D_N^q(X,Y) - D_N^p(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq 72 N^6 \gamma^2 \sum_{p<K^2<q} \frac{1}{(N-K)^4}. \tag{5.9}$$

This proves that $D_N^m(X,Y)$ converges in $L^2(P_{\alpha,\gamma})$ as $m \to \infty$. Similarly the other terms can be handled. This implies that $A_N^m(X,Y)$ converges to a limit $A_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$. Similarly it can also be proved that $B_N^m(X,Y)$ converges to a limit $B_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$ as $m \to \infty$. □

With $A_N^R(X,Y)$ and $B_N^R(X,Y)$ as defined above, in place of equations (3.1) and (3.2) we consider their regularized version, namely

$$\frac{\partial X_N}{\partial t} = A_N^R(X,Y), \tag{5.10}$$

$$\frac{\partial Y_N}{\partial t} = B_N^R(X,Y). \tag{5.11}$$

6. Dynamics

The dynamics of the model can be conveniently described in the Liouville–Koopman framework. Consider the Hilbert space $L^2(P_{\alpha,\gamma})$. Define the Liouville operator L as follows:

$$iL(X,Y) = \sum_K \left\{ A_K^R(X,Y) \frac{\partial}{\partial X_K} + B_K^R(X,Y) \frac{\partial}{\partial Y_K} \right\}. \tag{6.1}$$

With domain $\Im = \{F \in L^2(P_{\alpha,\gamma})/F$ depends on finitely many X_K's and Y_K's, is once continuously differentiable and vanishes at infinity$\}$.

Clearly $\Im$ is a dense subspace of $L^2(P_{\alpha,\gamma})$. L is a differential operator in countably infinite number of variables and is well-defined on $\Im$.

PROPOSITION 6.1

The operator L is a symmetric operator in $L^2(P_{\alpha,\gamma})$.

Proof. Consider $F, G \in \Im$ then

$$\int F \frac{\partial G}{\partial X_K} \mathrm{d}P_{\alpha,\gamma} = -\int G \left\{ \frac{\partial}{\partial X_K} - X_K \left(\frac{1}{\gamma} + \frac{\alpha}{K^2} \right) \right\} F \mathrm{d}P_{\alpha,\gamma} \tag{6.2}$$

and

$$\int F\frac{\partial G}{\partial Y_K}\mathrm{d}P_{\alpha,\gamma} = -\int G\left\{\frac{\partial}{\partial Y_K} - Y_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2+F}\right)\right\}F\mathrm{d}P_{\alpha,\gamma}. \tag{6.3}$$

From equations (3.5) and (3.6) it follows that

$$\sum_K\left\{X_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2}\right)A_K(X,Y) + Y_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2+F}\right)B_K(X,Y)\right\} = 0. \tag{6.4}$$

For each $K \in Z^2, A_K(X,Y)$ and $B_K(X,Y)$ are independent of X_K and Y_K. Hence from equations (6.2), (6.3) and (6.4) it follows that L is a symmetric operator. □

L is a well-defined symmetric operator, with $L^*1 = 0$. This implies that $P_{\alpha,\gamma}$ is locally invariant. If L is either self-adjoint or at least essentially self-adjoint, one could conclude that the probability measures $\{P_{\alpha,\gamma}\}$ for $\alpha, \gamma > 0$ are globally invariant. However all that can be proved is that it has self-adjoint extensions.

PROPOSITION 6.2

L is a symmetric operator with equal deficiency indices and therefore has self-adjoint extensions.

Proof. In order to prove that L has equal deficiency indices, it is enough to prove that there exists a conjugation operator J on the Hilbert space commutes with the operator L.

Define the conjugation $J : L^2(P_{\alpha,\gamma}) \to L^2(P_{\alpha,\gamma})$ by $J(f) = f$ – the complex conjugate of the function. Then clearly $J(\Im) = \Im$ and $AJ = JA$. This implies that A has equal deficiency indices and therefore self adjoint extensions. □

The main result of this investigation can be stated as

Theorem 6.1 *There exists a two-parameter family of probability measures $\{P_{\alpha,\gamma}\}\alpha,\gamma > 0$ on $H_{-\infty}$, the extended phase space of the quasi-geostrophic two-layer model, which are locally invariant under evolution and if $\gamma_1 \neq \gamma_2$ then $P_{\alpha,\gamma} \perp P_{\alpha,\gamma}$.* □

7. Discussion

While we have presented a construction of a class of locally invariant probability measures for a two-layer quasi-geostrophic fluid, it should be noted that in a sense these measures are trivial and cannot be expected to reveal any interesting features that may be observed in real flows. Clearly the barotropic and baroclinic modes are statistically independent. Further the unrenormalized energy is almost surely infinite. This is to some extent quenched by renormalizing it. The renormalization of energy depends on the pseudo-temperature γ. What has not been noticed is that the total enstrophy is also almost surely infinite. Again note the fact that the enstrophy is equipartitioned among the modes. Except for mathematical difficulties involved, there is really no good ground for not considering other functions of vortices. All quantities of the form $\int \omega^k \mathrm{d}x, k \in \mathbb{N}$ are also conserved. It is conceivable that non-quadratic conserved quantities may reveal more interesting phenomena, as in the case of quantum fields.

There have been various attempts to link these states with geostrophic turbulence (see [16]). This link is extremely tenuous as the system considered has no dissipation at all.

These locally invariant statistical states are distinct from the statistical stationary solutions considered in [20].

After completing this work we became aware of the work carried out in [15] on 2-D Euler flows, where the entire emphasis is based on obtaining a descritization.

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Jacobian of meromorphic curves

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Abstract. The contact structure of two meromorphic curves gives a factorization of their jacobian.

Keywords. jacobian; factorization; deformation; contact.

1. Introduction

Let $J(F,G) = J_{(X,Y)}(F,G)$ be the jacobian of $F = F(X,Y)$ and $G = G(X,Y)$ with respect to X and Y, i.e., let $J(F,G) = F_X G_Y - F_Y G_X$ where subscripts denote partial derivatives. Here, to begin with, F and G are plane curves, i.e., polynomials in X and Y over an algebraically closed ground field k of characteristic zero. More generally, we let F and G be meromorphic curves, i.e., polynomials in Y over the (formal) meromorphic series field $k((X))$.

In terms of the the contact structure of F and G, we shall produce a factorization of $J(F,G)$. Note that if $G = -X$ then $J(F,G) = F_Y$; in this special case, our results generalize some results of Merle [Me], Delgado [De], and Kuo–Lu [KL] who studied the situation when F has one (Merle) or two (Delgado) or more (Kuo–Lu) branches. These authors restricted their attention to the analytic case, i.e., when F is a polynomial in Y over the (formal) power series ring $k[[X]]$. With an eye on the Jacobian conjecture, we are particularly interested in the meromorphic case.

The main technique we use is the method of Newton polygon, i.e., the method of deformations, characteristic sequences, truncations, and contact sets given in Abhyankar's 1977 Kyoto paper [Ab]. In § 2–5 we shall review the relevant material from [Ab]. In § 6 we shall introduce the tree of contacts and in § 7–9 we shall show how this gives rise to the factorizations.

The said Jacobian conjecture predicts that if the jacobian of two bivariate polynomials $F(X,Y)$ and $G(X,Y)$ is a nonzero constant then the variables X and Y can be expressed as polynomials in F and G, i.e., if $0 \neq J(F,G) \in k$ for F and G in $k[X,Y]$ then $k[F,G] = k[X,Y]$. We hope that the results of this paper may contribute towards a better understanding of this bivariate conjecture, and hence also of its obvious multivariate incarnation.

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2. Deformations

We are interested in studying polynomials in indeterminates X and Y over an algebraically closed ground field k of characteristic zero. To have more elbow room to manoeuver, we consider the larger ring $R=k((X))[Y]$ of polynomials in Y over $k((X))$, i.e., with coefficients in $k((X))$, where $k((X))$ is the meromorphic series field in X over k.

Given any

$$g=g(X,Y)=\sum_{i\in\mathbb{Z}} g^{[i]}X^i=\sum_{j\in\mathbb{Z}} g^{((j))}Y^j\in R,$$

with

$$g^{[i]}=g^{[i]}(Y)\in k[Y] \quad\text{and}\quad g^{((j))}=g^{((j))}(X)\in k((X)),$$

we put

$$\mathrm{Supp}_X g=\{i\in\mathbb{Z}: g^{[i]}\neq 0\} \quad\text{and}\quad \mathrm{Supp}_Y g=\{j\in\mathbb{Z}: g^{((j))}\neq 0\},$$

and we call this the X-support and the Y-support of g respectively. Note that these supports are bounded from below and above respectively, and upon letting

$$\gamma^{\sharp}=\mathrm{ord}_X g=\text{the } X\text{-order of } g \quad\text{and}\quad \gamma=\deg_Y g=\text{the } Y\text{-degree of } g$$

we have

$$\gamma^{\sharp}=\begin{cases}\min(\mathrm{Supp}_X g) & \text{if } g\neq 0\\ \infty & \text{if } g=0\end{cases} \quad\text{and}\quad \gamma=\begin{cases}\max(\mathrm{Supp}_Y g) & \text{if } g\neq 0\\ -\infty & \text{if } g=0.\end{cases}$$

Now

$$g^{[i]}(Y)=\sum_{j\in\mathbb{Z}} g^{(i,j)}Y^j \quad\text{and}\quad g^{((j))}(X)=\sum_{i\in\mathbb{Z}} g^{(i,j)}X^i \quad\text{with}\quad g^{(i,j)}\in k$$

and we put

$$\mathrm{Supp}(g)=\mathrm{Supp}_{(X,Y)}g=\{(i,j)\in\mathbb{Z}\times\mathbb{Z}: g^{(i,j)}\neq 0\}$$

and we call this the support, or the (X,Y)-support, of g. We put

$$\mathrm{inco}_X g=\begin{cases}g^{[\gamma^{\sharp}]} & \text{if } g\neq 0\\ 0 & \text{if } g=0\end{cases} \quad\text{and}\quad \mathrm{deco}_Y g=\begin{cases}g^{((\gamma))} & \text{if } g\neq 0\\ 0 & \text{if } g=0\end{cases}$$

and we call this the X-initial-coefficient and the Y-degree-coefficient of g respectively. Upon letting

$$\begin{aligned}\hat{\gamma}^{\sharp}&=\mathrm{ord}(g)=\text{the (total) order of } g\\ &=\mathrm{ord}_{(X,Y)}g=\text{the } (X,Y)\text{-order of } g\end{aligned}$$

we have

$$\hat{\gamma}^{\sharp}=\begin{cases}\min\{i+j:(i,j)\in\mathrm{Supp}(g)\} & \text{if } g\neq 0\\ \infty & \text{if } g=0\end{cases}$$

and we put

$$\text{info}(g) = \text{info}_{(X,Y)}g = \begin{cases} \sum\limits_{i+j=\hat{\gamma}^{\natural}} g^{(i,j)}X^iY^j & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

and we call this the initial-form, or the (X, Y)-initial-form, of g.[1] If $g \in k[X, Y]$ then upon letting

$$\begin{aligned} \hat{\gamma} &= \deg(g) = \text{the (total) degree of } g \\ &= \deg_{(X,Y)}g = \text{the } (X, Y)\text{-degree of } g \end{aligned}$$

we have

$$\hat{\gamma} = \begin{cases} \max\{i + j : (i, j) \in \text{Supp}(g)\} & \text{if } g \neq 0 \\ -\infty & \text{if } g = 0 \end{cases}$$

and we put

$$\text{defo}(g) = \text{defo}_{(X,Y)}g = \begin{cases} \sum\limits_{i+j=\hat{\gamma}} g^{(i,j)}X^iY^j & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

and we call this the degree-form, or the (X, Y)-degree-form, of g.[2]

Given any $z = z(X) \in k((X))$, we write

$$z = z(X) = \sum_{i \in \mathbb{Z}} z[i]X^i \quad \text{with} \quad z[i] \in k,$$

and we put

$$z[i] = 0 \text{ for all } i \in \mathbb{Q} \setminus \mathbb{Z},$$

and we let

$$\epsilon(z) = \begin{cases} \text{the set of all } (U, V, W) \in \mathbb{Z}^3 \text{ such that } U > 0 < V \\ \text{and } iV/U \in \mathbb{Z} \text{ for all } i \in \text{Supp}_X z \text{ with } i < WU/V \end{cases}$$

and we call this the edge of z, and for any $(U, V, W) \in \epsilon(z)$ we let

$$z^{\dagger}(X, U, V, W) = \sum_{i<WU/V} z[i]X^{iV/U} \in k((X))$$

and

$$z^{\ddagger}(X, U, V, W, Y) = z^{\dagger}(X, U, V, W) + X^W Y \in R$$

[1] In an obvious manner, the definitions of $\text{Supp}_X g$, $\text{ord}_X g$, $\text{inco}_X g$, $\text{Supp}_Y g$, $\deg_Y g$, $\text{deco}_Y g$, $\text{Supp}_{(X,Y)} g$, $\text{ord}_{(X,Y)} g$, and $\text{info}_{(X,Y)} g$, can be extended to any g in $k((X))[Y, Y^{-1}]$, and for any such g we can also define $\text{ord}_Y g$ and $\text{inco}_Y g$, and then we have: $g = 0 \Leftrightarrow \text{Supp}_X g = \emptyset \Leftrightarrow \text{ord}_X g = \infty \Leftrightarrow \text{inco}_X g = 0 \Leftrightarrow \text{Supp}_Y g = \emptyset \Leftrightarrow \text{ord}_Y g = \infty \Leftrightarrow \deg_Y g = -\infty \Leftrightarrow \text{inco}_Y g = 0 \Leftrightarrow \text{deco}_Y g = 0 \Leftrightarrow \text{Supp}_{(X,Y)} g = \emptyset \Leftrightarrow \text{ord}_{(X,Y)} g = \infty \Leftrightarrow \text{info}_{(X,Y)} g = 0$.

[2] Again, in an obvious manner, the definitions of $\deg_{(X,Y)} g$ and $\text{defo}_{(X,Y)} g$ can be extended to any g in $k[X, X^{-1}, Y, Y^{-1}]$, and for any such g we can also define $\deg_X g$ and $\text{deco}_X g$, and then we have: $g = 0 \Leftrightarrow \deg_X g = -\infty \Leftrightarrow \text{deco}_X g = 0 \Leftrightarrow \deg_{(X,Y)} g = -\infty \Leftrightarrow \text{defo}_{(X,Y)} g = 0$.

and

$$z^{\dagger *}(X,U,V,W)=\sum_{i\le WU/V} z[i]X^{iV/U}\in k((X))$$

and we call these the (U,V,W)-truncation, the (U,V,W)-deformation, and the strict (U,V,W)-truncation of $z(X)$ respectively. Given any $H=H(X,Y)\in R$, we are interested in calculating $\mathrm{ord}_X H(X^V,z^\ddagger(X,U,V,W,Y))$ and $\mathrm{inco}_X H(X^V,z^\ddagger(X,U,V,W,Y))$.[3] For this purpose we proceed to give a review on characteristic sequences.

3. Characteristic sequences

Let $\hat{R}^\natural$ be the set of all monic polynomials in Y over $k((X))$, i.e., those nonzero members of R in whom the coefficient of the highest Y-degree term is 1. Let $R^\natural$ be the set of all irreducible monic polynomials in Y over $k((X))$, i.e., those members of $\hat{R}^\natural$ which generate prime ideals in R; note that their Y-degrees are positive integers.

Given any $f=f(X,Y)\in R^\natural$ of Y-degree n, by Newton's theorem

$$f(X^n,Y)=\prod_{1\le j\le n}[Y-z_j(X)] \quad \text{with} \quad z_j(X)\in k((X)),$$

where we note that $\mathrm{Supp}_X z_j$ is independent of j. Let $m(f)=m_i(f)_{0\le i\le h(m(f))+1}$ be the newtonian sequence of characteristic exponents of f relative to n as defined on page 300 of [Ab], let $d(m(f))=d_i(m(f))_{0\le i\le h(d(m(f)))+2}$ be the GCD-sequence of $m(f)$ as defined on page 297 of [Ab], let $q(m(f))=q_i(m(f))_{0\le i\le h(q(m(f)))+1}$ be the difference sequence of $m(f)$ as defined on page 301 of [Ab], let $s(q(m(f)))=s_i(q(m(f)))_{0\le i\le h(s(q(m(f))))+1}$ be the inner product sequence of $q(m(f))$ as defined on page 302 of [Ab], and let $r(q(m(f)))=r_i(q(m(f)))_{0\le i\le h(r(q(m(f))))+1}$ be the normalized inner product sequence of $q(m(f))$ as defined on p. 302 of [Ab].[4] Note that then

$$\begin{aligned} h(d(m(f))) &= h(m(f)) = h(q(m(f))) \\ &= h(s(q(m(f)))) = h(r(q(m(f)))) = \text{a nonnegative integer} \end{aligned}$$

and

$$d_0(m(f))=0 \quad \text{and} \quad d_{h(d(m(f)))+1}(m(f))=1$$

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[3] To motivate the definitions of $\epsilon(z)$ and $z^\ddagger$, given any $H=H(X,Y)=\sum_{i,j}H^{(i,j)}X^iY^j\in R$ with $H^{(i,j)}\in k$, let $\Gamma^\sharp$ and $\Theta^\sharp(Y)$ be the values of $\mathrm{ord}_X H(X^V,z^\ddagger(X,U,V,W,Y))$ and $\mathrm{inco}_X H(X^V,z^\ddagger(X,U,V,W,Y))$ when $z=0$, i.e., let $\Gamma^\sharp=\mathrm{ord}_X H(X^V,X^WY)$ and $\Theta^\sharp(Y)=\mathrm{inco}_X H(X^V,X^WY)$. Also let Γ and $\Theta(X,Y)$ be the weighted order and the weighted initial form of $H(X,Y)$, when we give weights (V,W) to (X,Y), i.e., let $\Gamma=\min\{iV+jW:(i,j)\in\mathrm{Supp}_{(X,Y)}H(X,Y)\}$ and $\Theta(X,Y)=\sum_{iV+jW=\Gamma}H^{(i,j)}X^iY^j$. Then $\Gamma^\sharp=\Gamma=\mathrm{ord}_{(X,Y)}H(X^V,Y^W)$ and $\Theta(X,Y)=\mathrm{info}_{(X,Y)}H(X^V,Y^W)$. Moreover, $\Theta^\sharp(Y)$ and $\Theta(X,Y)$ determine each other by the formulas $\Theta^\sharp(Y)=\Theta(1,Y)$ and $\Theta(X,Y)=X^{\Gamma/V}\Theta^\sharp(X^{-W/V}Y)$. The parameter U is a normalizing parameter which essentially says that we want to intersect the "meromorphic curve" $H(X,Y)=0$ with a deformation of the "irreducible meromorphic curve" $f(X,Y)=0$ where $f(X,Y)$ is a monic irreducible polynomial of degree $U=n$ in Y over $k((X))$; to do this we take a "fractional meromorphic" root $y(X)$ of $f(X,Y)=0$ with $y(X^n)=z(X)\in k((X))$, and then after "deforming" $y(X)$ at $X^{W/U}$ we substitute the deformation in $f(X,Y)$ for Y. For further motivation see the definitions of $\epsilon(f,\lambda)$ and $t(f,\lambda)$ displayed in the middle of the next section.

[4] It is really not necessary to look up [Ab] for the definitions of the sequences m,d,q,s,r, since they are completely redefined in the next three sentences ending with the displayed item $(\bullet)$.

and

$$d_1(m(f)) = m_0(f) = q_0(m(f)) = s_0(q(m(f))) = r_0(q(m(f))) = n,$$

and

$$\begin{aligned} d_{h(m(f))+2}(m(f)) &= m_{h(m(f))+1}(f) = q_{h(m(f))+1}(m(f)) \\ &= s_{h(m(f))+1}(q(m(f))) = r_{h(m(f))+1}(q(m(f))) = \infty, \end{aligned}$$

and

$$m_1(f) = q_1(m(f)) = s_1(q(m(f)))/n = r_1(q(m(f))) = \min(\mathrm{Supp}_X z_1),$$

with the understanding that the min of the empty set is ∞. Also note that: $h(m(f)) = 0 \Leftrightarrow f(X,Y) = Y$. Finally note that if $f(X,Y) \neq Y$ then for $2 \leq i \leq h(m(f))$ we have that $d_i(m(f))$, $m_i(f)$, $q_i(m(f))$, $s_i(q(m(f)))$, $r_i(q(m(f)))$ are integers with $d_i(m(f)) > 0$ such that:

$$(\bullet) \qquad \begin{cases} d_i(m(f)) = \mathrm{GCD}(m_0(f), m_1(f), \ldots, m_{i-1}(f)), \\ m_i(f) = \min(\mathrm{Supp}_X z_1 \setminus d_i(m(f))\mathbb{Z}), \\ q_i(m(f)) = m_i(f) - m_{i-1}(f), \\ s_i(q(m(f))) = q_1(m(f))d_1(m(f)) + \cdots + q_i(m(f))d_i(m(f)), \\ \text{and } r_i(q(m(f))) = s_i(q(m(f)))/d_i(m(f)). \end{cases}$$

In the rest of this section we shall use the abbreviations

$$d_i = d_i(m(f)) \quad \text{and} \quad s_i = s_i(q(m(f)))$$

for all relevant values of i. Let the sequence $c(f) = c_i(f)_{1 \leq i \leq h(c(f))}$ be defined by putting

$$h(c(f)) = h(m(f)) \quad \text{and} \quad c_i(f) = m_i(f)/n \text{ for } 1 \leq i \leq h(c(f))$$

and let us call this the normalized characteristic sequence of f. Note that then $c_1(f) < c_2(f) < \cdots < c_{h(c(f))}(f)$ are rational numbers, out of which only $c_1(f)$ could be an integer. To obtain an alternative characterization of the noninteger members of this sequence, for any rational number λ, we let

$$p(f,\lambda) = \begin{cases} \text{the unique nonnegative integer } \leq h(c(f)) \text{ such that} \\ c_i(f) < \lambda \leq c_j(f) \text{ for } 1 \leq i \leq p(f,\lambda) < j \leq h(c(f)) \end{cases}$$

and

$$p^*(f,\lambda) = \begin{cases} \text{the unique nonnegative integer } \leq h(c(f)) \text{ such that} \\ c_i(f) \leq \lambda < c_j(f) \text{ for } 1 \leq i \leq p^*(f,\lambda) < j \leq h(c(f)) \end{cases}$$

and

$$D(f,\lambda) = n/d_{p+1} \quad \text{with} \quad p = p(f,\lambda)$$

and

$$D^*(f,\lambda) = n/d_{p^*+1} \quad \text{with} \quad p^* = p^*(f,\lambda)$$

and

$$S(f,\lambda) = \begin{cases} (s_p + (n\lambda - m_p(f))d_{p+1})/n^2 & \text{if } p = p(f,\lambda) \neq 0 \\ \lambda & \text{if } p = p(f,\lambda) = 0 \end{cases}$$

and, for any $z \in k((X))$, we let

$$A(f,\lambda,z) = \prod_{i=1}^{p(f,\lambda)} ((d_i/d_{i+1})z[m_i(f)]^{(d_i/d_{i+1})-1})^{d_{i+1}D(f,\lambda)/n}$$

and

$$\hat{A}(f,\lambda,z) = A(f,\lambda,z)^{n/D(f,\lambda)}$$

and

$$E(f,\lambda,z,Y) = Y^{D^*(f,\lambda)/D(f,\lambda)} - z[n\lambda]^{D^*(f,\lambda)/D(f,\lambda)}$$

and

$$\hat{E}(f,\lambda,z,Y) = E(f,\lambda,z,Y)^{n/D^*(f,\lambda)}$$

and we call these the λ-position, the strict λ-position, the λ-degree, the strict λ-degree, the λ-strength, the (λ,z)-reduced-constant, the (λ,z)-constant, the (λ,z)-reduced-polynomial, and the(λ,z)-polynomial of f respectively; note that the above objects p, p^*, D, D^*, S, A, $\hat{A}$, E, and $\hat{E}$ respectively correspond to the objects $p(<)$, $p(\leq)$, D, E, s, B, $\hat{B}$, P, and $\hat{P}$ introduced on pp. 326–328 of [Ab]. We also define the sequence $m(f,\lambda) = m_i(f,\lambda)_{0\leq i\leq h(m(f,\lambda))+1}$ by putting

$$h(m(f,\lambda)) = p(f,\lambda) \text{ and } m_i(f,\lambda) = m_i(f)D(f,\lambda)/n \text{ for } 0\leq i\leq p(f,\lambda)+1$$

with the understanding that $m_i(f,\lambda) = \infty$ for $i = p(f,\lambda)+1$, and we define the sequence $m^*(f,\lambda) = m_i^*(f,\lambda)_{0\leq i\leq h(m^*(f,\lambda))+1}$ by putting

$$h(m^*(f,\lambda)) = p^*(f,\lambda) \quad \text{and}$$
$$m_i^*(f,\lambda) = m_i(f)D^*(f,\lambda)/n \text{ for } 0\leq i\leq p^*(f,\lambda)+1$$

with the understanding that $m_i^*(f,\lambda) = \infty$ for $i = p^*(f,\lambda)+1$, and we define the sequence $c(f,\lambda) = c_i(f,\lambda)_{1\leq i\leq h(c(f,\lambda))}$ by putting

$$h(c(f,\lambda)) = p(f,\lambda) \quad \text{and} \quad c_i(f,\lambda) = c_i(f) \text{ for } 1\leq i\leq p(f,\lambda)$$

and we define the sequence $c^*(f,\lambda) = c_i^*(f,\lambda)_{1\leq i\leq h(c^*(f,\lambda))}$ by putting

$$h(c^*(f,\lambda)) = p^*(f,\lambda) \quad \text{and} \quad c_i^*(f,\lambda) = c_i(f) \text{ for } 1\leq i\leq p^*(f,\lambda),$$

and we call these sequences the λ-characteristic-sequence, the strict λ-characteristic-sequence, the λ-normalized-characteristic-sequence, and the strict λ-normalized-characteristic-sequence of f respectively. We also let

$$\epsilon(f,\lambda) = \begin{cases} \text{the set of all } (z,U,V,W)\in k((X))\times\mathbb{Z}^3 \text{ such that } U=n, W/V=\lambda, \\ \text{and } (U,V,W)\in\epsilon(z) \text{ where } z=z(X)\in k((X)) \text{ with } f(X^n,z(X))=0 \end{cases}$$

and we call this the λ-edge of f. Finally we define $t(f,\lambda) = t(f,\lambda)(X,Y)$ to be the unique member of $R^\natural$ such that

$$t(f,\lambda)(X^V, z^\dagger(X,U,V,W)) = 0 \text{ for some}$$
$$\text{(and hence for all)}(z,U,V,W)\in\epsilon(f,\lambda)$$

and we call this the λ-normalized-truncation of f, and we define $t^*(f,\lambda) = t^*(f,\lambda)(X,Y)$ to be the unique member of $R^\natural$ such that

$$t^*(f,\lambda)(X^V, z^{\dagger *}(X,U,V,W)) = 0 \text{ for some (and hence for all) } (z,U,V,W) \in \epsilon(f,\lambda)$$

and we call this the strict λ-normalized-truncation of f; note that on p. 294 of [Ab] we have called these the open and closed $(n\lambda)$-truncations of f respectively.

From the above definitions of the various objects, we see that

$$\begin{cases} p(f,\lambda) \text{ and } p^*(f,\lambda) \text{ are integers with} \\ 0 \le p(f,\lambda) \le p^*(f,\lambda) \le h(m(f)), \text{ and} \\ D(f,\lambda) \text{ and } D^*(f,\lambda) \text{ are positive integers with} \\ n/D^*(f,\lambda) \in \mathbb{Z} \text{ and } D^*(f,\lambda)/D(f,\lambda) \in \mathbb{Z} \end{cases} \tag{NP1}$$

and

$$\begin{cases} t(f,\lambda) \text{ and } t^*(f,\lambda) \text{ are elements of } R^\natural \text{ such that:} \\ m(t(f,\lambda)) = m(f,\lambda) \text{ and } m(t^*(f,\lambda)) = m^*(f,\lambda), \\ c(t(f,\lambda)) = c(f,\lambda) \text{ and } c(t^*(f,\lambda)) = c^*(f,\lambda), \\ \deg_Y t(f,\lambda) = D(f,\lambda) \text{ and } \deg_Y t^*(f,\lambda) = D^*(f,\lambda), \text{ and} \\ h(m(t(f,\lambda))) = h(c(t(f,\lambda))) = p(f,\lambda) \text{ and } h(m(t^*(f,\lambda))) \\ \qquad = h(c(t^*(f,\lambda))) = p^*(f,\lambda) \end{cases} \tag{NP2}$$

and

$$\begin{cases} A(f,\lambda,z) \in k \text{ and } \hat{A}(f,\lambda,z) = A(f,\lambda,z)^{n/D(f,\lambda)} \in k \text{ are such that:} \\ \text{if } f(X^n, z(X)) = 0 \text{ then } A(f,\lambda,z) \neq 0 \neq \hat{A}(f,\lambda,z) \end{cases} \tag{NP3}$$

and

$$\begin{cases} E(f,\lambda,z,Y) = Y^{D^*(f,\lambda)/D(f,\lambda)} - z[n\lambda]^{D^*(f,\lambda)/D(f,\lambda)} \in k[Y] \\ \text{and } \hat{E}(f,\lambda,z,Y) = E(f,\lambda,z,Y)^{n/D^*(f,\lambda)} \in k[Y] \\ \text{are monic polynomials of degrees } D^*(f,\lambda)/D(f,\lambda) \text{ and } n/D(f,\lambda) \\ \text{respectively, where } z[n\lambda] \in k \text{ is such that: } z[n\lambda] \neq 0 \Leftrightarrow n\lambda \in \mathrm{Supp}_X z \end{cases} \tag{NP4}$$

and

$$\begin{cases} D^*(f,\lambda)/D(f,\lambda) > 1 \\ \Leftrightarrow \lambda = c_i(f) \notin \mathbb{Z} \text{ for some } i \in \{1,\dots,h(c(f))\} \\ \Leftrightarrow \hat{E}(f,\lambda,z,Y) \text{ has more than one root in } k \\ \qquad \text{for any } z = z(X) \in k((X)) \text{ with } f(X^n, z(X)) = 0 \end{cases} \tag{NP5}$$

and

$$\begin{cases} S(f,\lambda) \in \mathbb{Q} \text{ is such that:} \\ \text{if } (z,U,V,W) \in \epsilon(f,\lambda) \text{ then } S(f,\lambda)nV \in \mathbb{Z}. \end{cases} \tag{NP6}$$

With this preparation, what we have called Newton polygon (3) on p. 334 of [Ab] can be restated by saying that:

$$\begin{cases}\text{if } (z,U,V,W)\in\epsilon(f,\lambda)\\ \text{then } \mathrm{ord}_X f(X^V, z^{\ddagger}(X,U,V,W,Y)) = S(f,\lambda)nV\\ \text{and } \mathrm{inco}_X f(X^V, z^{\ddagger}(X,U,V,W,Y)) = \hat{A}(f,\lambda,z)\hat{E}(f,\lambda,z,Y).\end{cases} \tag{NP7}$$

In view of (NP3) and (NP5), the last line of (NP7) tells us that the noninteger members of the sequence $c_i(f)_{1\le i\le h(c(f))}$ are exactly those values of λ for which $\mathrm{inco}_X(f(X^V, z(X, U, V, W, Y)))$ has more than one root in k; this then is the alternative characterization we spoke of.

Given any other $f' = f'(X,Y)\in R^{\natural}$ of Y-degree n', by Newton's theorem

$$f'(X^{n'},Y) = \prod_{1\le j\le n'} [Y - z'_j(X)] \quad \text{with} \quad z'_j(X)\in k((X)).$$

Recall that on p. 287 of [Ab] the contact $\mathrm{cont}(f,f')$ of f with f' is defined by putting

$$\mathrm{cont}(f,f') = \max\{(1/n')\mathrm{ord}_X[z_j(X^{n'}) - z'_{j'}(X^n)] : 1\le j\le n \text{ and } 1\le j'\le n'\}.$$

We define the normalized contact $\mathrm{noc}(f,f')$ of f with f' by putting $\mathrm{noc}(f,f') = (1/n)\,\mathrm{cont}(f,f')$, i.e., equivalently, by putting

$$\mathrm{noc}(f,f') = \max\{(1/(nn'))\mathrm{ord}_X[z_j(X^{n'}) - z'_{j'}(X^n)] : 1\le j\le n \text{ and } 1\le j'\le n'\}.$$

We note that if $f\neq f'$ then $\mathrm{noc}(f,f')$ is a rational number, and if $f=f'$ then $\mathrm{noc}(f,f')=\infty$. We also note the isosceles triangle property which we shall tacitly use and which says that

$$\begin{cases} f''\in R^{\natural} \Rightarrow \mathrm{noc}(f,f'')\ge \min(\mathrm{noc}(f,f'),\mathrm{noc}(f',f''))\\ \text{and}\\ f''\in R^{\natural} \text{ with } \mathrm{noc}(f,f')\neq\mathrm{noc}(f',f'')\Rightarrow \mathrm{noc}(f,f'')\\ \quad = \min(\mathrm{noc}(f,f'),\mathrm{noc}(f',f'')).\end{cases} \tag{ITP}$$

In view of the confluence lemmas given on pp. 338–344 of [Ab] we see that

$$\begin{cases}\text{if } \lambda\le\lambda' = \mathrm{noc}(f,f')\\ \text{then } p(f',\lambda) = p(f,\lambda), D(f',\lambda) = D(f,\lambda), S(f',\lambda) = S(f,\lambda),\\ m(f',\lambda) = m(f,\lambda), c(f',\lambda) = c(f,\lambda), t(f',\lambda) = t(f,\lambda),\\ \text{and } A(f',\lambda,z') = A(f,\lambda,z)\\ \text{where we have chosen } z = z(X) \text{ and } z' = z'(X) \text{ in } k((X)) \text{ such that}\\ f(X^n,z(X)) = 0 = f'(X^{n'},z'(X)) \text{ and } (1/(nn'))\mathrm{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda'\end{cases} \tag{GNP1}$$

and

$$\begin{cases}\text{if } \lambda<\lambda' = \mathrm{noc}(f,f')\\ \text{then } p^*(f',\lambda) = p^*(f,\lambda), D^*(f',\lambda) = D^*(f,\lambda),\\ m^*(f',\lambda) = m^*(f,\lambda), c^*(f',\lambda) = c^*(f,\lambda), t^*(f',\lambda) = t^*(f,\lambda),\\ \text{and } E(f',\lambda,z',Y) = E(f,\lambda,z,Y)\\ \text{where we have chosen } z = z(X) \text{ and } z' = z'(X) \text{ in } k((X)) \text{ such that}\\ f(X^n,z(X)) = 0 = f'(X^{n'},z'(X)) \text{ and } (1/(nn'))\,\mathrm{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda'\end{cases} \tag{GNP2}$$

and

$$\begin{cases} \text{if } \lambda = \lambda' = \mathrm{noc}(f, f') \\ \text{then } E(f', \lambda, z', Y) \text{ and } E(f, \lambda, z, Y) \text{ do not have a common root} \\ \text{where we have chosen } z = z(X) \text{ and } z' = z'(X) \text{ in } k((X)) \text{ such that} \\ f(X^n, z(X)) = 0 = f'(X^{n'}, z'(X)) \text{ and } (1/(nn'))\mathrm{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \tag{GNP3}$$

and

$$\begin{cases} \text{if } \lambda \leq \lambda' = \mathrm{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } S(f, \lambda)n'V \in \mathbb{Z}. \end{cases} \tag{GNP4}$$

What we have called generalized Newton polygon (6) on pp. 346–347 of [Ab] can now be restated by saying that:

$$\begin{cases} \text{if } \lambda \leq \lambda' = \mathrm{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } \mathrm{ord}_X f'(X^V, z^\ddagger(X, U, V, W)) = S(f, \lambda)n'V \\ \text{and } \mathrm{inco}_X f'(X^V, z^\ddagger(U, V, W, Y)) = \hat{A}(f', \lambda, z')\hat{E}(f', \lambda, z', Y) \\ \text{where we have chosen } z' = z'(X) \in k((X)) \text{ such that} \\ f'(X^{n'}, z'(X)) = 0 \text{ and } (1/(nn'))\mathrm{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \tag{GNP5}$$

and

$$\begin{cases} \text{if } \lambda > \lambda' = \mathrm{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } \mathrm{ord}_X f'(X^V, z^\ddagger(X, U, V, W)) = S(f, \lambda')n'V \\ \text{and } 0 \neq \mathrm{inco}_X f'(X^V, z^\ddagger(U, V, W, Y)) \\ \quad = \hat{A}(f', \lambda', z')\hat{E}(f', \lambda', z', z[n\lambda]) \in k \\ \text{where we have chosen } z' = z'(X) \in k((X)) \text{ such that} \\ f'(X^{n'}, z'(X)) = 0 \text{ and } (1/(nn'))\mathrm{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \tag{GNP6}$$

and

$$\begin{cases} \text{if } \lambda = \lambda' = \mathrm{noc}(f, f') \text{ and } f(X^n, z(X)) = 0 \text{ with } z(X) \in k((X)) \\ \text{then } \mathrm{ord}_X f'(X^n, z(X)) = S(f, \lambda)n'n. \end{cases} \tag{GNP7}$$

Finally we note that, for the truncations $t(f, \lambda)$ and $t^*(f, \lambda)$, we obviously have

$$\begin{cases} \mathrm{noc}(f, t(f, \lambda)) \geq \lambda \text{ and} \\ \mathrm{noc}(f, t^*(f, \lambda)) > \lambda. \end{cases} \tag{GNP8}$$

4. Truncations and buds

To continue discussing truncations, we let $R^\flat$ be the set of all buds in R, where by a bud we mean a pair $B = (\sigma(B), \lambda(B))$ with $\emptyset \neq \sigma(B) \subset R^\natural$ and $\lambda(B) \in \mathbb{Q}$ such that $\mathrm{noc}(f, f') \geq \lambda(B)$ for all f and f' in $\sigma(B)$; we call $\sigma(B)$ the stem of B, and $\lambda(B)$ the level of B; we also let $\tau(B) = \{f \in R^\natural : \mathrm{noc}(f, f') \geq \lambda(B) \text{ for all } f' \in \sigma(B)\}$, and we call $\tau(B)$ the flower

of B.[5] For any $f \in R^\natural$ and $B \in R^\flat$, we let $\mathrm{noc}(f,B)$ be the rational number defined by saying that if $f \in \tau(B)$ then $\mathrm{noc}(f,B) = \lambda(B)$, whereas if $f \notin \tau(B)$ then $\mathrm{noc}(f,B)$ equals the common value (see (ITP)) of $\mathrm{noc}(f,f')$ as f' varies in $\tau(B)$; we call $\mathrm{noc}(f,B)$ the normalized contact of f with B, and we note that: $\mathrm{noc}(f,B) \neq \lambda(B) \Leftrightarrow \mathrm{noc}(f,B) < \lambda(B) \Leftrightarrow f \notin \tau(B)$.[6] For any $f \in R^\natural$ and $\lambda \in \mathbb{Q}$, we let $\overline{R}(f,\lambda) = \{B' \in R^\flat : f \in \tau(B')$ and $\lambda(B') = \lambda\}$; members of $\overline{R}(f,\lambda)$ may be called λ-buddies of f. For any $f \in R^\natural$ and $B \in R^\flat$, we let $\overline{R}(f,B) = \overline{R}(f,\mathrm{noc}(f,B))$; members of $\overline{R}(f,B)$ may be called B-buddies of f. For any $B \in R^\flat$, we let $\overline{R}(B) = \{B' \in R^\flat : \tau(B') = \tau(B)$ and $\lambda(B') = \lambda(B)\}$; members of $\overline{R}(B)$ may be called buddies of B.

Given any bud B, by (GNP1) we see that there is a unique nonnegative integer $p(B)$, a unique positive integer $D(B)$, a unique rational number $S(B)$, a unique sequence of integers $m(B) = m_i(B)_{0 \le i \le p(B)+1}$ with the exception that $m_{p(B)+1} = \infty$, a unique sequence of rational numbers $c(B) = c_i(B)_{1 \le i \le p(B)}$, a unique member $t(B)$ of $R^\natural$, a unique nonzero element $A(B)$ of k, and a unique nonempty set $\epsilon(B)$ of triples (z, V, W) with $z = z(X) \in k((X))$ and $0 < V \in \mathbb{Z}$ and $W \in \mathbb{Z}$, having the bud properties which say that

$$\begin{cases} \text{for every } f \in \tau(B), \text{ upon letting } \deg_Y f = n, \text{ we have:} \\ p(f,\lambda(B)) = p(B), D(f,\lambda(B)) = D(B), S(f,\lambda(B)) = S(B), \\ m(f,\lambda(B)) = m(B), c(f,\lambda(B)) = c(B), t(f,\lambda(B)) = t(B), \\ A(f,\lambda(B),\tilde{z}) = A(B) \text{ for all } \tilde{z} = \tilde{z}(X) \in k((X)) \text{ with } f(X^n, \tilde{z}(X)) = 0, \\ \text{and } (z,n,V,W) \mapsto (\hat{z},V,W) \text{ gives a surjection of } \epsilon(f,\lambda(B)) \text{ onto } \epsilon(B) \\ \text{where } \hat{z}(X) = z^\dagger(X,n,V,W). \end{cases} \tag{BP1}$$

We call $p(B)$, $D(B)$, $S(B)$, $m(B)$, $c(B)$, $t(B)$, $A(B)$, and $\epsilon(B)$, the position, the degree, the strength, the characteristic sequence, the normalized characteristic sequence, the normalized truncation, the reduced constant, and the edge of B respectively, and we note that then for $t(B)$ we have

$$\begin{cases} t(B) \in \tau(B), \deg_Y t(B) = D(B), m(t(B)) = m(B), \\ c(t(B)) = c(B), h(m(t(B))) = h(c(t(B))) = p(B), \text{ and} \\ \epsilon(B) = \{(z,V,W) : 0 < V \in D(B)\mathbb{Z} \text{ and } W = \lambda(B)V \in \mathbb{Z} \text{ and} \\ \qquad z = z(X) \in k((X)) \text{ with } t(B)(X^V, z(X)) = 0\}. \end{cases} \tag{BP2}$$

Given any bud B, by (BP1) and (BP2) we see that

$$\begin{cases} \text{for any } B' \in R^\flat \text{ we have:} \\ B' \in \overline{R}(B) \Leftrightarrow \overline{R}(B') = \overline{R}(B) \\ \qquad \Leftrightarrow \tau(B') \cap \tau(B) \neq \emptyset \text{ and } \lambda(B') = \lambda(B) \\ \qquad \Rightarrow p(B') = p(B), D(B') = D(B), S(B') = S(B), m(B') = m(B), \\ \qquad\quad c(B') = c(B), t(B') = t(B), A(B') = A(B), \text{ and } \epsilon(B') = \epsilon(B) \end{cases} \tag{BP3}$$

[5] Basically, the stem $\sigma(B)$ of a bud $B = (\sigma(B), \lambda(B))$ is a nonempty set of irreducible meromorphic curves $f(X,Y) = 0$ whose fractional meromorphic roots mutually coincide up to $X^{\lambda(B)}$, and its flower $\tau(B)$ is the set of all irreducible meromorphic curves whose fractional meromorphic roots coincide with the fractional meromorphic roots of members of $\sigma(B)$ up to $X^{\lambda(B)}$.

[6] Equivalently, $\mathrm{noc}(f,B)$ can be defined by saying that, for any $f \in R^\natural$ and $B \in R^\flat$, we have $\mathrm{noc}(f,B) = \min\{\mathrm{noc}(f,f') : f' \in \tau(B)\}$.

and by (NP1), (GNP4), (GNP5) and (GNP6) we see that

$$\begin{cases}\text{for any } f\in\tau(B), \text{ with } \deg_Y f=n, \text{ we have } n/D(B)\in\mathbb{Z},\\ \text{and for any } (z,V,W)\in\epsilon(B) \text{ we have}\\ \operatorname{ord}_X f(X^V,z(X)+X^W Y)=S(B)nV\in\mathbb{Z}\\ \text{and } \deg_Y \operatorname{inco}_X f(X^V,z(X)+X^W Y)=n/D(B)\end{cases}\tag{BP4}$$

and

$$\begin{cases}\text{for any } f'\in R^{\natural}\setminus\tau(B) \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ 0\neq \operatorname{inco}_X f'(X^V,z(X)+X^W Y)\in k,\\ \text{and for any } B'\in\overline{R}(f',B), \text{ upon letting } \deg_Y f'=n', \text{ we have}\\ \operatorname{ord}_X f'(X^V,z(X)+X^W Y)=S(B')n'V\in\mathbb{Z}.\end{cases}\tag{BP5}$$

Next we let $R^{\flat *}$ be the set of all strict buds in R, where by a strict bud we mean a bud B such that $\operatorname{noc}(f,f')>\lambda(B)$ for all f and f' in $\sigma(B)$; we also let $\tau^*(B)=\{f\in R^{\natural}: \operatorname{noc}(f,f')>\lambda(B)$ for all $f'\in\sigma(B)\}$, and we call $\tau^*(B)$ the strict flower of B.[7] For any $f\in R^{\natural}$ and $\lambda\in\mathbb{Q}$ we let $\overline{R}^*(f,\lambda)=\{B'\in R^{\flat *}: f\in\tau^*(B')$ and $\lambda(B')=\lambda\}$; members of $\overline{R}^*(f,\lambda)$ may be called strict λ-buddies of f. For any $f\in R^{\natural}$ and $B\in R^{\flat}$, we let $\overline{R}^*(f,B)=\overline{R}^*(f,\operatorname{noc}(f,B))$; members of $\overline{R}^*(f,B)$ may be called strict B-buddies of f. For any $B\in R^{\flat}$, we let $\overline{R}^*(B)=R^{\flat *}\cap\overline{R}(B)$; members of $\overline{R}^*(B)$ may be called strict buddies of B. Finally, for any $B\in R^{\flat *}$, we let $\overline{R}^{**}(B)=\{B'\in\overline{R}^*(B):\tau^*(B')=\tau^*(B)\}$; members of $\overline{R}^{**}(B)$ may be called doubly strict buddies of B.

Given any strict bud B, by (GNP2) we see that there is a unique nonnegative integer $p^*(B)$, a unique positive integer $D^*(B)$, a unique sequence of integers $m^*(B)=m_i^*(B)_{0\le i\le p^*(B)+1}$ with the exception that $m^*_{p^*(B)+1}=\infty$, and a unique sequence of rational numbers $c^*(B)=c_i^*(B)_{1\le i\le p^*(B)}$, a unique member $t^*(B)$ of $R^{\natural}$, a unique monic polynomial $E(B,Y)$ in $k[Y]$, a unique element $E_0(B)$ in k, and a unique nonempty set $\epsilon^*(B)$ of triples (z,V,W) with $z=z(X)\in k((X))$ and $0<V\in\mathbb{Z}$ and $W\in\mathbb{Z}$, having the strict bud properties which say that

$$\begin{cases}\text{for every } f\in\tau^*(B), \text{ upon letting } \deg_Y f=n, \text{ we have:}\\ p^*(f,\lambda(B))=p^*(B)\ge p(B), D^*(f,\lambda(B))=D^*(B)\in D(B)\mathbb{Z},\\ m^*(f,\lambda(B))=m^*(B), c^*(f,\lambda(B))=c^*(B), t^*(f,\lambda(B))=t^*(B),\\ E(f,\lambda(B),\tilde{z},Y)=E(B,Y)=Y^{D^*(B)/D(B)}-E_0(B)\\ \text{for all } \tilde{z}=\tilde{z}(X)\in k((X)) \text{ with } f(X^n,\tilde{z}(X))=0,\\ \text{and } (z,n,V,W)\mapsto(\hat{z},V,W) \text{ gives a surjection of } \epsilon(f,\lambda(B)) \text{ onto } \epsilon^*(B)\\ \text{where } \hat{z}(X)=z^{\dagger *}(X,n,V,W).\end{cases}\tag{SBP1}$$

We call $p^*(B)$, $D^*(B)$, $m^*(B)$, $c^*(B)$, $t^*(B)$, $E(B,Y)$, $E_0(B)$, and $\epsilon^*(B)$, the strict position, the strict degree, the strict characteristic sequence, the strict normalized characteristic sequence, the strict normalized truncation, the reduced polynomial, the polynomial

[7] Again, basically, the stem $\sigma(B)$ of a strict bud $B=(\sigma(B),\lambda(B))$ is a nonempty set of irreducible meromorphic curves $f(X,Y)=0$ whose fractional meromorphic roots mutually coincide through $X^{\lambda(B)}$, and its strict flower $\tau^*(B)$ is the set of all irreducible meromorphic curves whose fractional meromorphic roots coincide with the fractional meromorphic roots of members of $\sigma(B)$ through $X^{\lambda(B)}$.

constant, and the strict edge of B respectively, and we note that then for $t^*(B)$ we have

$$\begin{cases} t^*(B)\in\tau^*(B), \deg_Y t^*(B)=D^*(B), m(t^*(B))=m^*(B), \\ c(t^*(B))=c^*(B), h(m(t^*(B)))=h(c(t^*(B)))=p^*(B), \text{ and} \\ \epsilon^*(B)=\{(z,V,W): 0<V\in D^*(B)\mathbb{Z} \text{ and } W=\lambda(B)V\in\mathbb{Z} \text{ and} \\ \quad z=z(X)\in k((X)) \text{ with } t^*(B)(X^V,z(X))=0\}. \end{cases} \tag{SBP2}$$

Given any strict bud B, by (SBP1) and (SBP2) we see that

$$\begin{cases} \text{for any } B'\in R^{\flat *} \text{ we have:} \\ B'\in\overline{R}^{**}(B) \Leftrightarrow \overline{R}^{**}(B')=\overline{R}^{**}(B) \\ \qquad\qquad \Leftrightarrow \tau^*(B')\cap\tau^*(B)\neq\emptyset \text{ and } \lambda(B')=\lambda(B) \\ \qquad\qquad \Rightarrow p^*(B')=p^*(B), D^*(B')=D^*(B), m^*(B')=m^*(B), \\ \qquad\qquad\quad c^*(B')=c^*(B), t^*(B')=t^*(B), E(B',Y)=E(B,Y), \\ \qquad\qquad\quad \text{and } \epsilon^*(B')=\epsilon^*(B) \end{cases} \tag{SBP3}$$

and by (NP1) and (GNP5) we see that

$$\begin{cases} \text{for any } f\in\tau^*(B), \text{ with } \deg_Y f=n, \text{ we have } n/D^*(B)\in\mathbb{Z}, \\ \text{and for any } (z,V,W)\in\epsilon(B) \text{ we have} \\ \operatorname{inco}_X f(X^V, z(X)+X^W Y)=A(B)^{n/D(B)}E(B,Y)^{n/D^*(B)}. \end{cases} \tag{SBP4}$$

Given any bud B, by (NP2), (NP4) and (GNP3) we get the mixed bud properties which say that

$$\begin{cases} \text{for any } B'\in\overline{R}^*(B) \text{ and } B''\in\overline{R}^*(B) \text{ we have :} \\ E(B',Y)\neq E(B'',Y)\Leftrightarrow\tau^*(B')\neq\tau^*(B'') \\ \quad\Leftrightarrow\tau^*(B')\cap\tau^*(B'')=\emptyset \\ \quad\Leftrightarrow E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k \end{cases} \tag{MBP1}$$

and

$$\begin{cases} \text{for any } B'\in\overline{R}^*(B) \text{ we have:} \\ E_0(B')=0\Leftrightarrow B'\in\overline{R}^*(t(B),B) \\ \qquad\qquad \Rightarrow p^*(B')=p(B'), D^*(B')=D(B), m^*(B')=m(B'), \\ \qquad\qquad\quad c^*(B')=c(B'), t^*(B')=t(B'), \text{ and } \epsilon^*(B')=\epsilon(B') \end{cases} \tag{MBP2}$$

and

$$\begin{cases} \text{for any } B'\in\overline{R}^*(B)\setminus\overline{R}^*(t(B),B) \text{ and } B''\in\overline{R}^*(B)\setminus\overline{R}^*(t(B),B) \\ \text{we have : } p^*(B')=p^*(B''), D^*(B')=D^*(B''), m^*(B')=m^*(B''), \\ \text{and } c^*(B')=c^*(B''). \end{cases} \tag{MBP3}$$

5. Contact sets

Given any $F=F(X,Y)\in R$ of Y-degree N, we can write

$$F=\prod_{0\le j\le\chi(F)} F_j \quad \text{where} \quad F_0=F_0(X)\in K((X))$$

and

$$F_j = F_j(X,Y) \in R^\natural \quad \text{with} \quad \deg_Y F_j = N_j \text{ for } 1 \leq j \leq \chi(F)$$

and $\chi(F)$ is a nonnegative integer such that: $\chi(F) = 0 \Leftrightarrow F \in k((X))$.[8] We define the contact set $C(F)$ of F by putting

$$\begin{aligned} C(F) = & \{c_i(F_j) : 1 \leq j \leq \chi(F) \text{ and } 1 \leq i \leq h(c(F_j)) \text{ and } c_i(F_j) \notin \mathbb{Z}\} \\ & \cup \{\mathrm{noc}(F_j, F_{j'}) : 1 \leq j < j' \leq \chi(F) \text{ with } F_j \neq F_{j'}\}. \end{aligned}$$

Upon letting

$$N^\natural = \prod_{1 \leq j \leq \chi(F)} N_j$$

(with the usual convention that the product of an empty family is 1), by Newton's theorem we have

$$F(X^{N^\natural}, Y) = F_0(X^{N^\natural}) \prod_{1 \leq j \leq N} [Y - z_j^\natural(X)] \quad \text{with} \quad z_j^\natural(X) \in k((X))$$

and by the material on p. 300 of [Ab], as an alternative characterization of $C(F)$, we get

$$C(F) = \{(1/N^\natural)\,\mathrm{ord}_X[z_j^\natural(X) - z_{j'}^\natural(X)] : 1 \leq j < j' \leq N \text{ with } z_j^\natural(X) \neq z_{j'}^\natural(X)\}.$$

Note that

$$C(F) = \emptyset \Leftrightarrow N_j = 1 \text{ for } 1 \leq j \leq \chi(F) \text{ and } F_j = F_{j'} \text{ for } 1 \leq j < j' \leq \chi(F).$$

Given any $G = G(X,Y) \in R$ of Y-degree M, we can write

$$G = \prod_{0 \leq j \leq \chi(G)} G_j \quad \text{where} \quad G_0 = G_0(X) \in K((X))$$

and

$$G_j = G_j(X,Y) \in R^\natural \quad \text{with} \quad \deg_Y G_j = M_j \text{ for } 1 \leq j \leq \chi(G)$$

and $\chi(G)$ is a nonnegative integer such that: $\chi(G) = 0 \Leftrightarrow G \in k((X))$. Note that now

$$\begin{aligned} C(FG) = C(F) \cup C(G) \cup \{\mathrm{noc}(F_j, G_{j'}) : & \ 1 \leq j \leq \chi(F) \text{ and} \\ & \ 1 \leq j' \leq \chi(G) \text{ with } F_j \neq G_{j'}\}. \end{aligned}$$

Let

$$J(F,G) = J_{(X,Y)}(F,G)$$

be the jacobian of $F = F(X,Y)$ and $G = G(X,Y)$ with respect to X and Y, i.e., let

$$J(F,G) = F_X G_Y - G_X F_Y$$

where subscripts denote partial derivatives. Our aim is to produce a factorization of $J(F,G)$ in terms of the contact set $C(FG)$.

[8] In other words, if $F \in k((X))$ then $\chi(F) = 0$, whereas if $F \notin k((X))$ then $\chi(F)$ equals the number of irreducible factors of F in R.

In case of $F \neq 0$ we can write

$$F = X^{N^\sharp} P + (\text{terms of } X\text{-degree} > N^\sharp)$$

where

$$N^\sharp = \mathrm{ord}_X(F) \quad \text{and} \quad 0 \neq P = P(Y) = \mathrm{inco}_X(F) \in k[Y] \quad \text{with} \quad \deg_Y(P) = \nu.$$

Likewise, in case of $G \neq 0$ we can write

$$G = X^{M^\sharp} Q + (\text{terms of } X\text{-degree} > M^\sharp)$$

where

$$M^\sharp = \mathrm{ord}_X(G) \quad \text{and} \quad 0 \neq Q = Q(Y) = \mathrm{inco}_X(G) \in k[Y] \quad \text{with} \quad \deg_Y(Q) = \mu.$$

Now in case of $F \neq 0 \neq G$ we get

$$\begin{aligned} J(F,G) &= X^{N^\sharp + M^\sharp - 1}(N^\sharp P Q_Y - M^\sharp P_Y Q) \\ &\quad + (\text{terms of } X\text{-degree} > N^\sharp + M^\sharp - 1) \end{aligned} \tag{JE1}$$

and hence

$$\mathrm{ord}_X J(F,G) \geq N^\sharp + M^\sharp - 1 \tag{JE2}$$

and

$$\begin{aligned} \mathrm{ord}_X J(F,G) = N^\sharp + M^\sharp - 1 &\Leftrightarrow N^\sharp P Q_Y - M^\sharp P_Y Q \neq 0 \\ &\Rightarrow \mathrm{inco}_X J(F,G) = N^\sharp P Q_Y - M^\sharp P_Y Q. \end{aligned} \tag{JE3}$$

These Jacobian estimates are basic in getting a factorization of $J(F,G)$ out of $C(FG)$ or, more precisely, out of the "tree" $T(FG)$ which, in § 6, we shall build from $C(FG)$. Moreover, as we shall explain in § 7, most of this set-up works in getting a factorization of any $H \in R$ out of any tree T. In § 8 we shall apply it to the situation when $H = J(F,G) = F_Y$ with $G = -X$. In § 9 we shall consider the general case of $H = J(F,G)$.

6. Trees

By allowing the level $\lambda(B)$ of a bud B to be $-\infty$ we get the set $R^\flat_\infty$ of all improper buds B; note that any nonempty subset of $R^\natural$ can be the stem $\sigma(B)$ of an improper bud B; moreover, for any improper bud B we have $\lambda(B) = -\infty$ and $\tau(B) = R^\natural$. We put $\hat{R}^\flat = R^\flat \cup R^\flat_\infty$, and we call a member of $\hat{R}^\flat$ a generalized bud. For any $B \in \hat{R}^\flat$ we let $\tau^*(B) = \{f \in \tau(B) : \mathrm{noc}(f, f') > \lambda(B)$ for some $f' \in \sigma(B)\}$, and we call $\tau^*(B)$ the strict flower of B; note that for any $B \in R^{\flat *}$ this definition coincides with the definition made earlier; also note that for any $B \in R^\flat_\infty$ we have $\tau^*(B) = R^\natural$. For any $B \in \hat{R}^\flat$ we let $\tau'(B) = \tau(B) \setminus \tau^*(B)$, and we call $\tau'(B)$ the primitive flower of B. Previously we have defined the normalized contact $\mathrm{noc}(f, B)$ for all $f \in R^\natural$ and $B \in R^\flat$; now we extend this by putting $\mathrm{noc}(f, B) = -\infty$ for all $f \in R^\natural$ and $B \in R^\flat_\infty$. For any $f \in R^\natural$ and $B \in \hat{R}^\flat$ we define $R^*(f, B)$ to be the unique member of $\hat{R}^\flat$ whose stem is $\{f\}$ and whose level is $\mathrm{noc}(f, B)$, and we call $R^*(f, B)$ the strict B-friend of f; note that then $R^*(f, B)$ belongs to $R^{\flat *}$ or $R^\flat_\infty$ according as $B \in R^\flat$ or $B \in R^\flat_\infty$. For any $B \in \hat{R}^\flat$ we define $R^*(B)$ to be the set of all $B' \in R^{\flat *} \cup R^\flat_\infty$ such that $\lambda(B') = \lambda(B)$ and $\sigma(B') = \tau^*(B') \cap \sigma(B)$, and we call members $R^*(B)$ strict friends of B; note that $\sigma(B) = \coprod_{B' \in R^*(B)} \sigma(B')$ gives a partition of $\sigma(B)$ into pairwise disjoint nonempty subsets.

Now the set $\hat{R}^\flat$ is prepartially ordered by defining $B' \geq B$ to mean $\lambda(B') \geq \lambda(B)$ and $\tau(B') \subset \tau(B)$.[9] For any B' and B in $\hat{R}^\flat$ we write $B' \gg B$ or $B \ll B'$ to mean $B' > B$ and $\lambda(B') > \lambda(B)$, i.e., to mean $\lambda(B') > \lambda(B)$ and $\tau(B') \subset \tau(B)$. For any $B' \gg B$ in $\hat{R}^\flat$ we define $R^*(B', B)$ to be the unique member of $\hat{R}^\flat$ whose stem is $\sigma(B')$ and whose level is $\lambda(B)$, and we call $R^*(B', B)$ the strict B-friend of B'; note that then $R^*(B', B)$ belongs to $R^{\flat *}$ or $R^\flat_\infty$ according as $\lambda(B) \neq -\infty$ or $\lambda(B) = -\infty$. For any $B' \gg B$ in $\hat{R}^\flat$ we also put $\tau^*(B', B) = \tau^*(R^*(B', B)) \setminus \tau(B')$, and we call $\tau^*(B', B)$ the strict B-flower of B'.

Let $\hat{R}^\sharp$ be the set of all trees in R, where by a tree we mean a subset T of $\hat{R}^\flat$ such that T contains an improper bud, and for any $B' \neq B$ in T with $\lambda(B') = \lambda(B)$ we have $\tau(B') \cap \tau(B) = \emptyset$; note that then the prepartial order $\geq$ induces a partial order on T, and hence in particular T has a unique improper bud; we call this improper bud the root of T and denote it by $R_\infty(T)$; also note that for any B' and B in T we have: $B' > B \Leftrightarrow B' \gg B$. For any tree T, we put $\Lambda(T) = \{\lambda(B) : B \in T\}$ and we call $\Lambda(T)$ the level set of T; we define the height $h(T)$ of T by putting $h(T) = \infty$ if $\Lambda(T)$ is infinite, and $h(T) =$ the cardinality of $\Lambda(T)$ minus 1 if $\Lambda(T)$ is finite; moreover, in case $h(T)$ is a nonnegative integer, i.e., in case $\Lambda(T)$ is a finite set, we let $l(T) = l_i(T)_{0 \leq i \leq h(T)}$ be the strictly increasing sequence $l_0(T) < \cdots < l_{h(T)}(T)$ such that $\{l_0(T), \ldots, l_{h(T)}(T)\} = \Lambda(T)$, and we call $l(T)$ the level sequence of T. Note that a tree T is finite iff its level set $\Lambda(T)$ is finite and T has at most a finite number of generalized buds of any given level. We put

$$R^\sharp = \text{the set of all finite trees in } R.$$

For any generalized bud B in any tree T, we put $\pi(T, B) = \{B' \in T : B' > B\}$ and we call $\pi(T, B)$ the B-preroof of T, and we put $\rho(T, B) = \{B' \in \pi(T, B)$: there is no $B'' \in \pi(T, B)$ with $B' > B''\}$ and we call $\rho(T, B)$ the B-roof of T. For any generalized bud B in any tree T, we also put $\tau(T, B) = \tau(B) \setminus \cup\{\tau(B') : B' \in \rho(T, B)\}$ and we call $\tau(T, B)$ the B-flower of T, and we put $\tau^*(T, B) = \tau^*(B) \setminus \cup\{\tau(B') : B' \in \rho(T, B)\}$ and we call $\tau^*(T, B)$ the strict T-flower of B; note that then $\tau(T, B) = \tau(B) \setminus \cup\{\tau(B') : B' \in \pi(T, B)\}$ and $\tau^*(T, B) = \tau^*(B) \setminus \cup\{\tau(B') : B' \in \pi(T, B)\} = \tau(T, B) \setminus \tau'(B)$.[10]

A tree T is said to be strict if for every $\lambda \in \Lambda(T)$ we have $\sigma(R_\infty(T)) = \cup_{B \in T^{(\lambda)}} \sigma(B)$ where $T^{(\lambda)}$ is the set of all $B \in T$ with $\lambda(B) = \lambda$. Given any $\lambda \in \mathbb{Q} \cup \{-\infty\}$, by (ITP) we see that $f \sim_\lambda f'$ gives an equivalence relation on $R^\natural$ where $f \sim_\lambda f'$ means $\mathrm{noc}(f, f') \geq \lambda$. It follows that, given any $\hat{\sigma} \subset R^\natural$ and $\hat{\Lambda} \subset \mathbb{Q}$, there is a unique strict tree $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ with $\Lambda(\hat{T}(\hat{\sigma}, \hat{\Lambda})) = \{-\infty\} \cup \hat{\Lambda}$ such that $\sigma(R_\infty(\hat{T}(\hat{\sigma}, \hat{\Lambda}))) = \hat{\sigma}$ or $\{Y\}$ according as $\hat{\sigma}$ is nonempty or empty; we call $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ the $\hat{\Lambda}$-tree of $\hat{\sigma}$; note that, if $\hat{\sigma}$ is nonempty then, for every $\lambda \in \hat{\Lambda}$, the stems of the buds of $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ of level λ are the equivalence classes of $\hat{\sigma}$ under $\sim_\lambda$; likewise, if $\hat{\sigma}$ is empty then, for every $\lambda \in \hat{\Lambda}$, the stem of the unique bud of $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ of level λ is $\{Y\}$. We put

$$R^{\sharp *} = \text{the set of all finite strict trees in } R$$

and we note that for any $B \in T \in R^{\sharp *}$ with $\lambda(B) = l_i$ for some $i < h(T)$ we have $\rho(T, B) = \{B' \in T : \lambda(B') = l_{i+1}\}$ and $\sigma(B) = \coprod_{B' \in \rho(T,B)} \sigma(B')$ which is a partition of $\sigma(B)$ into pairwise disjoint nonempty subsets. For any $F \in R$, with its monic irreducible factors

[9] A set is prepartially ordered by $\geq$ means: $a \geq b$ and $b \geq c$ implies $a \geq c$. It is partially ordered if also: $a \geq b$ and $b \geq a$ implies $a = b$.

[10] For printing convenience we may write $\cup\{\tau(B') : B' \in \rho(T, B)\}$ instead of $\cup_{B' \in \rho(T,B)} \tau(B')$, with similar notation for $\cap$, $\sum$ and $\prod$.

$F_1, \ldots, F_{\chi(F)}$ as in the previous section, we put

$$T(F) = \hat{T}(\{F_1, \ldots, F_{\chi(F)}\}, C(F))$$

and we call $T(F)$ the tree of F, and we note that then $T(F) \in R^{\sharp *}$.

A tree T' is a subtree of a tree T if for every $B' \in T'$ there exists some (and hence a unique) $B \in T$ such that $\sigma(B') \subset \sigma(B)$ and $\lambda(B') = \lambda(B)$. Every tree is clearly a subtree of the universal tree $\hat{T}(R^{\natural}, \mathbb{Q})$, which is a strict tree of infinite height.[11]

Remark (TR1). Basically we are interested in comparing the tree $T(FG)$ of the product of two members F and G of R with the tree $T(J(F,G))$ of their jacobian. In case of $G = -X$, this reduces to comparing $T(F)$ with $T(F_Y)$.

Remark (TR2). For the benefit of the readers (and ourselves) we shall now describe three examples of the tree $T(F)$ of various types of $F \in \hat{R}^{\natural}$.

Example (TR3). First, here is an example of $F \in \hat{R}^{\natural}$ which is irreducible and has only one characteristic exponent, i.e., with $\chi(F) = 1$ and $h(m(F)) = 1$. Namely, let

$$1 < n \in \mathbb{Z} \text{ and } 0 \neq e \in \mathbb{Z} \text{ with } \mathrm{GCD}(n,e) = 1$$

and

$$F = f = f(X,Y) = Y^n + \sum_{1 \le i \le n} w_i(X) Y^{n-i}$$

where $w_i(X) \in k((X))$ is such that

$$\mathrm{ord}_X w_i(X) > ie/n \text{ for } 1 \le i \le n-1 \text{ and } \mathrm{ord}_X w_n(X) = e$$

and let κ be the coefficient of X^e in $w_n(X)$, i.e., let $0 \neq \kappa \in k$ be such that $\mathrm{ord}_X(w_n(X) - \kappa X^e) > e$. Then f is irreducible in $\hat{R}^{\natural}$, and we have the Newtonian factorization

$$f(X^n, Y) = \prod_{1 \le j \le n} [Y - z_j(X)]$$

where $z_j(X) \in k((X))$ is such that

$$z_j(X) = \omega^j \kappa^* X^e + (\text{terms of degree} > e \text{ in } X)$$

where ω is a primitive n-th root of 1 in k, and κ^* is an n-th root of $-\kappa$ in k.

To see this, first note that $f(X^n, X^e Y) = X^{ne} g(X,Y)$ where

$$g(X,Y) = Y^n + \sum_{1 \le i \le n} v_i(X) Y^{n-i}$$

and $v_i(X) = X^{-ie} w_i(X^n) \in k[[X]]$ is such that

$$v_i(0) = 0 \text{ for } 1 \le i \le n-1 \text{ and } v_n(0) = \kappa.$$

[11] This universal tree is like the Ashwattha Tree of the Bhagwad-Gita. The stem of its root contains the embryos of all the past, present and future creatures in nascent form. Its trunks travel upwards first comprising of large tribes and then of smaller and smaller clans. Its "ultimate" shoots reaching heaven are the individual souls eager to embrace their maker.

Now $g(0,Y)=Y^n-\kappa^{*n}$, and hence we get the desired factorization by applying Hensel's lemma. Since $\mathrm{GCD}(n,e)=1$, we see that f is irreducible in $\hat{R}^\natural$.

The above factorization of f yields $h(m(f))=1$ with

$$m_0(f)=q_0(m(f))=s_0(q(m(f)))=r_0(q(m(f)))=n \text{ and } d_1(m(f))=n$$

and

$$m_1(f)=q_1(m(f))=s_1(q(m(f)))/n=r_1(q(m(f)))=e \text{ and } d_2(m(f))=1.$$

Therefore

$$C(F)=C(f)=\{c_1(f)\} \text{ with } c_1(f)=e/n.$$

Hence $h(T(f))=1$ with

$$l_0(T(f))=-\infty \text{ and } l_1(T(f))=c_1(f)=e/n$$

and upon letting

$$B_i\in\hat{R}^\flat \text{ with } \sigma(B_i)=\{f\} \text{ and } \lambda(B_i)=l_i(T(f)) \text{ for } 0\le i\le 1$$

we have

$$T(F)=T(f)=\{B_0,B_1\}$$

with

$$D'(B_0)=0 \text{ and } D'(B_1)=n-1.$$

Note that for F to be analytic, i.e., for it to belong to the ring $k[[X]][Y]$, the condition $e>0$ is necessary and sufficient. However, for F to be pure meromorphic, i.e., for it to belong to the ring $k[X^{-1}][Y]$, i.e., for the existence of $\Phi(X,Y)\in k[X,Y]$ with $F(X,Y)=\Phi(X^{-1},Y)$, the condition $e<0$ is necessary but not sufficient. As a specific illustration of the analytic case we may take $(n,e)=(4,5)$ and $(w_1(X),\dots,w_{n-1}(X),w_n(X))=(0,\dots,0,X^5)$, giving us $F(X,Y)=Y^4+X^5$. Similarly, as a specific illustration of the pure meromorphic case we may take $(n,e)=(4,-3)$ and $(w_1(X),\dots,w_{n-1}(X),w_n(X))=(0,\dots,0,X^{-3})$, giving us $F(X,Y)=Y^4+X^{-3}$, i.e., $F(X,Y)=\Phi(X^{-1},Y)$ with $\Phi(X,Y)=Y^4+X^3$.

Example (TR4). Next, here is an example of $F\in\hat{R}^\natural$ which is irreducible and has two characteristic exponents, i.e., with $\chi(F)=1$ and $h(m(F))=2$. Namely, let

$$F=f=f(X,Y)=(Y^2-X^{2a+1})^2-X^{3a+b+2}Y \text{ with } a\in\mathbb{Z} \text{ and } 0\le b\in\mathbb{Z}.$$

Then f is irreducible in $\hat{R}^\natural$, and we have the Newtonian factorization

$$f(X^4,Y)=\prod_{1\le j\le 4}[Y-z_j(X)]$$

where $z_j(X)\in k((X))$ is such that

$$z_j(X)=(\iota^j X)^{4a+2}+\tfrac{1}{2}(\iota^j X)^{4a+2b+3}+(\text{terms of degree} > 4a+2b+3 \text{ in } X)$$

where ι is a primitive 4-th root of 1 in k (e.g., $\iota=$ the usual i).

To see this, first note that $f(X^4,X^{4a+2}Y)=X^{16a+8}g(X,Y)$ where

$$g(X,Y)=(Y^2-1)^2-X^{4b+2}Y.$$

Now for $\jmath = 1$ or -1, upon letting $g_\jmath(X,Y) = g(X, Y+\jmath)$ we have

$$g_\jmath(X,Y) = (2\jmath + Y)^2Y^2 - \jmath(1+\jmath Y)X^{4b+2}$$

and hence (say by the binomial theorem) we get

$$g_\jmath(X,Y) = [(2\jmath+Y)Y - \jmath^*\theta(Y)X^{2b+1}][(2\jmath+Y)Y + \jmath^*\theta(Y)X^{2b+1}]$$

where

$$\theta(Y) = 1 + (\jmath Y/2) - \sum_{i=2}^{\infty} 1\times 3\times\cdots\times(2i-3)\times(-\jmath Y/2)^i/i!$$

and $\jmath^* = \iota^2$ or ι according as $\jmath = 1$ or -1, and therefore (say by the Weierstrass preparati theorem) we have

$$g(X,Y) = \prod_{1\le j\le 4}[Y - y_j(X)]$$

where $y_j(X) \in k[[X]]$ is such that

$$y_j(X) = (-1)^j[1 + \tfrac{1}{2}(\iota^j X)^{2b+1} + (\text{terms of degree} > 2b+1 \text{ in } X)].$$

Since $f(X^4, X^{4a+2}Y) = X^{16a+8}g(X,Y)$, we get the above factorization of $f(X^4, Y)$. Sin the GCD of 4 with the support of $z_1(X)$ is 1, we conclude that f is irreducible in $\hat{R}^\natural$, i. $\chi(F) = 1$.

The above factorization of f yields $h(m(f)) = 2$ with

$$m_0(f) = q_0(m(f)) = s_0(q(m(f))) = r_0(q(m(f))) = 4 \text{ and } d_1(m(f)) = 4$$

and

$$m_1(f) = q_1(m(f)) = s_1(q(m(f)))/4 = r_1(q(m(f))) = 4a+2 \text{ and } d_2(m(f)) =$$

and

$$m_2(f) = 4a + 2b + 3 \text{ and } q_2(m(f)) = 2b+1$$

and

$$s_2(q(m(f))) = 16a+4b+10 \text{ and } r_2(q(m(f))) = 8a+2b+5 \text{ and } d_3(m(f)) =$$

Therefore

$$C(F) = C(f) = \{c_1(f), c_2(f)\}$$

with

$$c_1(f) = (2a+1)/2 \text{ and } c_2(f) = (4a+2b+3)/4.$$

Hence $h(T(f)) = 2$ with $l_0(T(f)) = -\infty$ and

$$l_1(T(f)) = c_1(f) = (2a+1)/2 \text{ and } l_2(T(f)) = c_2(f) = (4a+2b+3)/4$$

and upon letting

$$B_i \in \hat{R}^\flat \text{ with } \sigma(B_i) = \{f\} \text{ and } \lambda(B_i) = l_i(T(f)) \text{ for } 0 \le i \le 2$$

'e have

$$T(F) = T(f) = \{B_0, B_1, B_2\}.$$

with $D'(B_0)=0$ and

$$D'(B_1)=1 \text{ and } D'(B_2)=2.$$

As a specific illustration of the analytic case we may take $(a,b)=(1,0)$, giving us $F(X,Y)=(Y^2-X^3)^2-X^5Y$. Similarly, as a specific illustration of the pure meromorphic case we may take $(a,b)=(-1,1)$, giving us $F(X,Y)=(Y^2-X^{-1})^2-Y$, i.e., $F(X,Y)=\Phi(X^{-1},Y)$ with $\Phi(X,Y)=(Y^2-X)^2-Y$. Note that this Φ is a variable in the sense that $k[X,Y]=k[\Phi,\Psi]$ for some Ψ in $k[X,Y]$; in our situation we can take $\Psi(X,Y)=Y^2-X$.

Example (TR5). Finally, here is an example of $F\in\hat{R}^\natural$ which has two factors, i.e., with $\chi(F)=2$. Namely, let

$$0<n\in\mathbb{Z} \text{ and } a\in\mathbb{Z} \text{ and } 0\le b\in\mathbb{Z}$$

and

$$F=F(X,Y)=Y^{n+2}+\sum_{2\le i\le n+2} u_i(X)Y^{n+2-i}$$

where $u_i(X)\in k((X))$ is such that

$$\operatorname{ord}_X u_i(X)\ge i(a+1) \text{ for } 3\le i\le n+1$$

and

$$\operatorname{ord}_X u_2(X)=2a+1 \text{ and } \operatorname{ord}_X u_{n+2}(X)=(n+2)(a+1)+b$$

and let $0\ne\kappa'\in k$ and $0\ne\kappa\in k$ be the coefficients of X^{2a+1} and $X^{(n+2)(a+1)+b}$ in $u_2(X)$ and $u_{n+2}(X)$ respectively. Then

$$F(X,Y)=f(X,Y)f'(X,Y) \text{ with } f(X,Y)\ne f'(X,Y)$$

where

$$f(X,Y)=Y^n+\sum_{1\le i\le n} w_i(X)Y^{n-i}\in\hat{R}^\natural \text{ and } w_i(X)\in k((X))$$

with

$$\begin{cases}\operatorname{ord}_X w_i(X)>ie/n \text{ for } 1\le i\le n-1 \text{ and } \operatorname{ord}_X w_n(X)=e+b\\ \text{for the integer } e=na+n+1 \text{ for which } \mathrm{GCD}(n,e)=1\end{cases}$$

and

$$f'(X,Y)=Y^2+\sum_{1\le i\le 2} w_i'(X)Y^{n-i}\in R^\natural \text{ and } w_i'(X)\in k((X))$$

with

$$\begin{cases}\operatorname{ord}_X w_1'(X)>e'/2 \text{ and } \operatorname{ord}_X w_2'(X)=e'\\ \text{for the integer } e'=2a+1 \text{ for which } \mathrm{GCD}(2,e')=1\end{cases}$$

and $0\ne\kappa'\in k$ and $0\ne\kappa/\kappa'\in k$ are the coefficients of $X^{e'}$ and X^{e+b} in $w_2'(X)$ and $w_n(X)$ respectively. Moreover, if $b=0$ then we also have $f(X,Y)\in R^\natural$.

To see this, first note that $F(X,X^aY)=X^{na+2a}g(X,Y)$ where

$$g(X,Y)=Y^{n+2}+\sum_{2\le i\le n+2} v_i(X)Y^{n+2-i}$$

and $v_i(X) = X^{-ia}u_i(X) \in k[[X]]$ is such that

$$\mathrm{ord}_X\, v_i(X) \geq i \text{ for } 3 \leq i \leq n+1$$

and

$$\mathrm{ord}_X\, v_2(X) = 1 \text{ and } \mathrm{ord}_X\, v_{n+2}(X) = n+2+b$$

and $0 \neq \kappa' \in k$ and $0 \neq \kappa \in k$ are the coefficients of X and X^{n+2+b} in $v_2(X)$ and $v_{n+2}(X)$ respectively. Now the initial form $g(X,Y)$ is $\kappa' XY^n$ which factors into the coprime factors $\kappa' X$ and Y^n, and hence by the tangent lemma incarnation of Hensel's lemma (cf. pp. 140–141 of Abhyankar's 1990 AMS book "*Algebraic geometry for scientists and engineers*") we can find $\phi(X,Y)$ and $\phi'(X,Y)$ in $k[[X,Y]]$ such that $g(X,Y) = \phi(X,Y)\ \phi'(X,Y)$ and

$$\phi(X,Y) = Y^n + \phi_{n+1}(X,Y) + (\text{terms of degree} > n+1 \text{ in } X \text{ and } Y)$$

and

$$\phi'(X,Y) = \kappa' X + \phi'_2(X,Y) + (\text{terms of degree} > 2 \text{ in } X \text{ and } Y)$$

where $\phi_{n+1}(X,Y) \in k[X,Y]$ is homogeneous of degree $n+1$ and $\phi'_2(X,Y) \in k[X,Y]$ is homogeneous of degree 2 (with the understanding that the zero polynomial is homogeneous of any degree). Comparing terms of degree $n+2$ in the equation $g(X,Y) = \phi(X,Y)\ \phi'(X,Y)$ we get

$$\kappa' X\phi_{n+1}(X,Y) + Y^n\phi'_2(X,Y) = Y^{n+2} + \sum_{2 \leq i \leq n+2} \kappa_i X^i Y^{n+2-i}$$

where $\kappa_2 \in k$ is the coefficient of X^2 in $v_2(X) - \kappa' X$, and $\kappa_i \in k$ is the coefficient of X^i in $v_i(X)$ for $3 \leq i \leq n+2$. Successively putting $X = 0$ and $Y = 0$ in the above equation we see that $\phi'_2(0,Y) = Y^2$ and $\phi_{n+1}(X,0) = \kappa_{n+2}X^{n+1}$. Therefore, in view of the Weierstrass preparation theorem, we can find $\theta(X,Y)$ and $\theta'(X,Y)$ in $k[[X,Y]]$ with $\theta(0,0) \neq 0 \neq \theta'(0,0)$ such that upon letting $\tilde{f}(X,Y) = \theta(X,Y)\phi(X,Y)$ and $\tilde{f}'(X,Y) = \theta'(X,Y)\phi'(X,Y)$ we have $g(X,Y) = \tilde{f}(X,Y)\tilde{f}'(X,Y)$ and

$$\tilde{f}(X,Y) = Y^n + \sum_{1 \leq i \leq n} \tilde{w}_i(X) Y^{n-i} \text{ and } \tilde{w}_i(X) \in k[[X]]$$

with

$$\mathrm{ord}_X\, \tilde{w}_i(X) > i(n+1)/n \text{ for } 1 \leq i \leq n-1 \text{ and } \mathrm{ord}_X\, \tilde{w}_n(X) = n+1+b$$

and

$$\tilde{f}'(X,Y) = Y^2 + \sum_{1 \leq i \leq 2} \tilde{w}'_i(X) Y^{2-i} \text{ and } \tilde{w}'_i(X) \in k[[X]]$$

with

$$\mathrm{ord}_X\, \tilde{w}'_1(X) > 1/2 \text{ and } \mathrm{ord}_X\, \tilde{w}'_2(X) = 1$$

and $0 \neq \kappa' \in k$ and $0 \neq \kappa/\kappa' \in k$ are the coefficients of X and X^{n+1+b} in $\tilde{w}'_2(X)$ and $\tilde{w}_n(X)$ respectively. Now upon letting $f(X,Y) = X^{na}\tilde{f}(X, X^{-a}Y)$ and $f'(X,Y) = X^{2a}\tilde{f}'(X, X^{-a}Y)$, we get the desired factorization of $F(X,Y)$. Since $(n, e+b) \neq (2, e')$, we also get $f \neq f'$. By (TR3) it follows that f' is irreducible in $\hat{R}^{\natural}$, and if $b = 0$ then so is f.

Now assuming $b = 0$ and $n > 1$, in view of (TR3), the factorization of F tells us that $h(T(F)) = 2$ with $l_0(T(F)) = -\infty$ and

$$l_1(T(F)) = a + (1/2) \text{ and } l_2(T(F)) = a + 1 + (1/n)$$

and upon letting

$$\begin{cases} B_0 \in \hat{R}^\flat \text{ with } \sigma(B_0) = \{f, f'\} \text{ and } \lambda(B_0) = l_0(T(F)), \\ \text{and } B_1 \in \hat{R}^\flat \text{ with } \sigma(B_1) = \{f, f'\} \text{ and } \lambda(B_1) = l_1(T(F)), \\ \text{and } B_2 \in \hat{R}^\flat \text{ with } \sigma(B_2) = \{f\} \text{ and } \lambda(B_2) = l_2(T(F)), \\ \text{and } B_2' \in \hat{R}^\flat \text{ with } \sigma(B_2') = \{f'\} \text{ and } \lambda(B_2') = l_2(T(F)), \end{cases}$$

we have

$$T(F) = \{B_0, B_1, B_2, B_2'\}$$

with $D'(B_0) = 0$ and

$$D'(B_1) = 2 \text{ and } D'(B_2) = n-1 \text{ and } D'(B_2') = 0.$$

As a specific illustration of the pure meromorphic case, taking $a = -1$ and $(u_2(X), u_{n+2}(X)) = (\kappa' X^{-1}, \kappa)$ with $\kappa' \neq 0 \neq \kappa$ in k and $u_i(X) = \kappa_i \in k$ for $3 \le i \le n+1$, we get $F(X, Y) = \Phi(X^{-1}, Y)$ where

$$\Phi(X, Y) = Y^{n+2} + \kappa' X Y^n + \kappa + \sum_{3 \le i \le n+1} \kappa_i Y^{n+2-i} \in k[X, Y]$$

with

$$0 \neq \kappa' \in k \text{ and } 0 \neq \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \le i \le n+1.$$

7. Slices

Given any $H = H(X, Y) \in R$ of Y-degree O, we can write

$$H = \prod_{0 \le j \le \chi(H)} H_j \quad \text{where} \quad H_0 = H_0(X) \in k((X)) \tag{SP1}$$

and

$$H_j = H_j(X, Y) \in R^\natural \quad \text{with} \quad \deg_Y H_j = O_j \text{ for } 1 \le j \le \chi(H) \tag{SP2}$$

and $\chi(H)$ is a nonnegative integer such that: $\chi(H) = 0 \Leftrightarrow H \in k((X))$. Now

$$H = H_0 H_\infty \quad \text{with} \quad H_\infty = \prod_{1 \le j \le \chi(H)} H_j \tag{SP3}$$

where we note that $H_\infty \in \hat{R}^\natural$, and we call H_∞ the monic part of H.

We put

$$\Omega_B(H) = \prod_{1 \le j \le \chi(H) \text{ with } H_j \in \tau(B)} H_j \quad \text{for all } B \in \hat{R}^\flat \tag{SP4}$$

and we call $\Omega_B(H)$ the B-slice of H, and we note that then $\Omega_B(H) \in \hat{R}^\natural$, and we recall that

$$\begin{cases} \text{for all } B \in \hat{R}^\flat \text{ we have} \\ \tau(B) = \{f \in R^\natural : \operatorname{noc}(f, f') \ge \lambda(B) \text{ for all } f' \in \sigma(B)\}. \end{cases} \tag{SP5}$$

We also put

$$\Omega_{(T,B)}(H) = \prod_{1\le j\le\chi(H) \text{ with } H_j\in\tau(T,B)} H_j \quad \text{for all } B\in T\in R^\sharp \tag{SP6}$$

and we call $\Omega_{(T,B)}(H)$ the (T,B)-slice of H, and we note that then $\Omega_{(T,B)}(H)\in \hat{R}^\natural$, and we recall that

$$\begin{cases} \text{for all } B\in T\in \hat{R}^\sharp \text{ we have} \\ \tau(T,B) = \tau(B)\setminus \cup_{B'\in\rho(T,B)}\tau(B') = \tau(B)\setminus \cup_{B'\in\pi(T,B)}\tau(B') \\ \text{where } \pi(T,B) = \{B'\in T : B' > B\} \\ \text{and } \rho(T,B) = \{B'\in\pi(T,B)\text{: there is no } B''\in\pi(T,B) \text{ with } B' > B''\}. \end{cases} \tag{SP7}$$

Clearly we have the slice properties which say that

$$H_\infty = \Omega_B(H) \quad \text{for all } B\in R^\flat_\infty \tag{SP8}$$

and

$$\Omega_B(H) = \Omega_{(T,B)}(H) \prod_{B'\in\rho(T,B)} \Omega_{B'}(H) \quad \text{for all } B\in T\in R^\sharp \tag{SP9}$$

and hence[12]

$$H_\infty = \prod_{B\in T} \Omega_{(T,B)}(H) \quad \text{for all } T\in R^\sharp \tag{SP10}$$

where

$$\begin{cases} \text{for all } B\in T\in R^\sharp \text{ we have} \\ \deg_Y\Omega_{(T,B)}(H) = \deg_Y\Omega_B(H) - \sum_{B'\in\rho(T,B)} \deg_Y\Omega_{B'}(H). \end{cases} \tag{SP11}$$

By (BP4) we also see that

$$\begin{cases} \text{for all } B\in R^\flat \text{ and } (z,V,W)\in\epsilon(B) \text{ and } f\in\tau(B) \text{ we have} \\ \deg_Y f = D(B)\deg_Y \operatorname{inco}_X f(X^V, z(X)+X^W Y)\in D(B)\mathbb{Z} \end{cases} \tag{SP12}$$

[12] In the innocent looking formula (SP10), there is more than meets the eye. Indeed it is the central theme of the paper. It says that any finite tree T gives rise to a factorization of the monic part H_∞ of any meromorphic curve H into the pairwise coprime monic factors $\Omega_{(T,B)}(H)$ with B varying in T. Formula (SP20) gives a further factorization of $\Omega_{(T,B)}$ into the two coprime monic factors $\Omega'_B(H)$ and $\Omega^*_{(T,B)}(H)$. When the finite tree T is strict, formula (SP30) gives a still further factorization of $\Omega^*_{(T,B)}(H)$ into the pairwise coprime monic factors $\Omega^*_{(B',B)}(H)$. Item (SP50) gives a condition for the factorization of H_∞ to consist only of the factors $\Omega'_B(H)$, and item (SP80) gives a companion to this condition. The remaining items (SP1)–(SP9), (SP11)–(SP19), (SP21)–(SP29), (SP31)–(SP49), and (SP51)–(SP79), give us details about these factors, such as their Y-degrees, and hence in particular the information as to which of these factors are trivial (i.e., are reduced to 1) and which are not. Out of these items, the most noteworthy are labelled as (SP40), (SP60), and (SP70). Now roughly speaking, $\Omega_B(H)$ collects together those irreducible monic factors of H whose normalized contact with members of $\sigma(B)$ is at least $\lambda(B)$, and out of these only those are kept in $\Omega'_B(H)$ whose normalized contact with members of $\sigma(B)$ is exactly $\lambda(B')$, while the remaining are put in $\Omega^*_B(H)$. A similar description prevails for $\Omega_{(T,B)}(H)$, $\Omega^*_{(T,B)}(H)$, and $\Omega^*_{(B',B)}(H)$. As we shall see in the next two sections, more details about these factorizations can be given when T and H are somehow related.

and hence by (BP5) we see that

$$\begin{cases} \text{for all } B\in R^{\flat} \text{ and } (z,V,W)\in\epsilon(B), \text{ upon letting } H_B=\Omega_B(H),\\ \text{we have that } 0\neq \operatorname{inco}_X H_B(X^V,z(X)+X^WY)\in k[Y]\\ \text{with } \deg_Y\Omega_B(H)=D(B)\deg_Y \operatorname{inco}_X H_B(X^V,z(X)+X^WY)\in D(B)\mathbb{Z}\\ \text{and } \operatorname{inco}_X H(X^V,z(X)+X^WY)=\mu\operatorname{inco}_X H_B(X^V,z(X)+X^WY)\\ \text{where } \mu\in k \text{ is such that: } \mu=0\Leftrightarrow H=0.\end{cases} \tag{SP13}$$

The factorization (SP10) can be refined further. To see this we first put

$$\Omega'_B(H)=\prod_{1\le j\le\chi(H)\text{ with }H_j\in\tau'(B)} H_j \quad \text{for all } B\in\hat{R}^{\flat} \tag{SP14}$$

and we call $\Omega'_B(H)$ the primitive B-slice of H, and we note that then $\Omega'_B(H)\in\hat{R}^{\natural}$, and we recall that

$$\begin{cases}\text{for all } B\in\hat{R}^{\flat} \text{ we have}\\ \tau'(B)=\{f\in\tau(B): \operatorname{noc}(f,f')=\lambda(B) \text{ for all } f'\in\sigma(B)\}.\end{cases} \tag{SP15}$$

Next we put

$$\Omega^*_B(H)=\prod_{1\le j\le\chi(H)\text{ with }H_j\in\tau^*(B)} H_j \quad \text{for all } B\in\hat{R}^{\flat} \tag{SP16}$$

and we call $\Omega^*_B(H)$ the strict B-slice of H, and we note that then $\Omega^*_B(H)\in\hat{R}^{\natural}$, and we recall that

$$\begin{cases}\text{for all } B\in\hat{R}^{\flat} \text{ we have}\\ \tau^*(B)=\tau(B)\setminus\tau'(B)\\ \qquad\quad=\{f\in\tau(B): \operatorname{noc}(f,f')>\lambda(B) \text{ for some } f'\in\sigma(B)\}.\end{cases} \tag{SP17}$$

We also put

$$\Omega^*_{(T,B)}(H)=\prod_{1\le j\le\chi(H)\text{ with }H_j\in\tau^*(T,B)} H_j \quad \text{for all } B\in T\in R^{\sharp} \tag{SP18}$$

and we call $\Omega^*_{(T,B)}(H)$ the strict (T,B)-slice of H, and we note that then $\Omega^*_{(T,B)}(H)\in\hat{R}^{\natural}$, and we recall that

$$\begin{cases}\text{for all } B\in T\in\hat{R}^{\sharp} \text{ we have}\\ \tau^*(T,B)=\tau(T,B)\setminus\tau'(B)\\ \qquad\qquad=\tau^*(B)\setminus\cup_{B'\in\rho(T,B)}\tau(B')=\tau^*(B)\setminus\cup_{B'\in\pi(T,B)}\tau(B').\end{cases} \tag{SP19}$$

Now clearly

$$\Omega_{(T,B)}(H)=\Omega'_B(H)\Omega^*_{(T,B)}(H) \quad \text{for all } B\in T\in R^{\sharp} \tag{SP20}$$

where

$$\begin{cases}\text{for all } B\in R^{\flat} \text{ we have}\\ \deg_Y\Omega'_B(H)=\deg_Y\Omega_B(H)-\deg_Y\Omega^*_B(H).\end{cases} \tag{SP21}$$

and

$$\begin{cases}\text{for all } B\in T\in R^\sharp \text{ we have}\\ \deg_Y\Omega^*_{(T,B)}(H)=\deg_Y\Omega^*_B(H)-\sum\limits_{B'\in\rho(T,B)}\deg_Y\Omega_{B'}(H).\end{cases}\tag{SP22}$$

To describe the above Y-degrees more precisely, given any $z=z(X)\in k((X))$, $0<V\in\mathbb{Z}$, and $W\in\mathbb{Z}$, we define the modified X-initial-coefficient of H relative to $[z,V,W]$, to be denoted by $\mathrm{minco}_X[z,V,W](H)$, by putting

$$\mathrm{minco}_X[z,V,W](H)=\mathrm{inco}_X H(X^V,z(X)+X^WY)$$

and, given any $\hat\sigma\subset R$, we define the strict X-initial-coefficient of $(H,\hat\sigma)$ relative to $[z,V,W]$, to be denoted by $\mathrm{sinco}_X[z,V,W](H,\hat\sigma)$, and the primitive X-initial-coefficient of $(H,\hat\sigma)$ relative to $[z,V,W]$, to be denoted by $\mathrm{pinco}_X[z,V,W](H,\hat\sigma)$, by saying that

$$\begin{cases}\text{if } H=0 \text{ then}\\ \text{we have } \mathrm{sinco}_X[z,V,W](H,\hat\sigma)=1=\mathrm{pinco}_X[z,V,W](H,\hat\sigma)\end{cases}$$

whereas

$$\begin{cases}\text{if } H\neq 0 \text{ then, upon letting}\\ \mathrm{minco}_X[z,V,W](H)=\mu_0\prod\limits_{1\le i\le\nu}(Y-\mu_i) \text{ with } 0\neq\mu_0\in k \text{ and } \mu_i\in k\\ \text{and } \Theta_f(Y)=\mathrm{minco}_X[z,V,W](f) \text{ for all } f\in\hat\sigma\\ \text{and } \sigma^*=\{i\in\{1,\dots,\nu\}:\Theta_f(\mu_i)=0 \text{ for some } f\in\hat\sigma\}\\ \text{and } \sigma'=\{i\in\{1,\dots,\nu\}:\Theta_f(\mu_i)\neq 0 \text{ for all } f\in\hat\sigma\},\\ \text{we have } \mathrm{sinco}_X[z,V,W](H,\hat\sigma)=\prod\limits_{i\in\sigma^*}(Y-\mu_i)\\ \text{and } \mathrm{pinco}_X[z,V,W](H,\hat\sigma)=\prod\limits_{i\in\sigma'}(Y-\mu_i).\end{cases}$$

Recall that

$$\begin{cases}\text{for all } B\in R^\flat \text{ we have } 0\neq A(B)\in k,\\ \text{and for all } B\in R^{\flat*} \text{ we have}\\ E(B,Y)=Y^{D^*(B)/D(B)}-E_0(B) \text{ with } E_0(B)\in k.\end{cases}\tag{SP23}$$

Now by (MBP2) we see that

$$\begin{cases}\text{for any } B\in R^{\flat*} \text{ we have:}\\ E_0(B)=0\Leftrightarrow B\in R^*(t(B),B)\\ \qquad\qquad\Rightarrow D^*(B)=D(B) \text{ and } t^*(B)=t(B)\\ \qquad\qquad\quad \text{and } \lambda(B)\neq c_i(B) \text{ for } 1\le i\le p^*(B)\end{cases}\tag{SP24}$$

and by (MBP1) we see that

$$\begin{cases}\text{for any } B' \text{ and } B'' \text{ in } R^{\flat*} \text{ with } \tau(B')=\tau(B'') \text{ and } \lambda(B')=\lambda(B'')\\ \text{we have: } \tau^*(B')\cap\tau^*(B'')=\emptyset\Leftrightarrow E(B',Y) \text{ and } E(B'',Y) \text{ have no}\\ \text{common root in } k\end{cases}\tag{SP25}$$

and by (SBP4) we see that

$$\begin{cases} \text{for all } B\in R^{\flat *} \text{ with } (z,V,W)\in\epsilon(B) \text{ and } f\in\tau^*(B) \text{ with } \deg_Y f=n \\ \text{we have } \operatorname{minco}_X[z,V,W](f)=A(B)E(B,Y)^{n/D^*(B)}. \end{cases} \tag{SP26}$$

By (SP25) and (SP26) we conclude that

$$\begin{cases} \text{for all } B\in R^{\flat} \text{ and } (z,V,W)\in\epsilon(B) \text{ we have} \\ \text{that } \operatorname{sinco}_X[z,V,W](H,\sigma(B))\in k[Y] \text{ is monic in } Y \\ \text{with } \deg_Y\Omega^*_B(H)=D(B)\ \deg_Y \operatorname{sinco}_X[z,V,W](H,\sigma(B))\in D(B)\mathbb{Z} \\ \text{and } \operatorname{pinco}_X[z,V,W](H,\sigma(B))\in k[Y] \text{ is monic in } Y \\ \text{with } \deg_Y\Omega'_B(H)=D(B)\ \deg_Y \operatorname{pinco}_X[z,V,W](H,\sigma(B))\in D(B)\mathbb{Z}. \end{cases} \tag{SP27}$$

The factorization (SP20) can be refined still further when the finite tree T is strict. To see this we put

$$\Omega^*_{(B',B)}(H)=\prod_{1\le j\le \chi(H) \text{ with } H_j\in\tau^*(B',B)} H_j \quad \text{for all } B'\gg B \text{ in } \hat{R}^{\flat} \tag{SP28}$$

and we call $\Omega^*_{(B',B)}(H)$ the strict (B',B)-slice of H, and we note that then $\Omega^*_{(B',B)}(H)\in\hat{R}^{\natural}$, and we recall that

$$\begin{cases} \text{for all } B'\gg B \text{ in } \hat{R}^{\flat} \text{ we have} \\ \tau^*(B',B)=\tau^*(R^*(B',B))\setminus\tau(B') \text{ where} \\ R^*(B',B)\in\hat{R}^{\flat} \text{ is given by } \sigma(R^*(B',B))=\sigma(B') \text{ and } \lambda(R^*(B',B))=\lambda(B). \end{cases} \tag{SP29}$$

Now clearly

$$\Omega^*_{(T,B)}(H)=\prod_{B'\in\rho(T,B)}\Omega^*_{(B',B)}(H) \quad \text{for all } B\in T\in R^{\sharp *} \tag{SP30}$$

where

$$\begin{cases} \text{for all } B\in T\in\hat{R}^{\sharp} \text{ and } B'\in\rho(T,B) \text{ we have } B'\gg B \text{ in } \hat{R}^{\flat}, \\ \text{and in turn for all } B'\gg B \text{ in } \hat{R}^{\flat} \text{ we have} \\ \deg_Y\Omega^*_{(B',B)}(H)=\deg_Y\Omega^*_{R^*(B',B)}(H)-\deg_Y\Omega_{B'}(H) \end{cases} \tag{SP31}$$

and

$$\begin{cases} \text{for all } B\in T\in R^{\sharp *} \text{ with } \lambda(B)=l_{h(T)} \text{ we have } \rho(T,B)=\emptyset, \\ \text{whereas for all } B\in T\in R^{\sharp *} \text{ with } \lambda(B)=l_i \text{ for some } i<h(T) \text{ we have} \\ \rho(T,B)=\{B'\in T:\lambda(B')=\lambda_{i+1}\} \text{ and } \sigma(B)=\coprod_{B'\in\rho(T,B)}\sigma(B') \\ \text{which is a partition of } \sigma(B) \text{ into pairwise disjoint nonempty subsets.} \end{cases} \tag{SP32}$$

To get more information about $\Omega_B(H)$, first we recall that

$$\begin{cases} \text{for any } f\in R^{\natural} \text{ and } B\in\hat{R}^{\flat}, \\ R^*(f,B) \text{ is the unique member of } R^{\flat *}\cup R^{\flat}_{\infty} \\ \quad \text{with } \sigma(R^*(f,B))=\{f\} \\ \text{such that } \lambda(R^*(f,B))=\min\{\operatorname{noc}(f,f'):f'\in\tau(B)\} \end{cases} \tag{SP33}$$

and we put

$$S(H,B)=\begin{cases}\mathrm{ord}_X H_0(X)+\sum\limits_{1\le j\le\chi(H)} O_j S(R^*(H_j,B)) & \text{in case } B\in R^\flat\\ \mathrm{ord}_X H_0(X)+\sum\limits_{1\le j\le\chi(H)} \mathrm{ord}_X H_j(X,Y) & \text{in case } B\in R^\flat_\infty\end{cases} \tag{SP34}$$

(with the understanding that if $H=0$ then $S(H,B)=\infty$), and we call $S(H,B)$ the B-strength of H, and by (BP4) and (BP5) we see that

$$\begin{cases}\text{for any } B\in R^\flat \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ \mathrm{ord}_X H(X^V, z(X)+X^W Y)=VS(H,B),\\ \text{and for any } B\in R^\flat_\infty \text{ we have}\\ \mathrm{ord}_X H(X,Y)=S(H,B).\end{cases} \tag{SP35}$$

We also put

$$A^{**}(H,B)=\prod_{1\le j\le\chi(H) \text{ with } H_j\in\tau^*(B)} A(R^*(H_j,B)) \quad \text{for all } B\in R^{\flat *} \tag{SP36}$$

and we call $A^{**}(H,B)$ the doubly strict B-constant of H, and we put

$$D^{**}(H,B)=(\deg_Y \Omega^*_B(H))/D^*(B) \quad \text{for all } B\in R^{\flat *} \tag{SP37}$$

and we call $D^{**}(H,B)$ the doubly strict B-degree of H, and we note that, in view of (SBP4),

$$\begin{cases}\text{for any } B\in R^{\flat *} \text{ we have}\\ 0\neq A^{**}(H,B)\in k \text{ and } 0\le D^{**}(H,B)\in\mathbb{Z}\\ \text{with: } D^{**}(H,B)>0 \Leftrightarrow \Omega^*_B(H)\neq 1\end{cases} \tag{SP38}$$

and

$$\begin{cases}\text{for any } B\in R^{\flat *} \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ \mathrm{minco}_X[z,V,W](\Omega^*_B(H))=A^{**}(H,B)E(B,Y)^{D^{**}(H,B)}.\end{cases} \tag{SP39}$$

To collect together information about the Y-degrees of $\Omega_B(H)$, $\Omega'_B(H)$, $\Omega^*_B(H)$, in view of (SP13) and (SP27), we see that

$$\begin{cases}\text{for any } B\in R^\flat \text{ we have}\\ \deg_Y\Omega_B(H)=\deg_Y\Omega'_B(H)+\deg_Y\Omega^*_B(H)\\ \text{and for any } (z,V,W)\in\epsilon(B) \text{ we have}\\ \deg_Y\Omega'_B(H)=D(B)\ \deg_Y \mathrm{pinco}_X[z,V,W](H,\sigma(B))\\ \text{and } \deg_Y\Omega^*_B(H)=D(B)\ \deg_Y \mathrm{sinco}_X[z,V,W](H,\sigma(B))\\ \text{and if } H\neq 0 \text{ then we also have}\\ \deg_Y \mathrm{minco}_X[z,V,W](H)=\deg_Y \mathrm{pinco}_X[z,V,W](H,\sigma(B))\\ \quad +\deg_Y \mathrm{sinco}_X[z,V,W](H,\sigma(B))\\ \text{and } \deg_Y\Omega_B(H)=D(B)\deg_Y \mathrm{minco}_X[z,V,W](H).\end{cases} \tag{SP40}$$

Next we recall that

$$\begin{cases}\text{for any } B\in \hat{R}^\flat \text{ we have}\\ R^*(B)=\{B'\in R^{\flat *}\cup R^\flat_\infty : \lambda(B')=\lambda(B) \text{ and } \sigma(B')=\tau^*(B')\cap\sigma(B)\}\\ \text{and } \sigma(B)=\coprod_{B'\in R^*(B)}\sigma(B')\\ \text{which is a partition of } \sigma(B) \text{ into pairwise disjoint nonempty subsets}\end{cases} \tag{SP41}$$

and we put

$$R^*(H,B)=\{B'\in R^*(B) : \Omega^*_{B'}(H)\neq 1\} \quad \text{for all } B\in \hat{R}^\flat \tag{SP42}$$

where we note that $R^*(H,B)$ is a finite set whose members may be called the strict B-friends of H. We also put

$$D'(B)=\begin{cases}-D(B)+\sum_{B'\in R^*(B)} D^*(B') & \text{for all } B\in R^\flat\\ -1+\sum_{B'\in R^*(B)} 1 & \text{for all } B\in R^\flat_\infty\end{cases} \tag{SP43}$$

(with the understanding that if $R^*(B)$ is an infinite set then $D'(B)=\infty$), and we call $D'(B)$ the primitive degree of B.[13] Moreover we put

$$D''(B)=\begin{cases}-D(B)+\sum_{f\in\sigma(B)} \deg_Y f & \text{for all } B\in R^\flat\\ -1+\sum_{f\in\sigma(B)} \deg_Y f & \text{for all } B\in R^\flat_\infty\end{cases} \tag{SP44}$$

(with the understanding that if $\sigma(B)$ is an infinite set then $D''(B)=\infty$), and we call $D''(B)$ the doubly primitive degree of B,[14] and we note that[15]

$$\begin{cases}\text{if } B\in \hat{R}^\flat \text{ is such that } R^*(H,B)=R^*(B)\\ \text{and } \Omega_B(H) \text{ is devoid of multiple factors in } R,\\ \text{then } \deg_Y\Omega_B(H)=\begin{cases}D(B)+D''(B) & \text{in case } B\in R^\flat\\ 1+D''(B) & \text{in case } B\in R^\flat_\infty.\end{cases}\end{cases} \tag{SP45}$$

Now clearly

$$\begin{cases}\text{for all } B\in \hat{R}^\flat\\ \text{we have } \Omega_B(H)=\Omega'_B(H)\Omega^*_B(H)\\ \text{and } \Omega^*_B(H)=\prod_{B'\in R^*(H,B)}\Omega^*_{B'}(H)\end{cases} \tag{SP46}$$

[13] Observe that if $R^*(B)$ is a finite set then $D'(B)$ is a nonnegative integer; in particular, if $B\in R^\flat_\infty$ then $\operatorname{card}(R^*(B))=1$, where card denotes cardinality, and hence $D'(B)=0$. We shall use $D'(B)$ mainly when $R^*(H,B)=R^*(B)$; in that situation, $R^*(B)$ is obviously a finite set and hence $D'(B)$ is a nonnegative integer. For the use of $D'(B)$ in such a situation, see (SP60) and (SP70).

[14] Observe that if $\sigma(B)$ is a finite set then $D''(B)$ is a nonnegative integer. Also observe that definitions (SP43) and (SP44) would look more natural if for every $B\in R^\flat_\infty$ we put $D(B)=D^*(B)=1$.

[15] An element Φ in R is devoid of multiple factors in R means that $\Phi\neq 0$ and the ideal ΦR is its own radical in R.

and, in view of (SP23) to (SP26),

$$\begin{cases}\text{for all } B' \in R^*(B) \text{ with } B \in R^\flat \text{ we know that}\\ E(B',Y) \in k[Y] \text{ is monic in } Y \text{ having no multiple root in } k\\ \text{and } \deg_Y E(B',Y) = D^*(B')/D(B') = D^*(B')/D(B) > 0\end{cases} \tag{SP47}$$

and

$$\begin{cases}\text{for all } B' \neq B'' \text{ in } R^*(B) \text{ with } B \in R^\flat \text{ we know that}\\ E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k\end{cases} \tag{SP48}$$

and

$$\begin{cases}\text{for all } B \in R^\flat \text{ with } (z,V,W) \in \epsilon(B) \text{ we know that}\\ \text{minco}_X[z,V,W](\Omega'_B(H)) \text{ is a nonzero member of } k[Y]\\ \text{which has no common root with } E(B',Y) \text{ in } k \text{ for any } B' \in R^*(B).\end{cases} \tag{SP49}$$

By (SP10) and (SP20) we see that

$$\begin{cases}\text{for any } T \in R^\sharp \text{ we have:}\\ H_\infty = \prod_{R_\infty(T) \neq B \in T} \Omega'_B(H) \Leftrightarrow \deg_Y H_\infty = \sum_{R_\infty(T) \neq B \in T} \deg_Y \Omega'_B(H).\end{cases} \tag{SP50}$$

In the next section we shall show that (SP50) is applicable when $H = F_Y$ where $F \in R$ is devoid of multiple factors. In the section after that we shall apply (SP30) to the case when H is the jacobian $J(F,G)$ of F and G in R. To prepare for all this, until further notice, given any $B \in R^\flat$ with $(z,V,W) \in \epsilon(B)$, for every $\Phi = \Phi(X,Y) \in R$ let us put

$$\begin{cases}\tilde\Phi = \tilde\Phi(X,Y) = \Phi(X^V, z(X) + X^W Y) \quad \text{and}\\ I(\Phi) = \text{minco}_X[z,V,W](\Phi) = \text{inco}_X \tilde\Phi.\end{cases} \tag{SP51}$$

Then by (SP13) we see that

$$\begin{cases}\deg_Y \Omega_B(H) = D(B)\deg_Y I(\Omega_B(H))\\ \text{and } I(H) = \mu I(\Omega_B(H))\\ \text{where } \mu \in k \text{ is such that} : \mu = 0 \Leftrightarrow H = 0\end{cases} \tag{SP52}$$

and by the chain rule for partial derivatives we see that

$$\begin{cases}\text{if } I(H) \notin k\\ \text{then } I(H_Y) = (I(H))_Y.\end{cases} \tag{SP53}$$

Also clearly

$$\begin{cases}\text{for any } \Psi(Y) \in k[Y] \setminus k \text{ we have}\\ 0 \neq \Psi_Y(Y) \in k[Y] \text{ with } \deg_Y \Psi_Y(Y) = -1 + \deg_Y \Psi(Y)\end{cases} \tag{SP54}$$

and

$$\begin{cases}\text{for any } \Psi(Y) \in k[Y] \text{ and } \mu \in k \text{ and } 0 < \nu \in \mathbb{Z} \text{ we have:}\\ \Psi(Y) = (Y-\mu)^\nu \Psi'(Y) \text{ where } \Psi'(Y) \in k[Y] \text{ with } \Psi'(\mu) \neq 0\\ \Rightarrow \Psi_Y(Y) = (Y-\mu)^{\nu-1}\Psi''(Y) \text{ where } \Psi''(Y) \in k[Y] \text{ with } \Psi''(\mu) \neq 0.\end{cases} \tag{SP55}$$

By (SP46) we see that

$$I(\Omega_B(H)) = I(\Omega'_B(H)) \prod_{B' \in R^*(H,B)} I(\Omega^*_{B'}(H)). \tag{SP56}$$

Moreover, by (SP38), (SP39) and (SP47),

$$\begin{cases} \text{for all } B' \in R^*(H,B) \text{ we know that} \\ I(\Omega^*_{B'}(H)) = A^{**}(H,B')E(B',Y)^{D^{**}(H,B')} \\ \text{where } 0 \neq A^{**}(H,B') \in k, 0 < D^{**}(H,B') \in \mathbb{Z}, \\ E(B',Y) \in k[Y] \text{ is monic in } Y \text{ having no multiple roots in } k, \\ \text{and } \deg_Y E(B',Y) = D^*(B')/D(B) > 0. \end{cases} \tag{SP57}$$

Likewise, by (SP48) and (SP49),

$$\begin{cases} \text{for all } B' \neq B'' \text{ in } R^*(H,B) \text{ we know that} \\ E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k, \\ \text{and we also know that } I(\Omega'_B(H)) \text{ is a nonzero member of } k[Y] \\ \text{which has no common root with } \displaystyle\prod_{B' \in R^*(H,B)} E(B',Y) \text{ in } k. \end{cases} \tag{SP58}$$

By (SP51) to (SP58), we conclude that

$$\begin{cases} \text{if } \Omega_B(H) \neq 1 \\ \text{then } \deg_Y I(\Omega_B(H_Y)) = -1 + \deg_Y I(\Omega_B(H)) \\ \text{and } I(\Omega_B(H_Y)) = L(Y) \displaystyle\prod_{B' \in R^*(H,B)} E(B',Y)^{D^{**}(H,B')-1} \\ \text{where } 0 \neq L(Y) \in k[Y] \text{ has no common root with} \\ \quad \displaystyle\prod_{B' \in R^*(H,B)} E(B',Y) \text{ in } k \\ \text{and } \deg_Y L(Y) = -1 + \deg_Y I(\Omega'_B(H)) + \displaystyle\sum_{B' \in R^*(H,B)} \deg_Y E(B',Y) \\ \text{with } \deg_Y E(B',Y) = D^*(B')/D(B) \text{ for all } B' \in R^*(H,B). \end{cases} \tag{SP59}$$

Now if $R^*(H,B) = R^*(B)$ then clearly $\Omega_B(H) \neq 1 = \Omega'_B(H)$ and hence in particular $\deg_Y I(\Omega'_B(H)) = 0$ and therefore in the situation of (SP59) we get

$$\deg_Y L(Y) = -1 + \sum_{B' \in R^*(B)} [D^*(B')/D(B)] = D'(B)/D(B)$$

and in view of (SP26) we have $\text{pinco}_X[z,V,W](H_Y, \sigma(B)) = \mu L(Y)$ with $0 \neq \mu \in k$, and hence

$$\deg_Y \text{pinco}_X[z,V,W](H_Y, \sigma(B)) = \deg_Y L(Y) = D'(B)/D(B).$$

Thus

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \Omega_B(H) \neq 1 \\ \text{and } \deg_Y \text{pinco}_X[z,V,W](H_Y, \sigma(B)) = D'(B)/D(B). \end{cases} \tag{SP60}$$

In view of (SP40) and (SP51), by (SP59) we see that

$$\begin{cases} \text{if } \Omega_B(H) \neq 1 \\ \text{then } \deg_Y \operatorname{minco}_X[z,V,W](H_Y) = -1 + \deg_Y \operatorname{minco}_X[z,V,W](H) \\ \text{and } \deg_Y \Omega_B(H_Y) = -D(B) + \deg_Y \Omega_B(H). \end{cases} \quad \text{(SP61)}$$

In view of (SP51), by (SP56) to (SP59) we also see that

$$D^{**}(H_Y, B') = -1 + D^{**}(H, B') \quad \text{for all } B' \in R^*(H,B). \quad \text{(SP62)}$$

By (SP62) it follows that

$$\begin{cases} \text{if } B \in R^{\flat *} \text{ and } D^{**}(H,B) > 0 \\ \text{then } D^{**}(H_Y, B) = -1 + D^{**}(H,B). \end{cases} \quad \text{(SP63)}$$

In view of (SP40), by (SP60) and (SP61) we see that

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \deg_Y \operatorname{sinco}_X[z,V,W](H_Y, \sigma(B)) = -1 \\ \qquad - [D'(B)/D(B)] + \deg_Y \operatorname{minco}_X[z,V,W](H). \end{cases} \quad \text{(SP64)}$$

In view of (SP40), by (SP60) and (SP64) we see that

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \deg_Y \Omega_B(H_Y) = -D(B) + \deg_Y \Omega_B(H) \\ \text{and } \deg_Y \Omega'_B(H_Y) = D'(B) \\ \text{and } \deg_Y \Omega^*_B(H_Y) = -D(B) - D'(B) + \deg_Y \Omega_B(H). \end{cases} \quad \text{(SP65)}$$

Turning to the jacobian, upon letting

$$\Delta = J(H,G) \quad \text{and} \quad \overline{\Delta} = J(\tilde{H}, \tilde{G}) \quad \text{with } G \in R$$

by the chain rule for jacobians we get

$$\overline{\Delta}(X,Y) = VX^{V+W-1} \tilde{\Delta}(X,Y)$$

and now, assuming that $\Omega_B(H) \neq 1 = \Omega_B(G)$ and $G \neq 0 \neq S(G,B)$, we have

$$\tilde{H}(X,Y) = I(H) X^{VS(H,B)} + (\text{terms of degree} > VS(H,B) \text{ in } X)$$

with $I(H) \in k[Y] \setminus k$, and

$$\tilde{G}(X,Y) = I(G) X^{VS(G,B)} + (\text{terms of degree} > VS(G,B) \text{ in } X)$$

with $0 \neq I(G) \in k$ and $VS(G,B) \neq 0$, and hence we get

$$\begin{aligned} \overline{\Delta}(X,Y) = & - VS(G,B) I(G) (I(H))_Y X^{VS(H,B)+VS(G,B)-1} \\ & + (\text{terms of degree} \geq [VS(H,B) + VS(G,B)] \text{ in } X) \end{aligned}$$

and therefore by (SP52) we have

$$I(J(H,G)) = \mu I(H_Y) \quad \text{with } 0 \neq \mu \in k$$

and hence by (SP51) we get

$$\operatorname{minco}_X[z,V,W](J(H,G)) = \mu \operatorname{minco}_X[z,V,W](H_Y) \quad \text{with } 0 \neq \mu \in k.$$

Thus

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1 \text{ and } S(G,B)\neq 0\\ \text{then } \mathrm{minco}_X[z,V,W](J(H,G))=\mu\ \mathrm{minco}_X[z,V,W](H_Y) \text{ with } 0\neq\mu\in k\\ \text{and hence } \mathrm{pinco}_X[z,V,W](J(H,G),\sigma(B))=\mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \mathrm{sinco}_X[z,V,W](J(H,G),\sigma(B))=\mathrm{sinco}_X[z,V,W](H_Y,\sigma(B)).\end{cases}\tag{SP66}$$

In view of (SP40), by (SP66) we see that

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1 \text{ and } S(G,B)\neq 0\\ \text{then } \deg_Y \mathrm{minco}_X[z,V,W](J(H,G))=\deg_Y \mathrm{minco}_X[z,V,W](H_Y)\\ \text{and } \deg_Y \mathrm{pinco}_X[z,V,W](J(H,G),\sigma(B))=\deg_Y \mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \deg_Y \mathrm{sinco}_X[z,V,W](J(H,G),\sigma(B))\\ \quad=\deg_Y \mathrm{sinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \deg_Y\Omega_B(J(H,G))=\deg_Y\Omega_B(H_Y)\\ \text{and } \deg_Y\Omega'_B(J(H,G))=\deg_Y\Omega'_B(H_Y)\\ \text{and } \deg_Y\Omega^*_B(J(H,G))=\deg_Y\Omega^*_B(H_Y).\end{cases}\tag{SP67}$$

To consider another similar case, just for a moment let $j(H,G)$ stand for $(HG)_Y$; then upon letting

$$\delta=j(H,G)\quad\text{and}\quad\bar{\delta}=j(\tilde{H},\tilde{G})\quad\text{with } G\in R$$

by the product rule for derivatives we get

$$\bar{\delta}(X,Y)=X^W\tilde{\delta}(X,Y)$$

and now, assuming that $\Omega_B(H)\neq 1=\Omega_B(G)$ and $G\neq 0$, we have

$$\tilde{H}(X,Y)=I(H)X^{VS(H,B)}+(\text{terms of degree} > VS(H,B) \text{ in } X)$$

with $I(H)\in k[Y]\setminus k$, and

$$\tilde{G}(X,Y)=I(G)X^{VS(G,B)}+(\text{terms of degree} > VS(G,B) \text{ in } X)$$

with $0\neq I(G)\in k$, and hence we get

$$\begin{aligned}\bar{\delta}(X,Y)&=I(G)(I(H))_Y X^{VS(H,B)+VS(G,B)}\\ &\qquad+(\text{terms of degree} > [VS(H,B)+VS(G,B)] \text{ in } X)\end{aligned}$$

and therefore by (SP52) we have

$$I((HG)_Y)=\mu I(H_Y)\quad\text{with } 0\neq\mu\in k$$

and hence by (SP51) we get

$$\mathrm{minco}_X[z,V,W]((HG)_Y)=\mu\ \mathrm{minco}_X[z,V,W](H_Y)\quad\text{with } 0\neq\mu\in k.$$

Thus

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1\\ \text{then } \mathrm{minco}_X[z,V,W]((HG)_Y)=\mu\ \mathrm{minco}_X[z,V,W](H_Y) \text{ with } 0\neq\mu\in k\\ \text{and hence } \mathrm{pinco}_X[z,V,W]((HG)_Y,\sigma(B))\\ \quad=\mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \mathrm{sinco}_X[z,V,W]((HG)_Y,\sigma(B))=\mathrm{sinco}_X[z,V,W](H_Y,\sigma(B)).\end{cases}\tag{SP68}$$

In view of (SP40), by (SP68) we see that

$$\begin{cases} \text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1 \\ \text{then } \deg_Y \operatorname{minco}_X[z,V,W]((HG)_Y)=\deg_Y \operatorname{minco}_X[z,V,W](H_Y) \\ \text{and } \deg_Y \operatorname{pinco}_X[z,V,W]((HG)_Y,\sigma(B))=\deg_Y \operatorname{pinco}_X[z,V,W](H_Y,\sigma(B)) \\ \text{and } \deg_Y \operatorname{sinco}_X[z,V,W]((HG)_Y,\sigma(B)) \\ \quad =\deg_Y \operatorname{sinco}_X[z,V,W](H_Y,\sigma(B)) \\ \text{and } \deg_Y \Omega_B((HG)_Y)=\deg_Y \Omega_B(H_Y) \\ \text{and } \deg_Y \Omega'_B((HG)_Y)=\deg_Y \Omega'_B(H_Y) \\ \text{and } \deg_Y \Omega^*_B((HG)_Y)=\deg_Y \Omega^*_B(H_Y). \end{cases} \tag{SP69}$$

Abandoning notation (SP51), let us summarize results (SP60) to (SP69) as lemmas (SP70) to (SP75) stated below.

Lemma (SP70). *If* $B\in R^\flat$ *and* $F\in R$ *are such that* $R^*(F,B)=R^*(B)$, *then* $\Omega_B(F)\neq 1$ *and we have*

$$\begin{cases} \deg_Y \Omega_B(F_Y)=-D(B)+\deg_Y \Omega_B(F) \\ \textit{and } \deg_Y \Omega'_B(F_Y)=D'(B) \\ \textit{and } \deg_Y \Omega^*_B(F_Y)=-D(B)-D'(B)+\deg_Y \Omega_B(F) \end{cases}$$

and for every $(z,V,W)\in\epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \operatorname{minco}_X[z,V,W](F_Y)=-1+\deg_Y \operatorname{minco}_X[z,V,W](F) \\ \textit{and } \deg_Y \operatorname{pinco}_X[z,V,W](F_Y,\sigma(B))=D'(B)/D(B) \\ \textit{and } \deg_Y \operatorname{sinco}_X[z,V,W](F_Y,\sigma(B)) \\ \quad =-1-[D'(B)/D(B)]+\deg_Y \operatorname{minco}_X[z,V,W](F). \end{cases}$$

Lemma (SP71). *If* $B\in R^\flat$ *and* $F\in R$ *are such that* $\Omega_B(F)\neq 1$, *then we have*

$$\deg_Y \Omega_B(F_Y)=-D(B)+\deg_Y \Omega_B(F)$$

and for every $(z,V,W)\in\epsilon(B)$ *we have*

$$\deg_Y \operatorname{minco}_X[z,V,W](F_Y)=-1+\deg_Y \operatorname{minco}_X[z,V,W](F).$$

Lemma (SP72). *Given any* $B\in R^\flat$ *and* $F\in R$, *for every* $B'\in R^*(F,B)$ *we have*

$$D^{**}(F_Y,B)=-1+D^{**}(F,B).$$

Lemma (SP73). *If* $B\in R^{\flat *}$ *and* $F\in R$ *are such that* $D^{**}(F,B)>0$, *then we have*

$$D^{**}(F_Y,B)=-1+D^{**}(F,B).$$

Lemma (SP74). *If* $B\in R^\flat$ *and* $F\in R$ *are such that* $\Omega_B(F)\neq 1$, *and* $0\neq G\in R$ *is such that* $\Omega_B(G)=1$ *and* $S(G,B)\neq 0$, *then we have*

$$\begin{cases} \deg_Y \Omega_B(J(F,G))=\deg_Y \Omega_B(F_Y) \\ \textit{and } \deg_Y \Omega'_B(J(F,G))=\deg_Y \Omega'_B(F_Y) \\ \textit{and } \deg_Y \Omega^*_B(J(F,G))=\deg_Y \Omega^*_B(F_Y) \end{cases}$$

and for every $(z, V, W) \in \epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \operatorname{minco}_X[z, V, W](J(F, G), \sigma(B)) = \deg_Y \operatorname{minco}_X[z, V, W](F_Y, \sigma(B)) \\ \textit{and } \deg_Y \operatorname{pinco}_X[z, V, W](J(F, G), \sigma(B)) = \deg_Y \operatorname{pinco}_X[z, V, W](F_Y, \sigma(B)) \\ \textit{and } \deg_Y \operatorname{sinco}_X[z, V, W](J(F, G), \sigma(B)) = \deg_Y \operatorname{sinco}_X[z, V, W](F_Y, \sigma(B)) \end{cases}$$

and actually we have

$$\begin{cases} \operatorname{minco}_X[z, V, W](J(F, G)) = \mu \operatorname{minco}_X[z, V, W](F_Y) \textit{ with } 0 \neq \mu \in k, \\ \textit{and } \operatorname{pinco}_X[z, V, W](J(F, G), \sigma(B)) = \operatorname{pinco}_X[z, V, W](F_Y, \sigma(B)), \\ \textit{and } \operatorname{sinco}_X[z, V, W](J(F, G), \sigma(B)) = \operatorname{sinco}_X[z, V, W](F_Y, \sigma(B)). \end{cases}$$

Lemma (SP75). *If* $B \in R^\flat$ *and* $F \in R$ *are such that* $\Omega_B(F) \neq 1$, *and* $0 \neq G \in R$ *is such that* $\Omega_B(G) = 1$, *then we have*

$$\begin{cases} \deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) \\ \textit{and } \deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) \\ \textit{and } \deg_Y \Omega^*_B((FG)_Y) = \deg_Y \Omega^*_B(F_Y) \end{cases}$$

and for every $(z, V, W) \in \epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \operatorname{minco}_X[z, V, W]((FG)_Y) = \deg_Y \operatorname{minco}_X[z, V, W](F_Y) \\ \textit{and } \deg_Y \operatorname{pinco}_X[z, V, W]((FG)_Y, \sigma(B)) = \deg_Y \operatorname{pinco}_X[z, V, W](F_Y, \sigma(B)) \\ \textit{and } \deg_Y \operatorname{sinco}_X[z, V, W]((FG)_Y, \sigma(B)) = \deg_Y \operatorname{sinco}_X[z, V, W](F_Y, \sigma(B)) \end{cases}$$

and actually we have

$$\begin{cases} \operatorname{minco}_X[z, V, W]((FG)_Y) = \mu \operatorname{minco}_X[z, V, W](F_Y) \textit{ with } 0 \neq \mu \in k, \\ \textit{and } \operatorname{pinco}_X[z, V, W]((FG)_Y, \sigma(B)) = \operatorname{pinco}_X[z, V, W](F_Y, \sigma(B)), \\ \textit{and } \operatorname{sinco}_X[z, V, W]((FG)_Y, \sigma(B)) = \operatorname{sinco}_X[z, V, W](F_Y, \sigma(B)). \end{cases}$$

Now, as a consequence of (SP45) and (SP70) we shall prove the following lemma:

Lemma (SP76). *If* $B \in \hat{R}^\flat$ *and* $F \in R$ *are such that* $R^*(F, B) = R^*(B)$ *and* F *is devoid of multiple factors in* R, *then* $\Omega_B(F) \neq 1$ *and* $\deg_Y \Omega_B(F_Y) = D''(B)$ *and* $\deg_Y \Omega'_B(F_Y) = D'(B)$.

Namely, if $B \in R^\flat$ and $F \in R$ are such that $R^*(F, B) = R^*(B)$ then by (SP70) we get $\Omega_B(F) \neq 1$ and $\deg_Y \Omega_B(F_Y) = -D(B) + \deg_Y \Omega_B(F)$ and $\deg_Y \Omega'_B(F_Y) = D'(B)$; if F is also devoid of multiple factors in R, then by (SP45) we know that $\deg_Y \Omega_B(F) = D(B) + D''(B)$ and hence we get $\deg_Y \Omega_B(F_Y) = D''(B)$. Likewise, if $B \in R^\flat_\infty$ and $F \in R$ are such that $\Omega_B(F) \neq 1$ then clearly $\deg_Y \Omega_B(F_Y) = -1 + \deg_Y \Omega_B(F)$; if F is also devoid of multiple factors in R, then by (SP45) we know that $\deg_Y \Omega_B(F) = 1 + D''(B)$ and hence we get $\deg_Y \Omega_B(F_Y) = D''(B)$. This completes the proof of (SP76).[16]

Next, as a consequence of (SP74) and (SP75) we shall prove the following lemma:

Lemma (SP77). *Given any* $F \in R \setminus k((X))$ *and* $0 \neq G \in R$, *upon letting* $T = T(FG)$, *we have the following.*

[16] We may tacitly use the obvious facts that: (1) if $B \in \hat{R}^\flat$ and $F \in R$ are such that $R^*(F, B) = R^*(B)$ then $\Omega_B(F) \neq 1$; (2) for every $B \in R^\flat_\infty$ we have $D'(B) = 0$; (3) for every $B \in R^\flat_\infty$ and every $G \in R$ we have $\Omega_B(G) = 1$ and hence $\deg_Y \Omega'_B(G) = 0$.

(SP77.1) *If* $B\in T$ *is such that* $\Omega_B(G)=1$ *and* $S(G,B)\neq 0$ *then*

$$\Omega_B(F)\neq 1$$

and

$$\deg_Y\Omega_B(J(F,G))=\deg_Y\Omega_B(F_Y) \quad \text{and} \quad \deg_Y\Omega'_B(J(F,G))=\deg_Y\Omega'_B(F_Y)$$

and for every $B'\in\pi(T,B)$ *we have*

$$\Omega_{B'}(F)\neq 1=\Omega_{B'}(G) \quad \text{and} \quad S(G,B')\neq 0$$

and

$$\deg_Y\Omega_{B'}(J(F,G))=\deg_Y\Omega_{B'}(F_Y) \quad \text{and} \quad \deg_Y\Omega'_{B'}(J(F,G))=\deg_Y\Omega'_{B'}(F_Y).$$

(SP77.2) *If* $B\in T$ *is such that* $\Omega_B(G)=1$ *then*

$$\Omega_B(F)\neq 1$$

and

$$\deg_Y\Omega_B((FG)_Y)=\deg_Y\Omega_B(F_Y) \quad \text{and} \quad \deg_Y\Omega'_B((FG)_Y)=\deg_Y\Omega'_B(F_Y)$$

and for every $B'\in\pi(T,B)$ *we have*

$$\Omega_{B'}(F)\neq 1=\Omega_{B'}(G)$$

and

$$\deg_Y\Omega_{B'}((FG)_Y)=\deg_Y\Omega_{B'}(F_Y) \quad \text{and} \quad \deg_Y\Omega'_{B'}((FG)_Y)=\deg_Y\Omega'_{B'}(F_Y).$$

Namely, if $B\in T$ is such that $\Omega_B(G)=1$ then obviously $\Omega_B(F)\neq 1$ and for every $B'\in\pi(T,B)$ we have $\Omega_{B'}(F)\neq 1=\Omega_{B'}$, and by (BP5) we also see that if $B\in T$ is such that $\Omega_B(G)=1$ then for every $B'\in\pi(T,B)$ we have $S(G,B')=S(G,B)$. Therefore, in case of $B\in T\setminus\{R_\infty(T)\}$, our assertions follow from (SP74) and (SP75). Moreover, if $B=R_\infty(T)$ and $\Omega_B(G)=1$ and $S(G,B)\neq 0$, then by the equation $J(F,G)=F_XG_Y-F_YG_X$ we see that $J(F,G)=-F_YG_X$ with $0\neq G_X\in k((X))$ and hence $\deg_Y\Omega_B(J(F,G))=\deg_Y\Omega_B(F_Y)$ and $\deg_Y\Omega'_B(J(F,G))=0=\deg_Y\Omega'_B(F_Y)$. Likewise, if $B=R_\infty(T)$ and $\Omega_B(G)=1$, then by the equation $(FG)_Y=FG_Y+F_YG$ we see that $(FG)_Y=F_YG$ with $0\neq G\in k((X))$ and hence $\deg_Y\Omega_B((FG)_Y)=\deg_Y\Omega_B(F_Y)$ and $\deg_Y\Omega'_B((FG)_Y)=0=\deg_Y\Omega'_B(F_Y)$. This completes the proof of (SP77).

Now we shall prove the following lemma:

Lemma (SP78). *For any* $\Phi\in R$, *upon letting* $T=T(\Phi)$, *we have the following.*

(SP78.1) *Given any* $B\in T$ *with* $\pi(T,B)\neq\emptyset$, *for every* $B'\in\rho(T,B)$ *there is a unique* $\alpha(B')\in R^*(B)$ *with* $\sigma(\alpha(B'))=\sigma(B')$, *and* $B'\mapsto\alpha(B')$ *gives a bijection of* $\rho(T,B)$ *onto* $R^*(B)$. *Moreover, for every* $B'\in\rho(T,B)$ *we have*

$$D(B')=\begin{cases} D^*(\alpha(B')) & \text{in case } B\in R^\flat \\ 1 & \text{in case } B\in R^\flat_\infty \end{cases}$$

and hence we have

$$D'(B)=\begin{cases} -D(B)+\sum\limits_{B'\in\rho(T,B)} D(B') & \text{in case } B\in R^\flat \\ -1+\sum\limits_{B'\in\rho(T,B)} D(B') & \text{in case } B\in R^\flat_\infty \end{cases}$$

where we note that if $B\in R^\flat_\infty$ *then* $\operatorname{card}(\rho(T,B))=\operatorname{card}(R^*(B))=1$.

(SP78.2) *Given any* $B \in T$ *with* $\pi(T,B)=\emptyset$, *for every* $f \in \sigma(B)$ *there is a unique* $\beta(f) \in R^*(B)$ *with* $f \in \sigma(\beta(f))$, *and* $f \mapsto \beta(f)$ *gives a bijection of* $\sigma(B)$ *onto* $R^*(B)$. *Moreover, for every* $f \in \sigma(B)$ *we have*

$$\deg_Y f = \begin{cases} D^*(\beta(f)) & \text{in case } B \in R^\flat \\ 1 & \text{in case } B \in R^\flat_\infty \end{cases}$$

and hence we have

$$D''(B) = D'(B).$$

(SP78.3) *For every* $B \in T$ *we have*

$$D''(B) = D'(B) + \sum_{B' \in \pi(T,B)} D'(B').$$

Namely, the proofs of (SP78.1) and (SP78.2) are straightforward. We shall prove (SP78.3) by induction on $\operatorname{card}(\pi(T,B))$. In case of $\operatorname{card}(\pi(T,B)) = 0$ our assertion follows from (SP78.2). So let $\operatorname{card}(\pi(T,B)) > 0$ and assume true for all smaller values of $\operatorname{card}(\pi(T,B))$. Then, by the induction hypothesis, for every $B' \in \rho(T,B)$ we have

$$D''(B') = D'(B') + \sum_{B'' \in \pi(T,B')} D'(B'')$$

and hence by the definition of $D''(B')$ we get

$$D'(B') + \sum_{B'' \in \pi(T,B')} D'(B'') = -D(B') + \sum_{f \in \sigma(B')} \deg_Y f.$$

Summing both sides of the above equation as B' varies over $\rho(T,B)$, we get

$$\sum_{B' \in \pi(T,B)} D'(B') = -\left[\sum_{B' \in \rho(T,B)} D(B')\right] + \left[\sum_{f \in \sigma(B)} \deg_Y f\right]$$

and by (SP78.1) we have

$$D'(B) = \begin{cases} -D(B) + \sum_{B' \in \rho(T,B)} D(B') & \text{in case } B \in R^\flat \\ -1 + \sum_{B' \in \rho(T,B)} D(B') & \text{in case } B \in R^\flat_\infty. \end{cases}$$

By adding the above two equations we get

$$D'(B) + \sum_{B' \in \pi(T,B)} D'(B') = \begin{cases} -D(B) + \sum_{f \in \sigma(B)} \deg_Y f & \text{in case } B \in R^\flat \\ -1 + \sum_{f \in \sigma(B)} \deg_Y f & \text{in case } B \in R^\flat_\infty \end{cases}$$

and hence by the definition of $D''(B)$ we conclude that

$$D''(B) = D'(B) + \sum_{B' \in \pi(T,B)} D'(B').$$

This completes the proof of (SP78.3).

As an immediate consequence of (SP78.3) we get the following lemma:

Lemma (SP79). *Let* $B\in T=T(\Phi)$ *with* $\Phi\in R$ *be such that:* $\deg_Y\Omega_B(H)=D''(B)$, $\deg_Y\Omega'_B(H)=D'(B)$, *and* $\deg_Y\Omega'_{B'}(H)=D'(B')$ *for all* $B'\in\pi(T,B)$. *Then*

$$\deg_Y\Omega_B(H)=\deg_Y\Omega'_B(H)+\sum_{B'\in\pi(T,B)}\deg_Y\Omega'_{B'}(H).$$

For any $B\in T=T(\Phi)$ with $\Phi\in R$, it is clear that $\Omega'_B(H)\prod_{B'\in\pi(T,B)}\Omega'_{B'}(H)$ divides $\Omega_B(H)$ in R and hence, as a companion to (SP50), and as a principle applicable in the situation of (SP79), we get the following lemma:

Thus Lemma (SP80). *For any* $B\in T=T(\Phi)$ *with* $\Phi\in R$ *we have:*

$$\begin{cases}\Omega_B(H)=\Omega'_B(H)\displaystyle\prod_{B'\in\pi(T,B)}\Omega'_{B'}(H)\\ \Leftrightarrow \deg_Y\Omega_B(H)=\deg_Y\Omega'_B(H)+\displaystyle\sum_{B'\in\pi(T,B)}\deg_Y\Omega'_{B'}(H).\end{cases}$$

8. Factorization of the derivative

If $T=T(F)$ where $F\in R\setminus k((X))$, then for every $B\in T$ we clearly have $R^*(F,B)=R^*(B)$. Therefore by (SP76), (SP79) and (SP80) we get the following derivative factorization theorem.

Theorem (DF1). *Let* $T=T(F)$ *where* $F\in R\setminus k((X))$ *is devoid of multiple factors in R. Then we have the following.*

(DF1.1) *For any* $B\in T$ *we have*

$$\deg_Y\Omega_B(F_Y)=D''(B)\quad\textit{and}\quad\deg_Y\Omega'_B(F_Y)=D'(B)$$

and

$$\Omega_B(F_Y)=\Omega'_B(F_Y)\prod_{B'\in\pi(T,B)}\Omega'_{B'}(F_Y).$$

where for every $B'\in\pi(T,B)$ *we have*

$$\deg_Y\Omega_{B'}(F_Y)=D''(B')\quad\textit{and}\quad\deg_Y\Omega'_{B'}(F_Y)=D'(B').$$

(DF1.2) *By taking* $B=R_\infty(T)$ *in* (DF1.1), *for the monic part* $(F_Y)_\infty$ *of* F_Y *we get*

$$(F_Y)_\infty=\prod_{B\in T\setminus\{R_\infty(T)\}}\Omega'_B(F_Y).$$

Remark (DF2). In the factorization (DF1.2), the factor $\Omega'_B(F_Y)$ really occurs, i.e., its Y-degree $D'(B)$ is nonzero, if and only if either: (*) $\mathrm{card}(R^*(B))=1$ and for the unique $B'\in R^*(B)$ we have $D^*(B')>D(B)$, or: (**) $\mathrm{card}(R^*(B))>1$. Note that in the irreducible case, i.e., when $F=f\in R^\natural$, (*) is always satisfied. Moreover, in the nontrivial irreducible case, i.e., when $F=f\in R^\natural$ with $\deg_Y f=n>1$, let us put

$$\hat{h}=\begin{cases}h(c(f)) & \text{if } c_1(f)\notin\mathbb{Z}\\ h(c(f))-1 & \text{if } c_1(f)\in\mathbb{Z}\end{cases}$$

and for $1 \le i \le \hat{h}$ let us put

$$\hat{c}_i = \begin{cases} c_i(f) & \text{if } c_1(f) \notin \mathbb{Z} \\ c_{i+1}(f) & \text{if } c_1(f) \in \mathbb{Z} \end{cases}$$

and

$$\hat{r}_i = \begin{cases} r_i(q(m(f))) & \text{if } r_1(f) \notin \mathbb{Z} \\ r_{i+1}(q(m(f))) & \text{if } r_1(f) \in \mathbb{Z} \end{cases}$$

and for $1 \le i \le \hat{h}+1$ let us put

$$\hat{d}_i = \begin{cases} d_i(m(f)) & \text{if } c_1(f) \notin \mathbb{Z} \\ d_{i+1}(m(f)) & \text{if } c_1(f) \in \mathbb{Z}. \end{cases}$$

Then $\hat{h}$ is a positive integer, $\hat{c}_1 < \hat{c}_2 < \cdots < \hat{c}_{\hat{h}}$ are in $\mathbb{Q} \setminus \mathbb{Z}$, and $n = \hat{d}_1 > \hat{d}_2 > \cdots > \hat{d}_{\hat{h}+1} = 1$ are integers with $\hat{d}_i \equiv 0 \,(\mathrm{mod}\, \hat{d}_{i+1})$ for $1 \le i \le \hat{h}$. Let $B_0 = (\sigma(B_0), \lambda(B_0)) \in R^{\flat}_{\infty}$ with $\sigma(B_0) = \{f\}$ and $\lambda(B_0) = -\infty$. For $1 \le i \le \hat{h}$ let $B_i = (\sigma(B_i), \lambda(B_i)) \in R^{\flat}$ with $\sigma(B_i) = \{f\}$ and $\lambda(B_i) = \hat{c}_i$. Then $T = T(f) = \{B_0, B_1, \ldots, B_{\hat{h}}\}$ with $R_\infty(T) = B_0 < B_1 < \cdots < B_{\hat{h}}$, and for $1 \le i \le \hat{h}$ we have

$$S(B_i) = (\hat{d}_i \hat{r}_i)/n^2 \quad \text{and} \quad D(B_i) = n/\hat{d}_i$$

and

$$D^*(B_i') = n/\hat{d}_{i+1} \quad \text{where} \quad \{B_i'\} = R^*(B_i).$$

Now

$$(f_Y)_\infty = (1/n) f_Y = \prod_{1 \le i \le \hat{h}} \Omega'_{B_i}(f_Y)$$

and for $1 \le i \le \hat{h}$ we have

$$\deg_Y \Omega'_{B_i}(f_Y) = D'(B_i) = -D(B_i) + D^*(B_i') = (n/\hat{d}_{i+1}) - (n/\hat{d}_i) > 0.$$

Let us factor f_Y into irreducible factors by writing

$$f_Y = n \prod_{1 \le j \le \chi} f^{(j)} \quad \text{with } f^{(j)} \in R^{\natural}$$

and for $1 \le i \le \hat{h}$ let us put

$$i^* = \{j \in \{1, \ldots, \chi\} : \mathrm{noc}(f, f^{(j)}) = \hat{c}_i\}.$$

Then

$$\{1, \ldots, \chi\} = \coprod_{1 \le i \le \hat{h}} i^*$$

is a partition into pairwise disjoint nonempty sets, and for $1 \le i \le \hat{h}$ we have

$$\Omega'_{B_i}(f_Y) = \prod_{j \in i^*} f^{(j)} \quad \text{with} \quad 0 < \deg_Y f^{(j)} \in (n/\hat{d}_i)\mathbb{Z} \quad \text{for all } j \in i^*$$

and

$$\mathrm{int}(f, \Omega'_{B_i}(f_Y)) = nS(B_i)\deg_Y \Omega'_{B_i}(f_Y) = [(\hat{d}_i/\hat{d}_{i+1}) - 1]\hat{r}_i$$

where int denotes intersection multiplicity.[17]

Example (DF3). Now, if we are in the nontrivial irreducible case of $F = f \in R^\natural$ with $\deg_Y f = n > 1$, and if $h(c(f)) = 1$ with $c_1(f) \notin \mathbb{Z}$, then the conclusions of the above Remark (DF2) say that $\Omega'_{B_1}(f_Y) = (1/n)f_Y$ with $\mathrm{int}(f, f_Y) = (n-1)m_1(f)$, and $\mathrm{noc}(f, f^{(j)}) = c_1(f) = m_1(f)/n$ for every irreducible factor $f^{(j)}$ of f_Y. To verify this in a particular situation, by taking $(w_1(X), \ldots, w_{n-1}(X), w_n(X)) = (0, \ldots, 0, X^e)$ in Example (TR3) of § 6 we have

$$F(X,Y) = f(X,Y) = Y^n + X^e \in R^\natural \text{ where } 0 \neq e \in \mathbb{Z} \text{ with } \mathrm{GCD}(n,e) = 1$$

and hence $h(T(f)) = h(f) = 1$ with $m_1(f) = e$ and

$$l_0(T(f)) = -\infty \text{ and } l_1(T(f)) = c_1(f) = e/n$$

and upon letting

$$B_i \in \hat{R}^\flat \text{ with } \sigma(B_i) = \{f\} \text{ and } \lambda(B_i) = l_i(T(f)) \text{ for } 0 \leq i \leq 1$$

we have

$$T(F) = T(f) = \{B_0, B_1\}$$

with

$$D'(B_0) = 0 \text{ and } D'(B_1) = n - 1.$$

Now clearly $f_Y = nY^{n-1}$, and hence $\mathrm{Res}_Y(f, f_Y) = n^n X^{(n-1)e}$ and $f^{(j)} = Y$ for $1 \leq j \leq \chi = n-1$, and therefore $\mathrm{int}(f, f_Y) = \mathrm{ord}_X \mathrm{Res}_Y(f, f_Y) = (n-1)e = (n-1)m_1(f)$ and $\mathrm{noc}(f, f^{(j)}) = (1/n)\mathrm{ord}_X f(X,0) = m_1(f)/n$ for $1 \leq j \leq \chi = n-1$. This completes the verification.

Example (DF4). Next, if we are in the nontrivial irreducible case of $F = f \in R^\natural$ with $\deg_Y f = n > 1$, and if $h(c(f)) = 2$ with $c_1(f) \notin \mathbb{Z}$, then the conclusions of the above Remark (DF2) say that $\Omega'_{B_1}(f_Y)\Omega'_{B_2}(f_Y) = (1/n)f_Y$ and for $1 \leq i \leq 2$ we have that: $\deg_Y \Omega'_{B_i}(f_Y) = D'(B_i) = (n/d_{i+1}) - (n/d_i) > 0$ and $\mathrm{int}(f, \Omega'_{B_i}(f_Y)) = [(d_i/d_{i+1}) - 1]r_i$ where $\Omega'_{B_i}(f_Y)$ is the product of all those irreducible factors $f^{(j)}$ of f_Y for which $\mathrm{noc}(f, f^{(j)}) = c_i(f)$, and moreover for each of these $f^{(j)}$ we have $0 < \deg_Y f^{(j)} \in (n/d_i)\mathbb{Z}$. To verify this in a particular situation, in Example (TR4) of § 6 we have

$$F(X,Y) = f(X,Y) = (Y^2 - X^{2a+1})^2 - X^{3a+b+2}Y \in R^\natural \text{ with } a \in \mathbb{Z} \text{ and } 0 \leq b \in \mathbb{Z}$$

with

$$\begin{cases} n = d_1 = 4 \text{ and } d_2 = 2 \text{ and } d_3 = 1, \text{ and} \\ c_1(f) = (2a+1)/2 \text{ and } c_2(f) = (4a+2b+3)/4, \text{ and} \\ [(d_1/d_2) - 1]r_1 = 4a + 2 \text{ and } [(d_2/d_3) - 1]r_2 = 8a + 2b + 5 \end{cases}$$

[17] The intersection multiplicity $\mathrm{int}(f, g)$ of $f \in R^\natural$ with $g \in R$ is defined by putting $\mathrm{int}(f,g) = \mathrm{ord}_X \mathrm{Res}_Y(f, g)$, where $\mathrm{Res}_Y(f,g)$ denotes the Y-resultant of f and g; equivalently, for any $z(X) \in k((X))$ with $f(X^n, z(X)) = 0$ where $\deg_Y f = n$, we have $\mathrm{int}(f,g) = \mathrm{ord}_X g(X^n, z(X))$; see pp. 286–287 of [Ab]. By (GNP7) we see that, given any $B \in R^\flat$ and $H \in R$, for every $f \in \sigma(B)$ we have $\mathrm{int}(f, \Omega'_B(H)) = nS(B)\deg_Y \Omega'_B(H)$ where $\deg_Y f = n$.

and $h(T(f)) = 2$ with $l_0(T(f)) = -\infty$ and

$$l_1(T(f)) = c_1(f) = (2a+1)/2 \text{ and } l_2(T(f)) = c_2(f) = (4a+2b+3)/4$$

and upon letting

$$B_i \in \hat{R}^\flat \text{ with } \sigma(B_i) = \{f\} \text{ and } \lambda(B_i) = l_i(T(f)) \text{ for } 0 \le i \le 2$$

we have

$$T(f) = T(F) = \{B_0, B_1, B_2\}.$$

with $D'(B_0) = 0$ and

$$D'(B_1) = 1 \text{ and } D'(B_2) = 2.$$

Now

$$f_Y = 4Y(Y^2 - X^{2a+1}) - X^{3a+b+2}$$

and hence by (TR5) of § 6 we see that $f_Y = 4Yf^{(1)}f^{(2)}$ where

$$f^{(1)}(X,Y) = Y - v(X) \in R^\natural \text{ and } v(X) \in k((X))$$

with

$$\mathrm{ord}_X v(X) = a + b + 1$$

and

$$f^{(2)}(X,Y) = Y^2 + \sum_{1 \le i \le 2} v_i'(X) Y^{2-i} \in R^\natural \text{ and } v_i'(X) \in k((X))$$

with

$$\mathrm{ord}_X v_1'(X) > (2a+1)/2 \text{ and } \mathrm{ord}_X v_2'(X) = 2a+1.$$

Comparing coefficients of Y^2 and Y in the equation $f_Y = 4f^{(1)}f^{(2)}$ we see that $v_1'(X) - v(X) = 0$ and $v_2'(X) - v_1'(X)v(X) = -X^{2a+1}$, and hence

$$f^{(2)}(X,Y) = Y^2 + v(X)Y - X^{2a+1} + v(X)^2.$$

Applying the quadratic equation formula to the above equation we get the roots of $f^{(2)}(X^4, Y)$ to be

$$\begin{aligned}
Y &= (-1/2)v(X^4) \pm (1/2)\sqrt{4X^{8a+4} - 3v(X^4)^2} \\
&= (-1/2)v(X^4) \pm X^{4a+2}\sqrt{1 - (3/4)X^{-8a-4}v(X^4)^2} \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree } > 4a+4b+4 \text{ in } X)] \\
&\quad \pm X^{4a+2}\sqrt{1 - (3/4)\mu^2 X^{8b+4} + (\text{terms of degree} > 8b+4 \text{ in } X)} \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)] \\
&\quad \pm X^{4a+2}[1 - (3/8)\mu^2 X^{8b+4} + (\text{terms of degree} > 8b+4 \text{ in } X)] \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)] \\
&\quad \pm [X^{4a+2} - (3/8)\mu^2 X^{4a+8b+6} + (\text{terms of degree} > 4a+8b+6 \text{ in } X)]
\end{aligned}$$

where for the third equality we are using the fact that $\mathrm{ord}_X v(X) = a+b+1$ and hence

$$v(X^4) = \mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X) \text{ with } 0 \neq \mu \in k$$

and for the fourth equality we are using the binomial theorem for exponent $1/2$. It follows that

$$f^{(2)}(X^4, Y) = [Y - y_1(X)][Y - y_2(X)] \text{ where } y_1 \in k((X)) \text{ and } y_2(X) \in k((X))$$

are such that

$$y_1(X) = X^{4a+2} - (\mu/2)X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)$$

and

$$y_2(X) = -X^{4a+2} - (\mu/2)X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X).$$

We also have

$$f^{(1)}(X^4, Y) = Y - v(X) \text{ where } v(X) \in k((X)) \text{ with } \mathrm{ord}_X = 4a+4b+4.$$

Finally by (TR4) of §6 we have

$$f(X^4, Y) = \prod_{1 \le j \le 4} [Y - z_j(X)]$$

with

$$z_j(X) = (\iota^j X)^{4a+2} + \tfrac{1}{2}(\iota^j X)^{4a+2b+3} + (\text{terms of degree} > 4a+2b+3 \text{ in } X)$$

where ι is a primitive 4-th root of 1 in k. By the above expressions for the roots of f and $f^{(1)}$ we get

$$\mathrm{int}(f, f^{(1)}) = 4a+2 \text{ and } \mathrm{noc}(f, f^{(1)}) = (2a+1)/2.$$

Likewise, by the above expressions for the roots of f and $f^{(2)}$ we get

$$\mathrm{int}\,(f, f^{(2)}) = 8a+2b+5 \text{ and } \mathrm{noc}(f, f^{(2)}) = (4a+2b+3)/4.$$

It follows that

$$\Omega'_{B_i}(f_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(B_i) \text{ for } 1 \le i \le 2$$

and this completes the verification.

Example (DF5). Finally, let us turn to the case of $F \in \hat{R}^\natural$ having two factors, i.e., such that $F = ff'$ with $f \in R^\natural$ and $f' \in R^\natural$. At the same time let us arrange matters so that F is pure meromorphic, i.e.,

$$F(X, Y) = \Phi(X^{-1}, Y) \text{ with } \Phi(X, Y) \in k[X, Y].$$

To do this, in Example (TR5) of §6, let us take $n > 1$ with $b = 0$ and $a = -1$, and

$$\Phi(X, Y) = Y^{n+2} + \kappa' XY^n + \hat{\kappa} Y + \kappa + \sum_{3 \le i \le n} \kappa_i Y^{n+2-i} \in k[X, Y]$$

with

$$0 \neq \kappa' \in k \text{ and } 0 \neq \hat{\kappa} \in k \text{ and } 0 \neq \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \le i \le n.$$

As explained in (TR5) of § 6, we then have

$$F(X,Y)=f(X,Y)f'(X,Y) \text{ with } f(X,Y)\neq f'(X,Y)$$

where

$$f(X,Y)=Y^n+\sum_{1\le i\le n} w_i(X)Y^{n-i}\in R^\natural \text{ and } w_i(X)\in k((X))$$

with

$$\operatorname{ord}_X w_i(X)>ie/n \text{ for } 1\le i\le n-1 \text{ and } \operatorname{ord}_X w_n(X)=e=1$$

and

$$f'(X,Y)=Y^2+\sum_{1\le i\le 2} w_i'(X)Y^{2-i}\in R^\natural \text{ and } w_i'(X)\in k((X))$$

with

$$\operatorname{ord}_X w_1'(X)>e'/2 \text{ and } \operatorname{ord}_X w_2'(X)=e'=-1$$

and $0\neq\kappa'\in k$ and $0\neq\kappa/\kappa'\in k$ are the coefficients of $X^{e'}$ and X^e in $w_2'(X)$ and $w_n(X)$ respectively. As explained in (TR5) of § 6, we also have $h(T(F))=2$ with $l_0(T(F))=-\infty$ and

$$l_1(T(F))=-1/2 \text{ and } l_2(T(F))=1/n$$

and upon letting

$$\begin{cases} B_0\in\hat{R}^\flat \text{ with } \sigma(B_0)=\{f,f'\} \text{ and } \lambda(B_0)=l_0(T(F)),\\ \text{and } B_1\in\hat{R}^\flat \text{ with } \sigma(B_1)=\{f,f'\} \text{ and } \lambda(B_1)=l_1(T(F)),\\ \text{and } B_2\in\hat{R}^\flat \text{ with } \sigma(B_2)=\{f\} \text{ and } \lambda(B_2)=l_2(T(F)),\\ \text{and } B_2'\in\hat{R}^\flat \text{ with } \sigma(B_2')=\{f'\} \text{ and } \lambda(B_2')=l_2(T(F)),\end{cases}$$

we have

$$T(F)=\{B_0,B_1,B_2,B_2'\}$$

with $D'(B_0)=0$ and

$$D'(B_1)=2 \text{ and } D'(B_2)=n-1 \text{ and } D'(B_2')=0.$$

Now

$$F_Y=(n+2)Y^{n+1}+n\kappa'X^{-1}Y^{n-1}+\hat{\kappa}+\sum_{3\le i\le n}(n+2-i)\kappa_iY^{n+1-i}$$

and hence by (TR5) of § 6 we see that $F_Y=(n+2)f^{(1)}f^{(2)}$ with $f^{(1)}\neq f^{(2)}$ where

$$f^{(2)}(X,Y)=Y^{n-1}+\sum_{1\le i\le n-1} v_i(X)Y^{n-1-i}\in R^\natural \text{ and } v_i(X)\in k((X))$$

with

$$\operatorname{ord}_X v_i(X)>ie/n \text{ for } 1\le i\le n-2 \text{ and } \operatorname{ord}_X v_{n-1}(X)=e=1$$

and

$$f^{(1)}(X,Y)=Y^2+\sum_{1\le i\le 2} v_i'(X)Y^{2-i}\in R^\natural \text{ and } v_i'(X)\in k((X))$$

with

$$\mathrm{ord}_X v_1'(X) > e'/2 \text{ and } \mathrm{ord}_X v_2'(X) = e' = -1$$

and $0 \neq n\kappa'/(n+2) \in k$ and $0 \neq \hat{\kappa}/(n\kappa') \in k$ are the coefficients of $X^{e'}$ and X^e in $v_2'(X)$ and $v_{n-1}(X)$ respectively. In view of (TR3) of § 6 we see that

$$f(X^n, Y) = \prod_{1 \le j \le n} [Y - z_j(X)]$$

where $z_j(X) \in k((X))$ is such that

$$z_j(X) = \omega^j \kappa^* X + (\text{terms of degree} > 1 \text{ in } X)$$

where ω is a primitive n-th root of 1 in k, and κ^* is an n-th root of $-\kappa/\kappa'$ in k, and

$$f'(X^2, Y) = \prod_{1 \le j \le 2} [Y - z_j'(X)]$$

where $z_j'(X) \in k((X))$ is such that

$$z_j'(X) = (-1)^j \kappa'^* X^{-1} + (\text{terms of degree} > -1 \text{ in } X)$$

where κ'^* is a square root of $-\kappa'$ in k. In view of (TR3) of § 6 we also see that

$$f^{(2)}(X^{n-1}, Y) = \prod_{1 \le j \le n-1} [Y - y_j(X)]$$

where $y_j(X) \in k((X))$ is such that

$$y_j(X) = \hat{\omega}^j \hat{\kappa}^* X + (\text{terms of degree} > 1 \text{ in } X)$$

where $\hat{\omega}$ is a primitive $(n-1)$-th root of 1 in k, and $\hat{\kappa}^*$ is an $(n-1)$-th root of $-\hat{\kappa}/(n\kappa')$ in k, and

$$f^{(1)}(X^2, Y) = \prod_{1 \le j \le 2} [Y - y_j'(X)]$$

where $y_j'(X) \in k((X))$ is such that

$$y_j'(X) = (-1)^j \hat{\kappa}'^* X^{-1} + (\text{terms of degree} > -1 \text{ in } X)$$

where $\hat{\kappa}'^*$ is a square root of $-n\kappa'/(n+2)$ in k. By the above expressions of the roots of $f, f', f^{(1)}, f^{(2)}$ we get

$$\begin{cases} \mathrm{int}\,(f, f^{(1)}) = -n \text{ and } \mathrm{noc}(f, f^{(1)}) = -1/2, \text{ and} \\ \mathrm{int}\,(f', f^{(1)}) = -2 \text{ and } \mathrm{noc}(f', f^{(1)}) = -1/2, \text{ and} \\ \mathrm{int}\,(f, f^{(2)}) = (n-1) \text{ and } \mathrm{noc}\,(f, f^{(2)}) = 1/n, \text{ and} \\ \mathrm{int}\,(f', f^{(2)}) = -(n-1) \text{ and } \mathrm{noc}(f', f^{(2)}) = -1/2, \end{cases}$$

and hence

$$\Omega_{B_i}'(F_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(B_i) \text{ for } 1 \le i \le 2$$

which verifies Theorem (DF1) in the present situation.

To get an example of $\hat{F} \in \hat{R}^\natural$ having three factors, we take $\hat{F}(X,Y) = \hat{\Phi}(X^{-1},Y)$ with $\hat{\Phi}(X,Y) = \Phi(X,Y) - \Phi(0,0) \in k[X,Y]$. Then by (TR5) of § 6 we get $\hat{F} = \hat{f}\hat{f}'\hat{f}''$ with $\hat{f}'' \neq \hat{f} \neq \hat{f}' \neq \hat{f}''$ where $\hat{f}''(X,Y) = Y \in R^\natural$ and

$$\hat{f}(X,Y) = Y^{n-1} + \sum_{1 \leq i \leq n-1} \hat{w}_i(X) Y^{n-1-i} \in R^\natural \text{ and } \hat{w}_i(X) \in k((X))$$

with

$$\mathrm{ord}_X\, \hat{w}_i(X) > ie/n \text{ for } 1 \leq i \leq n-2 \text{ and } \mathrm{ord}_X\, \hat{w}_{n-1}(X) = e = 1$$

and

$$\hat{f}'(X,Y) = Y^2 + \sum_{1 \leq i \leq 2} \hat{w}'_i(X) Y^{2-i} \in R^\natural \text{ and } \hat{w}'_i(X) \in k((X))$$

with

$$\mathrm{ord}_X\, \hat{w}'_1(X) > e'/2 \text{ and } \mathrm{ord}_X\, \hat{w}'_2(X) = e' = -1$$

and $0 \neq \kappa' \in k$ and $0 \neq \hat{\kappa}/\kappa' \in k$ are the coefficients of $X^{e'}$ and X^e in $\hat{w}'_2(X)$ and $\hat{w}_{n-1}(X)$ respectively. In view of (TR3) of § 6 we see that

$$\hat{f}(X^{n-1},Y) = \prod_{1 \leq j \leq n-1} [Y - \hat{z}_j(X)]$$

where $\hat{z}_j(X) \in k((X))$ is such that

$$\hat{z}_j(X) = \hat{\omega}^j \hat{\kappa}^* X + (\text{terms of degree} > 1 \text{ in } X)$$

where $\hat{\omega}$ is a primitive $(n-1)$-th root of 1 in k, and $\hat{\kappa}^*$ is an $(n-1)$-th root of $-\hat{\kappa}/\kappa'$ in k, and

$$\hat{f}'(X^2,Y) = \prod_{1 \leq j \leq 2} [Y - \hat{z}'_j(X)]$$

where $\hat{z}'_j(X) \in k((X))$ is such that

$$\hat{z}'_j(X) = (-1)^j \hat{\kappa}'^* X^{-1} + (\text{terms of degree} > -1 \text{ in } X)$$

where $\hat{\kappa}'^*$ is a square root of $-\kappa'$ in k. By the above expressions of the roots of $\hat{f}$ and $\hat{f}'$ it follows that $h(T(\hat{F})) = 2$ with $l_0(T(\hat{F})) = -\infty$ and

$$l_1(T(\hat{F})) = -1/2 \text{ and } l_2(T(\hat{F})) = 1/(n-1)$$

and upon letting

$$\begin{cases} \hat{B}_0 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_0) = \{\hat{f}, \hat{f}', \hat{f}''\} \text{ and } \lambda(\hat{B}_0) = l_0(T(\hat{F})), \\ \text{and } \hat{B}_1 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_1) = \{\hat{f}, \hat{f}', \hat{f}''\} \text{ and } \lambda(\hat{B}_1) = l_1(T(\hat{F})), \\ \text{and } \hat{B}_2 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_2) = \{\hat{f}, \hat{f}''\} \text{ and } \lambda(\hat{B}_2) = l_2(T(\hat{F})), \\ \text{and } \hat{B}'_2 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}'_2) = \{\hat{f}'\} \text{ and } \lambda(\hat{B}'_2) = l_2(T(\hat{F})), \end{cases}$$

we have

$$T(\hat{F}) = \{\hat{B}_0, \hat{B}_1, \hat{B}_2, \hat{B}'_2\}$$

with $D'(\hat{B}_0) = 0$ and

$$D'(\hat{B}_1) = 2 \text{ and } D'(\hat{B}_2) = n-1 \text{ and } D'(\hat{B}'_2) = 0.$$

Now $\hat{F}_Y = F_Y = (n+2) f^{(1)} f^{(2)}$, and by the above expressions of the roots of $\hat{f}, \hat{f}'$, $f^{(1)}, f^{(2)}$ we get

$$\begin{cases} \operatorname{int}(\hat{f}, f^{(1)}) = -(n-1) \text{ and } \operatorname{noc}(\hat{f}, f^{(1)}) = -1/2, \text{ and} \\ \operatorname{int}(\hat{f}', f^{(1)}) = -2 \text{ and } \operatorname{noc}(\hat{f}', f^{(1)}) = -1/2, \text{ and} \\ \operatorname{int}(\hat{f}'', f^{(1)}) = -1 \text{ and } \operatorname{noc}(\hat{f}'', f^{(1)}) = -1/2, \text{ and} \\ \operatorname{int}(\hat{f}, f^{(2)}) = (n-1) \text{ and } \operatorname{noc}(\hat{f}, f^{(2)}) = 1/(n-1), \text{ and} \\ \operatorname{int}(\hat{f}', f^{(2)}) = -(n-1) \text{ and } \operatorname{noc}(\hat{f}', f^{(2)}) = -1/2, \text{ and} \\ \operatorname{int}(\hat{f}'', f^{(2)}) = 1 \text{ and } \operatorname{noc}(\hat{f}'', f^{(2)}) = 1/(n-1), \end{cases}$$

and hence

$$\Omega'_{\hat{B}_i}(\hat{F}_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(\hat{B}_i) \text{ for } 1 \le i \le 2$$

which again verifies Theorem (DF1) in the present situation.

9. Factorization of the jacobian

If $T = T(FG)$ where $F \in R \setminus k((X))$ and $0 \neq G \in R$, then for every $B \in T$ with $\Omega_B(G) = 1$ we clearly have $R^*(F,B) = R^*(B)$. Therefore by (SP76), (SP77), (SP79) and (SP80) we get the following jacobian factorization theorem.

Theorem (JF1). *Let $T = T(FG)$ where $F \in R \setminus k((X))$ is devoid of multiple factors in R, and $0 \neq G \in R$. Then we have the following.*

(JF1.1) *If $B \in T$ is such that $\Omega_B(G) = 1$ and $S(G,B) \neq 0$ then we have*

$$\deg_Y \Omega_B(J(F,G)) = \deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) = D''(B)$$

and

$$\deg_Y \Omega'_B(J(F,G)) = \deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) = D'(B)$$

and

$$\Omega_B(J(F,G)) = \Omega'_B(J(F,G)) \prod_{B' \in \pi(T,B)} \Omega'_{B'}(J(F,G))$$

where for every $B' \in \pi(T,B)$ we have $\Omega_{B'}(G) = 1$ and $S(G,B') \neq 0$ and

$$\deg_Y \Omega_{B'}(J(F,G)) = \deg_Y \Omega_{B'}((FG)_Y) = \deg_Y \Omega_{B'}(F_Y) = D''(B')$$

and

$$\deg_Y \Omega'_{B'}(J(F,G)) = \deg_Y \Omega'_{B'}((FG)_Y) = \deg_Y \Omega'_{B'}(F_Y) = D'(B').$$

(JF1.2) *If $B \in T$ is such that $\Omega_B(G) = 1$ then we have*

$$\deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) = D''(B)$$

and

$$\deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) = D'(B)$$

and

$$\Omega_B((FG)_Y) = \Omega'_B((FG)_Y) \prod_{B' \in \pi(T,B)} \Omega'_{B'}((FG)_Y)$$

where for every $B' \in \pi(T,B)$ *we have* $\Omega_{B'}(G)=1$ *and*

$$\deg_Y \Omega_{B'}((FG)_Y) = \deg_Y \Omega_{B'}(F_Y) = D''(B')$$

and

$$\deg_Y \Omega'_{B'}((FG)_Y) = \deg_Y \Omega'_{B'}(F_Y) = D'(B').$$

Remark (JF2). The jacobian factorization (JF1.1) was based on (SP80), and it invoked the jacobian estimates (JE1) to (JE3) of § 5 only in the special case when $\deg_Y \mathrm{minco}_X [z,V,W](G)=0$. Elsewhere we shall discuss a more refined Jacobian factorization based on (SP30) by invoking the general case of (JE1) to (JE3).

Example (JF3). Now let us illustrate Theorem (JF1.1) by the example

$$F = F(X,Y) = Y^n + X^e \in R^\natural \text{ where } 0 \neq e \in \mathbb{Z} \text{ with } \mathrm{GCD}(n,e)=1$$

considered in (DF3) of § 8. For $G = X$ we have $J(F,G) = F_Y$ and we are reduced to (DF3)

Example (JF4). Next let us illustrate Theorem (JF1.1) by the example

$$F = F(X,Y) = (Y^2 - X^{2a+1})^2 - X^{3a+b+2}Y \in R^\natural \text{ with } a \in \mathbb{Z} \text{ and } 0 \leq b \in \mathbb{Z}$$

considered in (DF4) of § 8. Again, for $G = X$ we have $J(F,G) = F_Y$ and we are reduced to (DF4). Moreover, for

$$\hat{G} = \hat{G}(X,Y) = Y \in R^\natural$$

we have

$$J(F,\hat{G}) = F_X = -(4a+2)X^{2a}\hat{F}$$

with

$$\hat{F} = \hat{F}(X,Y) = Y^2 + (3a+b+2)(4a+2)^{-1}X^{a+b+1} - X^{2a+1}.$$

By (TR3) and (TR4) of § 6, it follows that $h(T(F\hat{G}))=2$ with $l_0(T(F\hat{G})) = -\infty$ and

$$l_1(T(F\hat{G})) = (2a+1)/2 \text{ and } l_2(T(F\hat{G})) = (4a+2b+3)/4$$

and upon letting

$$\begin{cases} \hat{B}_0 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_0) = \{F,\hat{G}\} \text{ and } \lambda(\hat{B}_0) = l_0(T(F\hat{G})), \\ \text{and } \hat{B}_1 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_1) = \{F,\hat{G}\} \text{ and } \lambda(\hat{B}_1) = l_1(T(F\hat{G})), \\ \text{and } \hat{B}_2 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}_2) = \{F\} \text{ and } \lambda(\hat{B}_2) = l_2(T(F\hat{G})), \\ \text{and } \hat{B}'_2 \in \hat{R}^\flat \text{ with } \sigma(\hat{B}'_2) = \{\hat{G}\} \text{ and } \lambda(\hat{B}'_2) = l_2(T(F\hat{G})), \end{cases}$$

we have

$$T(F\hat{G}) = \{\hat{B}_0, \hat{B}_1, \hat{B}_2, \hat{B}'_2\}$$

with $D'(\hat{B}_0) = 0$ and

$$D'(\hat{B}_1) = 2 \text{ and } D'(\hat{B}_2) = 2 \text{ and } D'(\hat{B}'_2) = 0.$$

By (TR3) and (TR4) we also see that $\hat{F} \in R^\natural$ with

$$\mathrm{noc}(\hat{G}, \hat{F}) = (2a+1)/4 \text{ and } \mathrm{noc}(F, \hat{F}) = (4a+2b+3)/4$$

and hence

$$\Omega'_B(J(F,\hat{G})) = \begin{cases} \hat{F} & \text{if } B = \hat{B}_2 \\ 1 & \text{if } B = \hat{B}_0 \text{ or } B = \hat{B}_1 \text{ or } B = \hat{B}'_2 \end{cases}$$

in accordance with Theorem (JF1.1). Likewise, for

$$\tilde{G} = \tilde{G}(X,Y) = Y^2 - X^{2a+1} \in R^\natural$$

we have

$$J(F,\tilde{G}) = J(-X^{3a+b+2}Y, \tilde{G}) = \begin{cases} -(6a+2b+4)X^{3a+b+1}\tilde{F} & \text{if } 3a+b+2 \neq 0 \\ -(2a+1)X^{2a}\tilde{F} & \text{if } 3a+b+2 = 0 \end{cases}$$

with

$$\tilde{F} = \tilde{F}(X,Y) = \begin{cases} Y^2 + (2a+1)(6a+2b+4)^{-1}X^{2a+1} & \text{if } 3a+b+2 \neq 0 \\ 1 & \text{if } 3a+b+2 = 0. \end{cases}$$

By (TR3) and (TR4) of § 6, it follows that $h(T(F\tilde{G})) = 2$ with $l_0(T(F\tilde{G})) = -\infty$ and

$$l_1(T(F\tilde{G})) = (2a+1)/2 \text{ and } l_2(T(F\tilde{G})) = (4a+2b+3)/4$$

and upon letting

$$\begin{cases} \tilde{B}_0 \in \hat{R}^\flat \text{ with } \sigma(\tilde{B}_0) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_0) = l_0(T(F\tilde{G})), \\ \text{and } \tilde{B}_1 \in \hat{R}^\flat \text{ with } \sigma(\tilde{B}_1) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_1) = l_1(T(F\tilde{G})), \\ \text{and } \tilde{B}_2 \in \hat{R}^\flat \text{ with } \sigma(\tilde{B}_2) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_2) = l_2(T(F\tilde{G})), \end{cases}$$

we have

$$T(F\tilde{G}) = \{\tilde{B}_0, \tilde{B}_1, \tilde{B}_2\}$$

with $D'(\tilde{B}_0) = 0$ and

$$D'(\tilde{B}_1) = 3 \text{ and } D'(\tilde{B}_2) = 4.$$

Thus the stem of every bud of $T(F\tilde{G})$ contains F as well as $\tilde{G}$, and hence Theorem (JF1.1) does not predict any factors of $J(F,\tilde{G})$. This is quite satisfactory when $3a+b+2=0$ because then $J(F,\tilde{G}) = -(2a+1)X^{2a+1}$ and so $J(F,\tilde{G})$ has no factor involving Y. A particularly interesting case of $3a+2b+2=0$ is the pure meromorphic case when $(a,b) = (-1,1)$. In that case, as noted in (TR4) of § 6, we have $F(X,Y) = \Phi(X^{-1}, Y)$ where $\Phi(X,Y) \in k[X,Y]$ is the variable $\Phi(X,Y) = (Y^2 - X)^2 - Y$; indeed, then $k[X,Y] = k[\Phi, \Psi]$ where $\Psi(X,Y) = Y^2 - X$ with $\tilde{G}(X,Y) = \Psi(X^{-1}, Y)$.

Example (JF5). Finally let us illustrate Theorem (JF1.1) by the example

$$F = F(X,Y) = \Phi(X^{-1}, Y)$$

where

$$\Phi(X,Y) = Y^{n+2} + \kappa' XY^n + \hat{\kappa}Y + \kappa + \sum_{3\le i\le n} \kappa_i Y^{n+2-i} \in k[X,Y]$$

with $n > 1$ and

$$0 \neq \kappa' \in k \text{ and } 0 \neq \hat{\kappa} \in k \text{ and } \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \le i \le n$$

considered in (DF5) of § 8 (where we used the notation $\hat{\Phi}$ and $\hat{F}$ for the special case of $\kappa = 0$). Once again, for $G = -X$ we have $J(F,G) = F_Y$ and we are reduced to (DF5). Note that the affine plane curve $\Phi = 0$ is nonsingular (at finite distance) for every κ; equivalently, Φ_X and Φ_Y have no solution in the affine plane $k \times k$. Moreover, Φ is irreducible for every $\kappa \neq 0$, but reducible for $\kappa = 0$. In (DF5) we have shown that Φ has two or three places at ∞, i.e., F has two or three factors in $R^\natural$, according as $\kappa \neq 0$ or $\kappa = 0$. Further interest in this nice family of bivariate polynomials Φ lies in the fact that it provides a convenient testing ground for the trivariate Jacobian conjecture. To elucidate this, given any $H_1 \in k[X_1,\dots,X_r]$ where r is any positive integer, let us say that H_1 is a variable in $k[X_1,\dots,X_r]$ to mean that $k[X_1,\dots,X_r] = k[H_1,\dots,H_r]$ for some $H_2,\dots,H_r$ in $k[X_1,\dots,X_r]$, and let us say that H_1 is a weak variable in $k[X_1,\dots,X_r]$ to mean that $0 \neq J(H_1,\dots,H_r) \in k$ for some $H_2,\dots,H_r$ in $k[X_1,\dots,X_r]$ where $J(H_1\dots,H_r)$ is the jacobian of $H_1,\dots,H_r$ with respect to $X_1,\dots,X_r$. The reducibility of Φ when $\kappa = 0$ shows that Φ is not a variable in $k[X,Y]$. The reducibility of Φ when $\kappa = 0$ also shows that Φ is not a variable in $k[X,Y,Z]$. It can be shown that Φ is not a weak variable in $k[X,Y]$, at least when $n+1$ is a prime number. However, as was pointed out to us by Ignacio Luengo, it is not known whether Φ is or is not a weak variable in $k[X,Y,Z]$, even when $n = 2$. To see that Φ is not a weak variable in $k[X,Y]$, first note that by the automorphism $(X,Y) \mapsto ((X - Y^2)/\kappa', Y)$ we can send Φ to the polynomial

$$XY^n + \hat{\kappa}Y + \kappa + \sum_{3\le i\le n} \kappa_i Y^{n+2-i} \in k[X,Y]$$

whose degree is $n+1$ and whose degree form XY^n has two coprime factors. On the other hand, it can easily be shown that if $H \in k[X,Y]$ is a weak variable in $k[X,Y]$ of prime degree then its degree form must be a power of a linear form.

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Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces)

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Abstract. Let X_0 be a projective curve whose singularity is one ordinary double point. We construct a birational model $G(n,d)$ of the moduli space $U(n,d)$ of stable torsion free sheaves in the case $(n,d)=1$, such that $G(n,d)$ has normal crossing singularities and behaves well under specialization i.e. if a smooth projective curve specializes to X_0, then the moduli space of stable vector bundles of rank n and degree d on X specializes to $G(n,d)$. This generalizes an earlier work of Gieseker in the rank two case.

Keywords. Torsion free sheaves; Gieseker functor; moduli.

1. Introduction

Let X_0 be an irreducible projective curve whose singularity is one ordinary double point and arithmetic genus $g \geq 2$. Then one has the moduli space $U(n,d)$ of stable torsion free sheaves of rank n and degree d, which is projective if $(n,d)=1$. Further it is reduced, its singularities are known and it has good specialization properties when $(n,d)=1$ i.e. if a smooth projective curve specializes to X_0, then the corresponding moduli space of the smooth curve specializes to $U(n,d)$ (cf. [10], [11], these constructions and properties hold more generally when X_0 has only double point singularities. It need not be irreducible). Let $U(n,d)^0$ denote the open subscheme of $U(n,d)$, corresponding to vector bundles on X_0 (i.e. locally free torsion free sheaves on X_0). Then $U(n,d)$ is a compactification of $U(n,d)^0$ and $U(n,d)\setminus U(n,d)^0$ is the singular locus of $U(n,d)$ when $(n,d)=1$. One knows that these singularities are *not* normal crossings.

Gieseker has constructed a compactification G of $G^0 = U(2,1)^0$ (i.e. for the case of rank 2 and degree 1) such that the singularities of G are (analytic) normal crossings and it has good specialization properties (cf. [5]). The points of $G\setminus G^0$ consist of vector bundles E on curves which are semi-stably equivalent to X_0, more precisely they are curves of the form X_k with a morphism $\pi : X_k \longrightarrow X_0$ such that π is an isomorphism over $X_0 \setminus \{p\}$ and $\pi^{-1}(p)$ is a chain R of projective lines (cf. Definition–Notation 2).

In this paper, we give a generalisation of Gieseker's construction for arbitrary rank. Gieseker's construction is based on m-Hilbert stability i.e. stability (in the GIT sense) of points of a Hilbert scheme corresponding to imbeddings of curves in Grassmannians. This approach is quite natural; however generalizing it to arbitrary rank seems complicated. Our method is different. It consists in establishing a relationship of the Gieseker moduli

with the moduli of torsion free sheaves. This allows us to deduce the construction of these new moduli spaces from those of torsion free sheaves.

Let $(n,d)=1$. We construct a projective variety $G(n,d)$ which is a compactification of $G(n,d)^0 = U(n,d)^0$ such that the singularities of $G(n,d)$ are (analytic) normal crossings (cf. Theorems 1 and 2). The points of $G(n,d) \setminus G(n,d)^0$ are again suitable vector bundles E on curves of the form X_k (modulo an equivalence relation, cf. Def. 3). We also get a canonical morphism

$$\pi_* : G(n,d) \longrightarrow U(n,d), \quad \text{defined by}$$
$$E \mapsto \pi_*(E) \ (\pi \text{ canonical morphism } X_k \longrightarrow X_0).$$

A crucial point in our construction is that a point of $G(n,d)$ which is a vector bundle E on X_k, is completely characterized by the following properties:

(1) the restriction of E to every component of R (which is $\simeq \mathbb{P}^1$) is of the form

$$\oplus \mathcal{O}(a_i), \ a_i \geq 0 \text{ and at least one } a_i > 0$$

or

(1′) the global sections of $E|_R$ give a closed immersion of R into a Grassmannian.

(By tensorisation by a line bundle from X_0, (1) $\Longleftrightarrow$ (1′), in fact by this tensorisation (1) $\Longrightarrow$ the global sections of E define a closed immersion of X_k into a Grassmannian (cf. Proposition 4))

(2) $\pi_*(E)$ is torsion free and stable.

In fact, we were led to this, among other things, by the observation that Gieseker's list of vector bundles (cf. p. 176, [5]) in rank two and degree one is characterized by the above properties.

Now π_* can be defined at the functorial level

$$\pi_* : \mathcal{G} \longrightarrow \mathcal{U}$$

or in more concrete terms we have to define families of objects in $G(n,d)$ parametrized by schemes T and check that π_* defines families of objects in $U(n,d)$ parametrized by T (cf. Proposition 7 and Lemma 4). Another crucial point in our construction is that π_* is proper (cf. Proposition 10). This comes to proving that a morphism (related to π_*) from a total family representing $G(n,d)$ into one representing $U(n,d)$ is proper. Once this is done the construction of the moduli space $G(n,d)$ results from that of $U(n,d)$ and standard geometric invariant theory. Total families representing $G(n,d)$ already figure in Gieseker's work (based on (1)′ above, cf. [5]). They are open subsets of Hilbert schemes associated to imbeddings of X_k in Grassmannians.

To prove the specialization properties, we have only to carry over our construction when $\mathcal{X} \longrightarrow S$ is a proper, flat family of curves ($S = \operatorname{Spec} A$, A discrete valuation ring) such that the generic fibre over S is smooth and the closed fibre $\simeq X_0$. The fact that $G(n,d)$ has only normal crossing singularities is essentially in Gieseker (cf. [5]).

Our construction seems to work in a far more general context, say a family of *stable* curves (in the sense of Deligne–Mumford, cf. [3]).

Further, if $(n,d) \neq 1$, it should also be possible to construct the generalized semi-stable Gieseker moduli spaces, which we have only briefly sketched (cf. Remark 6).

An interesting fact which we will take up later is about the fibres of the morphism π_*. They happen to be the wonderful compactifications of the projective group (in the sense of De Concini–Procesi [2], see Remark 9).

We believe that our set up gives a proper understanding of Gieseker's work (cf. [5]). The generalized Gieseker moduli spaces should be considered as solutions of the moduli problem associated to the following objects over X_0:

$$\{(\pi, E), \pi \text{ a proper map } X' \longrightarrow X_0 \text{ which is an isomorphism over } X_0 \setminus \{p\}$$
$$\text{and } E \text{ a vector bundle on } X' \text{ such that } \pi_*(E) \text{ is torsion free}\}.$$

We have of course to fix the "invariants" for the moduli. In this context it seems interesting to ask for generalisations when X_0 is replaced by a higher dimensional variety (say even a smooth surface).

Our set up seems also to give a tool for a systematic investigation of compactifications of the moduli spaces of principal bundles with reductive structure groups, on curves with singularities (ordinary double points).

We would also like to remark that attempts to generalize our earlier work (cf. [8]) to rank ≥ 3, by taking suitable reducible curves and showing that the moduli spaces of stable (or semi-stable) torsion free sheaves on these curves, would have normal crossings as singularities, do not seem to succeed.

After completing this paper, we came to know of a preprint of Teixidor i Bigas (cf. [15]), which is related to our work.

2. Vector bundles over the curves X_k

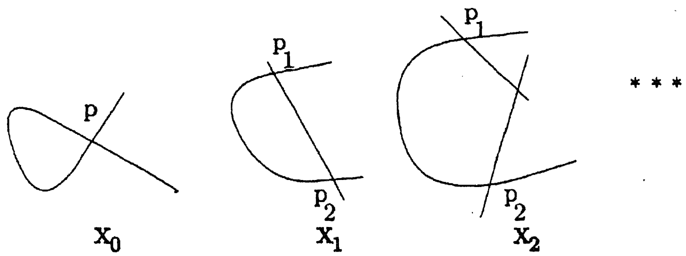
We work over an algebraic field k, which, for simplicity, we can assume to be the field of complex numbers.

DEFINITION-NOTATION 1

We call a scheme R, a chain of projective lines if $R = \cup_{i=1}^m R_i$, $R_i \simeq \mathbb{P}^1$, $R_i \cap R_j$ (for distinct i, j) is a single point if $|i-j| = 1$ and otherwise empty. We call m the length of R. Let E be a vector bundle of rank n on R. Then one knows that $E|_{R_i} \simeq \oplus_{j=1}^n \mathcal{O}(a_{ij})$, $a_{ij} \in \mathbb{Z}$. We say E is positive (≥ 0) if $a_{ij} \geq 0$, for all i and j. We say E is strictly positive (> 0) if it is positive and for every i, there is a j such that $a_{ij} > 0$. We say E is standard if $1 \geq a_{ij} \geq 0$ for all i,j and strictly standard if it is moreover strictly positive. If E is strictly standard, then $E_i = E|_{R_i} = L \oplus M$, where L is a direct sum of $\mathcal{O}(1)$'s and M is trivial. Then L is canonically defined and called the canonical sub-bundle of E_i and $E_i/L \simeq M$ is called the canonical quotient bundle of E_i.

DEFINITION-NOTATION 2

Let X_0 denote an irreducible projective curve which has just one ordinary double point 'p' as singular point. Let $\pi : X \longrightarrow X_0$ be the normalisation of X_0 and $\pi^{-1}(p) = \{p_1, p_2\}$. Let X_k be the curves which are "semi-stably equivalent to X_0":



i.e. X is a component of X_k $(k \geq 1)$ and if π denotes the canonical morphism $\pi : X_k \longrightarrow X_0$, $\pi^{-1}(p)$ is a chain R of projective lines of length k, passing through p_1 and p_2.

Let Z be a projective scheme with an ample line bundle $\mathcal{O}_Z(1)$ and E a vector bundle of rank n on Z. Then we see that if $H^0(E)$ generates E, through the evaluation map $H^0(E) \longrightarrow E_z$ (fibre of E at $z \in Z$) we get a canonical morphism

$$\phi_E = \phi : Z \longrightarrow \mathrm{Gr}(H^0(E), n) \text{ (Grassmannian of } n \text{ dim. quotients of } H^0(E))$$

such that the inverse image by ϕ of the canonical tautological quotient bundle on $\mathrm{Gr}(H^0(E), n)$ is isomorphic to E and conversely if this holds, H^0 generates E. If we choose further a basis of $H^0(E)$, we get in fact a canonical morphism of Z into $\mathrm{Gr}(m, n)$, the Grassmannian of rank n quotients of the standard vector space of rank m ($m =$ rank $H^0(E)$). Suppose then that ϕ is such a morphism. Then ϕ is injective if given $z_1, z_2 \in Z$ $(z_1 \neq z_2)$, there exist n sections $s_1, \ldots, s_n$ of E such that $(s_1 \wedge \cdots \wedge s_n)(z_1) = 0$ and $(s_1 \wedge \cdots \wedge s_n)(z_2) \neq 0$. Another sufficient condition for injectivity is that for the exact sequence

$$0 \longrightarrow I_{z_1, z_2} E \longrightarrow E \longrightarrow E_{z_1} \oplus E_{z_2} \longrightarrow 0 \quad (I_{z_1, z_2} \text{ ideal sheaf of } z_1, z_2)$$

the induced sequence

$$0 \longrightarrow H^0(I_{z_1, z_2} E) \longrightarrow H^0(E) \longrightarrow H^0(E_{z_1} \oplus E_{z_2}) (= E_{z_1} \oplus E_{z_2}) \longrightarrow 0 \tag{1}$$

is exact.

To give the differential of ϕ, note that we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(I_z E) & \longrightarrow & H^0(E) & \longrightarrow & E_z & \longrightarrow & 0, \text{ exact} \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & (I_z/I_z^2) \otimes E & \longrightarrow & E/I_z^2 E & \longrightarrow & E_z & \longrightarrow & 0, \text{ exact.} \end{array} \tag{2}$$

Then for every linear form $l : I_z/I_z^2 \longrightarrow k$ i.e. l belongs to the tangent space T_z of Z at z, we get a linear map $f_l : H^0(I_z E) \longrightarrow E_z$ i.e. $f_l \in \mathrm{Hom}(H^0(I_z E), E_z)$, which is the tangent space of $\mathrm{Gr}(H^0(E), n)$ at $\phi(z)$. We see that $d\phi(l) = f_l$. A sufficient condition for the injectivity of $d\phi$ at z is that the first vertical arrow in (2) is surjective and this is implied by the surjectivity of the second vertical arrow in (2).

We see also that for the bundle $E(k)$, k sufficiently large, all these sufficient conditions (also the one in (1) above) are satisfied for all $z \in Z$ so that the morphism $\phi_{E(k)}$ is a closed immersion.

PROPOSITION 1

Let R be a chain of projective lines and E a positive vector bundle of rank n on R. Then we have:

(i) *for $x \in R$, the evaluation map $H^0(R, E) \longrightarrow E_x$ is surjective,*
(ii) *if R_1 is a subchain (in the obvious sense) of projective lines of R, then the restriction map $H^0(R, E) \longrightarrow H^0(R_1, E)$ is surjective,*
(iii) $H^1(R, E) = 0$.
If moreover E is strictly positive, we have:
(iv) *Given $x, y \in R (x \neq y)$, $\exists\, s_1, \ldots, s_n \in H^0(R, E)$ such that $(s_1 \wedge \cdots \wedge s_n)(x) \neq 0$ and $(s_1 \wedge \cdots \wedge s_n)(y) = 0$.*

(v) *The canonical morphism*

$$\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$$

is a closed immersion.

Proof. The assertions are obvious when $R \simeq \mathbb{P}^1$. Then they are proved by induction on $l(R)$ = length of R. The proofs of (i) and (ii) are rather immediate. For (iii) we cut R into subchains R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \cap R_2$ reduces to a point q. Then we have the "patching" exact sequence (of sheaves)

$$0 \longrightarrow E \longrightarrow E|_{R_1} \oplus E|_{R_2} \xrightarrow{j} E_q \longrightarrow 0$$

where the map j is defined by $(s_1, s_2) \mapsto s_1(q) - s_2(q)$. By (ii) $H^0(E|_{R_1} \oplus E|_{R_2}) \longrightarrow E_q$ is surjective. Then since $H^1(E_q) = 0$, we see that

$$0 \longrightarrow H^1(E) \longrightarrow H^1(E|_{R_1}) \oplus H^1(E|_{R_2}) \longrightarrow 0$$

is exact. By the induction hypothesis the last term is zero. Hence $H^1(E) = 0$.

To prove (iv), we write $R = R_1 \cup R_2$, R_i subchains with $R_1 \cap R_2 = \{q\}$ and (in fact we can even assume $R_2 = \mathbb{P}^1$). Because of the induction hypothesis and (i), we see easily that if $x, y \in R_1$ or $x, y \in R_2$, we are done. Then we have only to consider the case $x \in R_1$, $y \in R_2$ and $x \neq q$. Then by the induction hypothesis, we can find $s'_1, \ldots, s'_n \in H^0(E|_{R_1})$ such that $(s'_1 \wedge \cdots \wedge s'_n)(x) \neq 0$ and $(s'_1 \wedge \cdots \wedge s'_n)(q) = 0$. Then $s'_1(q), \ldots, s'_n(q)$ are linearly dependent in E_q so that say

$$s'_1(q) = \sum_{i=2}^{n} a_i s'_i(q), \quad a_i \in k.$$

Then we can find $t_i \in H^0(E|_{R_2})$ such that $t_i(q) = s'_i(q)$ for $i \geq 2$ by (1). We set $t_1 = \sum_{i=2}^{n} a_i t_i$, $t_1 \in H^0(E|_{R_2})$. Then s'_i and t_i patch up to define s_i which have the required properties.

By (iv), the morphism ϕ is injective and hence to prove (v), we have only to show $d\phi$ is injective at all $x \in R$. Again we express $R = R_1 \cup R_2$ as above. Then by induction

$$\phi_i : R_i \longrightarrow \mathrm{Gr}(H^0(E|_{R_i}), n), \quad i = 1, 2$$

are closed immersions. Then only to show $(d\phi)_q$ is injective. Now by (ii), the canonical maps $H^0(E) \longrightarrow H^0(E|_{R_i})$, $i = 1, 2$, are surjective. Hence we get closed immersions

$$\mathrm{Gr}(H^0(E|_{R_i}), n) \hookrightarrow \mathrm{Gr}(H^0(E), n), \quad i = 1, 2.$$

Besides, we have the exact sequence

$$0 \longrightarrow H^0(E) \longrightarrow H^0(E|_{R_1}) \oplus H^0(E|_{R_2}) \longrightarrow E_q \longrightarrow 0.$$

This implies that the intersection

$$\mathrm{Gr}(H^0(E|_{R_1}), n) \cap \mathrm{Gr}(H^0(E|R_2), n) \quad \text{in } \mathrm{Gr}(H^0(E), n)$$

reduces to one point. It is not difficult to see that this intersection is transversal at this point. This implies that

$$(d\phi_1)|_q(T_{R_1,q}) \cap (d\phi_2)(T_{R_2,q}) = 0$$
$$(T_{R_i,q} \text{ is the tangent space to } R_i \text{ at } q)$$

from which it follows that $(d\phi)_q$ is injective. This proves Proposition 1.

Remark 1. Conversely, if E is a vector bundle on R such that the canonical morphism $\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$ is a closed immersion, we see that E is strictly positive.

PROPOSITION 2

Let E be a vector bundle of rank n on R (chain of projective lines) with $E|_{R_i} = \oplus_{j=1}^{n} \mathcal{O}(a_{ij})$ *(*R_i*-ith* $\mathbb{P}^1$ *component of R). Then we have*

(i) *(Riemann–Roch for E)*

$$\chi(E) = \sum_i \sum_j a_{ij} + n = \sum_i \deg E|_{R_i} + n = \deg E + n$$

where $\deg E = \sum_i \deg E|_{R_i}$ *(sometimes* $\deg E$ *is called the "total degree" of E).*

(ii) *if E is positive*

$$h^1(E) = 0 \quad \text{and } h^0(E) = \deg E + n.$$

The proof of this proposition is left as an easy exercise.

PROPOSITION 3

Let X_k *be the curve together with the canonical morphism* $\pi : X_k \longrightarrow X_0$ *so that* $\pi^{-1}(p)$ *is a chain R of projective lines of length k. Let E be a vector bundle of rank n on* X_k *such that* $E|_R$ *is positive. Then we have*

(i) $\pi_* \mathcal{O}_{X_k} \simeq \mathcal{O}_{X_0}$.
(ii) $R^i\pi_*(E) = 0$, $i > 0$.
(iii) $H^i(X_k, E) \simeq H^i(X_0, \pi_*(E))$.
(iv) *If E is trivial on R, then* $\pi_*(E)$ *is a vector bundle (i.e. locally trivial) and* $E \simeq \pi^*(\pi_*(E))$.
(v) *If* $H^1(X, I_{p_1,p_2}E|_X) = 0$, *then* $H^1(X_k, E) = 0$ *so that in this case we have also* $H^1(X_0, \pi_*(E)) = 0$.

Proof. The proof of (i) is rather immediate and we leave it. Let V be an affine neighbourhood of p, $U = \pi^{-1}(V)$ and $V' = U \cap X$. Then V' is affine and we have the "patching" exact sequence

$$0 \longrightarrow E|_U \longrightarrow E|_{V'} \oplus E|_R \longrightarrow T(= E_{p_1} \oplus E_{p_2}) \longrightarrow 0. \qquad (*)$$

Since V' is affine, the canonical (restriction) map $E|_{V'} \longrightarrow E_{p_1} \oplus E_{p_2}$ is surjective so that we get

$$H^i(E|_U) \simeq H^i(E|_{V'}) \oplus H^i(E|_R), \quad i > 0.$$

The RHS is zero and the assertions (ii) and (iii) follow. When $E|_R$ is trivial, $H^0(E|_R)$ is of dimension n and we get canonical isomorphisms of E_{p_i} with $H^0(E|_R)$, which leads to a canonical isomorphism $\theta : E_{p_1} \longrightarrow E_{p_2}$. We see that $H^0(E|_U)$ identifies with the subspace of $H^0(E|_{V'})$ consisting of elements s such that $\theta \cdot s(p_1) = s(p_2)$. This shows that $\pi_*(E)$

$(\pi_*(E)|_V = H^0(E|_U))$ identifies with the vector bundle on X_0 defined by $E|_X$ on the normalisation X and the patching condition $\theta : E_{p_1} \longrightarrow E_{p_2}$ and then (iv) follows:

To prove (v) note that the hypothesis implies that the canonical map $H^0(E|_X) \longrightarrow E_{p_1} \oplus E_{p_2}$ is surjective and $H^1(E|_X) = 0$. Consider the patching exact sequence:

$$0 \longrightarrow E \longrightarrow E|_X \oplus E|_R \longrightarrow T \longrightarrow 0.$$

Again we have $H^1(E) \simeq H^1(E|_X) \oplus H^1(E|_R)$, which implies $H^1(E) = 0$. This proves the proposition.

Remark 2. Let E be a positive vector bundle of rank n on a chain R of projective lines. Then we have a morphism $\pi : R \longrightarrow R'$ which contracts all the R_i in R ($R_i \simeq \mathbb{P}^1$) such that $E|_{R_i}$ is trivial and we have $E \simeq \pi^*(\pi_*(E))$ with $\pi_*(E)$ a strictly positive vector bundle on R' (for example by the same type of argument as for (v) of the above Proposition). By (i) of Proposition 1, we get a canonical morphism $\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$ and the image is again a chain of projective lines.

PROPOSITION 4

Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. If $F = E \otimes \pi^*(\mathcal{O}_{X_0}(l))$, then for $l \gg 0$ (more precisely if the conditions (a), (b), (c), (d) *in the proof below are satisfied), $H^0(F)$ generates F and the canonical morphism $\phi : X_k \longrightarrow \mathrm{Gr}$ $(H^0(F), n)$* is a closed immersion.

Further (for $l \gg 0$), $H^1(X_k, F) = 0$ so that by Prop. 3, we have

$$\begin{cases} H^0(X_k, F) \simeq H^0(X_0, \pi_*(F)), \text{ and} \\ H^i(X_k, F) = H^i(X_0, \pi_*(F)) = 0, \ i > 0. \end{cases}$$

Note that $E|_R \simeq F|_R$.

Proof. If $\pi^*(\mathcal{O}_{X_0}(1))$ were ample, this proposition is an easy consequence of the previous considerations, but this is not the case ($k \geq 1$). However, $\pi^*(\mathcal{O}_{X_0}(1)|_X$ is ample. Hence for $l \gg 0$, we see that $H^1(X, I_{p_1,p_2}F|_X) = 0$, so that by Prop. 3, $H^1(X_k, F) = 0$ and the last assertions of the proposition follow. Thus it remains only to show that ϕ is a closed immersion.

We can now suppose that if $l \gg 0$, the following conditions are satisfied:

(a) $H^1(X, I_{p_1,p_2}F|_X) = 0$, which implies that the canonical map $H^0(X, F|_X) \longrightarrow F_{p_1} \oplus F_{p_2}$ is surjective,

(b) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow I_{p_1,p_2}F|_X / I^2_{p_1,p_2}F|_X$$

is surjective,

(c) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow F|_X / I_x^2 F|_X$$

is surjective for $x \in X \setminus \{p_1, p_2\}$,

(d) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow F_{x_1} \oplus F_{x_2}$$

is surjective for $x_1, x_2 \in X \setminus \{p_1, p_2\}$, $x_1 \neq x_2$. We shall now see that these properties imply that ϕ is a closed immersion.

By (a) we see that the canonical restriction map

$$H^0(X_k, F) \longrightarrow H^0(R, F|_R) \tag{i}$$

is surjective. However the canonical map

$$H^0(X_k, F) \longrightarrow H^0(X, F|_X)$$

need not be surjective. If this were the case, the proof of the proposition would be straightforward. We have to do a little work to circumvent this minor difficulty.

We first observe that the image of the map $H^0(X_k, F) \longrightarrow H^0(X, F|_X)$ contains $H^0(X, I_{p_1, p_2} F|_X)$ i.e. sections of $F|_X$ vanishing at p_1, p_2. By this remark and (d), we see that

$$H^0(X_k, F) \longrightarrow F_{x_1} \oplus F_{x_2}, \quad x_1, x_2 \in X \setminus \{p_1, p_2\}, \quad x_1 \neq x_2$$

is surjective. Besides, because of (i) above and the fact that the canonical morphism

$$R \longrightarrow \mathrm{Gr}(H^0(F|_R), n) \tag{ii}$$

is a closed immersion, we see that $H^0(X_k, F)$ generates F and the canonical morphism

$$\phi : X_k \longrightarrow \mathrm{Gr}(H^0(X_k, F), n)$$

is in fact injective. In a similar manner, we see that (c) implies that the canonical map

$$H^0(X_k, F) \longrightarrow F/I_x^2 F, \quad x \in X \setminus \{p_1, p_2\}$$

is surjective. By the remarks on imbeddings into Grassmannians, these observations imply that $(d\phi)$ is injective at all $x \in X_k \setminus \{p_1, p_2\}$. Thus to complete the proof of the proposition, we have only to show that $(d\phi)$ is injective at p_1, p_2 respectively.

We have a canonical map

$$H^0(X_k, I_{X_k, p_1} F) \longrightarrow (I_{X_k, p_1}/I^2_{X_k, p_1}) \otimes E_{p_1}.$$

We have

$$(I_{X_k, p_1}/I^2_{X_k, p_1}) \otimes F_{p_1} \simeq (I_{X, p_1}/I^2_{X, p_1} \oplus I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1}.$$

Hence we get canonical linear maps

$$H^0(X_k, I_{X_k, p_1}, F) \xrightarrow{j_1} (I_{X, p_1}/I^2_{X, p_1}) \otimes F_{p_1}$$
$$H^0(X_k, I_{X_k, p_1} F) \xrightarrow{j_2} (I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1}.$$

Now $(I_{X, p_1}/I^2_{X, p_1})$ (resp. $I_{R, p_1}/I^2_{R, p_1}$) are 1-dimensional and therefore the RHS of the above maps can be identified with F_{p_1}. To prove injectivity of $d\phi$ at p_1, we have only to show $j_1 \neq 0$ and $j_2 \neq 0$ and they are not linearly dependent.

We have the following commutative diagram

$$\begin{tikzcd} H^0(X_k, I_{X_k, p_1} F) \arrow[r, "j_2"] \arrow[d, "f_2"'] & (I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1} \\ H^0(R, I_{R, p_1} F|_R) \arrow[ur, "g_2"'] & \end{tikzcd}$$

By (i) above, f_2 is surjective and by (ii), $g_2 \neq 0$. Hence $j_2 \neq 0$. We see that

$$H^0(X, I_{p_1, p_2}F|_X) \subset \ker f_2 \subset \ker j_2. \tag{iii}$$

Similarly for j_1, we have a commutative diagram

$$\begin{array}{lll} H^0(X_k, I_{X_k, p_1}F) & \xrightarrow{j_1} & (I_{X, p_1}/I^2_{X, p_1}) \otimes F_{p_1} \\ f_1 \downarrow & \nearrow g_1 & \\ H^0(X, I_{X, p_1}F|_X) & & \end{array}$$

Now the image of f_1 contains $H^0(X, I_{p_1, p_2}F|_X)$ and by (b) g_1 restricted to this subspace is surjective. This implies that $j_1 \neq 0$. We can identify $H^0(X, I_{p_1, p_2}F|_X)$ as also a subspace of $H^0(X_k, I_{X_k, p_1}F)$ and then f_1 is an isomorphism restricted to these spaces. Then we see that there are elements of $H^0(X, I_{p_1, p_2}F|_X)$ which are not in the kernel of j_1. Then by (iii), $\ker j_1 \neq \ker j_2$ which implies that j_1 and j_2 are linearly independent. This completes the proof of the proposition.

Consider the vector bundles E on X_k such that $E|_R$ are strictly positive. Our next aim is to characterize those E such that $\pi_*(E)$ are torsion free $(\pi : X_k \to X_0)$. This characterization involves only properties of $E|_R$.

Lemma 1. *Let E be a strictly standard vector bundle on $\mathbb{P}^1$ (see Def. 1) and $x, y \in \mathbb{P}^1$, $x \neq y$. Let L_x be a linear subspace of E_x. Let V be the linear subspace of $H^0(E)$ consisting of sections s such that $s(x) \in L_x$. Then we have the following:*

(i) *The canonical evaluation map $V \to L_x$ is surjective.*

(ii) *Let L_y be the image of V in E_y. Then $L_y \supset K_y$, K being the canonical subbundle of E. (cf. Def. 1). Further*

$$\dim(L_y/K_y) = \dim(\textit{Image of } L_x \textit{ in } V_x/K_x).$$

(iii) *Let $Q =$ the canonical quotient bundle E/K of E. Then we have a well-defined subbundle F of E such that $K \subset F$ and $F_x/K_x =$ Image of L_x in $Q_x = E_x/K_x$. Further, $F_y = L_y$. In fact the image of L_x in Q_x defines a subbundle Q' of Q such that $Q'_x =$ this image and F is the inverse image of Q' by the canonical homomorphism $E \to Q$.*

Proof. Recall that K is a direct sum of $\mathcal{O}(1)$ and $Q = E/K$ is trivial. The statement (iii) essentially gives the proof which is left as an easy exercise.

Remark 3. Let R be a chain of projective lines and E a strictly standard vector bundle on R. We denote by K_i the standard subbundle of $E_i = E|_{R_i}$ (R_i are the $\mathbb{P}^1$-components of R). We denote by q_i the point $R_i \cap R_{i+1}$. Let $p_1 = q_0 (\neq q_1)$ be a point on R_1 and $p_2 = q_k$ $(\neq q_{k-1})$ be a point on R_k.

Let S_i be the subchain $S_i = R_1 \cup \cdots \cup R_i$. Then by iterating the method in Lemma 1, given a linear subspace L of $E_{p_1} = E_{q_0}$, we get a linear subspace L_i of E_{q_i} such that if V_i is the subspace consisting of $s \in H^0(S_i, E|_{S_i})$ such that $s(q_0) \in L$, then the evaluation map $V_i \to L$ is surjective and the image of V_i in E_{q_i} is L_i.

In particular, if we take $L = (0)$, we denote the subspace L_k of $E_{q_k} = E_{p_2}$ by M.

We have the following:

Lemma 2. *We keep the notations as in Remark* 3. *Then the following are equivalent:*

(a) $\dim M = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_k$,

(b) *if $s \in H^0(R, E)$ and s vanishes at p_1, p_2, then s vanishes identically.*

Proof. We first observe that

$$(a) \iff \dim L_i = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_i, 1 \le i \le k.$$

where L_i are the spaces defined as above for $L = (0)$. By induction, we can assume that the lemma holds for $S_i, i \le k-1$. Suppose that (a) holds. Let $s \in H^0(R,E)$ be such that s vanishes at p_1 and p_2. Then $s(q_{k-1}) \in L_{k-1}$. But then by (a) we see that $(K_k)_{q_{k-1}} \cap L_{k-1} = (0)$. On the other hand, since $s(q_k) = 0$, we see that $s(q_{k-1}) \in (K_k)_{q_{k-1}}$. Hence $s(q_{k-1}) = 0$. This implies that s vanishes identically on R_k as well as on S_{k-1} by the induction hypothesis i.e. s vanishes identically on R.

Suppose now (b) holds. Then if (a) does not hold, we see that there is a j such that

$$\dim L_{j-1} = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_{j-1},$$
$$\text{and } (K_j)_{q_{j-1}} \cap L_{j-1} \ne (0).$$

We see that without loss of generality, we can suppose that $j = k$ i.e. we have

$$\dim L_{k-1} = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_{k-1},$$
$$\text{and } (K_k)_{q_{k-1}} \cap L_{k-1} \ne (0).$$

Then if $x \in (K_k)_{q_{k-1}} \cap L_{k-1}, x \ne 0$, we have a section s' of E on S_{k-1} such that $x = s'(q_{k-1})$ and $s'(q_0)$(i.e. $s'(p_1)) = 0$. On the other hand we have a section s'' of E on R_k such that $s''(q_{k-1}) = x$ and $s''(q_k) = 0$. Then s'' and s' patch up to define a section s of E on R such that s vanishes at p_1 and p_2 and s not identically zero. This gives a contradiction and the lemma follows.

Lemma 3. *Let R be a chain of projective lines and E a vector bundle on R such that $E|_{R_i} \simeq \oplus \mathcal{O}(a_{ij})$. If some $a_{ij} \ge 2$, say $a_{1j} \ge 2$ (without loss of generality). Then there is a section $s \in H^0(R,E)$ such that $s(p_1) = 0 = s(p_2)$ and s is not identically zero.*

Proof. There is a section θ of $\mathcal{O}(a_{1j})$ such that θ vanishes at q_0 and q_1 and θ not identically zero. Hence there is a section s' of $E|_{R_1}$ vanishing at q_0 and q_1 and which is not identically zero. We can obviously extend s' to a nonzero section s of E on R such that s vanishes at p_1 and p_2. This proves the lemma.

DEFINITION 3

Let F be a torsion free sheaf on X_0. Then one knows (cf. [8], [10]) that locally at the singular point p, F is of the form

$$F \simeq \bigoplus_{i=1}^{a} m \oplus \bigoplus_{i=1}^{b} \mathcal{O},$$

m being the maximal ideal of $\mathcal{O} = \mathcal{O}_{X,p}$. We refer to a as the type of F at p.

PROPOSITION 5

Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. Then we have the following:

(A) *$\pi_*(E)$ is torsion free on X_0 $(\pi : X_k \to X_0)$ if and only if a global section s of $E|_R$ vanishing at p_1 and p_2 vanishes identically. Thus by Lemmas 1, 2 and 3, we see that $\pi_*(E)$*

is torsion free if and only if

(1) $E|_R$ *is strictly standard, and*
(2) *the condition (a) of Lemma 2 holds.*

Note that (2) implies in particular that

(i) $k \leq n$, in fact
(ii) $k \leq \deg E|_R = \sum_i \deg E|_{R_i} \leq n$
(iii) if $k \geq 2$ and $R_i \cap R_j \neq \emptyset$, the intersection of the linear subspaces of $E|_{R_i \cap R_j}$, determined by the canonical sub-bundles of $E|_{R_i}$ and $E|_{R_j}$, is zero.

(B) *If* $\pi_*(E)$ *is torsion free, its type at 'p' is* $\deg E|_R$.

Proof. We have the following exact sequence of $\mathcal{O}_{X_k}$-modules

$$0 \longrightarrow I_X E \longrightarrow E \longrightarrow E|_X \longrightarrow 0,$$

I_X – ideal sheaf of X. Note that $I_X E$ can be identified with $I_{p_1,p_2} E|_R$ – the sheaf of sections of $E|_R$ vanishing at p_1, p_2. Then we have the exact sequence

$$0 \longrightarrow \pi_*(I_{p_1,p_2} E|R) \longrightarrow \pi_*(E) \longrightarrow \pi_*(E|_X).$$

Now $\pi_*(E|_X)$ is torsion free on X_0 and it is clear that $\pi_*(I_{p_1,p_2}E|R)$ is a torsion sheaf, in fact its support is at p. Hence it follows that the torsion subsheaf of $\pi_*(E)$ is precisely $\pi_*(I_{p_1,p_2}E|R)$. It is clear that $\pi_*(I_{p_1,p_2}E|R)$ is the sheaf determined by the k vector space $H^0(R, I_{p_1,p_2}E|R)$ considered as an $\mathcal{O}_{X_0,p}$-module. From these remarks the assertion (A) follows.

To prove (B) consider the exact sequence

$$0 \longrightarrow I_R E \longrightarrow E \longrightarrow E|_R \longrightarrow 0.$$

This gives the following exact sequence of $\mathcal{O}_{X_0,p}$-modules

$$0 \longrightarrow \pi_*(I_{p_1,p_2}E|_X)_{(p)} \longrightarrow \pi_*(E)_{(p)} \longrightarrow \pi_*(E|_R)_{(p)} \longrightarrow 0,$$

where the suffix 'p' indicates taking stalks of the sheaves at p (e.g. $\pi_*(E)_{(p)}$ is the stalk of $\pi_*(E)$ at p, and hence an $\mathcal{O}_{X_0,p}$-module).

We claim that

$$m_{X_0,p}(\pi_*(E)_{(p)}) = \pi_*(I_{p_1,p_2}E|_X)_{(p)}, \tag{$*$}$$

where $m_{X_0,p}$ is the maximal ideal of $\mathcal{O}_{X_0,p}$. We shall now show that $(*)$ implies the assertion (B).

Now $(*)$ implies that

$$\pi_*(E)_{(p)}/m_{X_0,p}(\pi_*(E)_{(p)}) \simeq \pi_*(E|_R)_{(p)}.$$

Now $\pi_*(E|_R)_{(p)}$ is annihilated by $m_{X_0,p}$ and in fact the k vector space $H^0(R, E|_R)$ considered as an $\mathcal{O}_{X_0,p}$-module. We have seen (cf. Proposition 2) that

$$\dim H^0(R, E|_R) = \deg E|_R + n.$$

Now if F is a torsion free $\mathcal{O}_{X_0,p}$-module of type a i.e. $F = \oplus_1^a m_{X_0,p} \oplus \oplus_1^{n-a} \mathcal{O}_{X_0,p}$, then

$$\dim F/m_{X_0,p}F = a + n,$$

since $\dim m_{X_0,p}/m^2_{X_0,p} = 2$. Now the assertion (B) follows. Thus it remains to prove the claim $(*)$ above. Choosing a trivialisation of $E|_X$ in a neighbourhood of p_1, p_2, we can consider $\pi_*(E)_{(p)}$ as the trivial module of rank n over the semi-local ring $\mathcal{O}_{X,p_1,p_2}$ of X at p_1, p_2. Then $\pi_*(I_{p_1,p_2}E|_X)_{(p)}$ is precisely its submodule vanishing at p_1, p_2. Then it suffices to prove the claim for $n=1$ and then the claim is just the statement that the radical of $\mathcal{O}_{X,p_1,p_2}$ identifies with $m_{X_0,p}$. This completes the proof of the proposition.

Remark 4. Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. Then we have seen in the proof of (B) above, that if $\pi_*(E)$ is torsion free we have

$$m_{X_0,p}(\pi_*(E)_{(p)}) = \pi_*(I_{p_1,p_2}E|_X)_{(p)}.$$

This is equivalent to saying that

$$I_{X_0,p}(\pi_*(E)) = \pi_*(I_{p_1,p_2}E|_X) \tag{$*$}$$

as on the R.H.S. π can be taken as the normalisation map $X \to X_0$ and then as π is an isomorphism over points outside p ($I_{X_0,p}$ is the ideal sheaf defined by p). Now we claim the following:

(i) $\pi_*(E)$ determines $E|_X$ i.e. if E_1 and E_2 on X_k (possibly for two different X_k with $E_i|_R$ strictly positive) are such that $\pi_*(E_i)$ are torsion free and $\pi_*(E_1) \simeq \pi_*(E_2)$, then $E_1|_X \simeq E_2|_X$.

(ii) if we have a family of vector bundles $\{E\}$ (on X_k's with $E|_R$ strictly positive) such that $\{\pi_*(E)\}$ is a bounded family of torsion free sheaves on X_0, then for $\ell \gg 0$ (independent of E) $\{E|_X\}$ is a bounded family and we have the properties of Proposition 4. i.e. for $F = E \otimes \pi^*(\mathcal{O}_{X_0}(\ell)), H^0(F)$ generates F and the canonical morphism $\phi: X_k \to \mathrm{Gr}(H^0(F), n)$ is a closed immersion. Besides $H^1(X_k, F) = 0$ and

(a) $H^0(X_k, F) \simeq H^0(X_0, \pi_*(F))$
(b) $H^i(X_k, F) = H^i(X_0, \pi_*(F)) = 0,\ i \geq 1$.

To prove these claims, we require

(iii) if $\pi: X \to X_0$ is the normalisation map, the functor π_* : (vector bundles on X) $\to$ (torsion free sheaves on X_0) is faithful i.e.

$$\mathrm{Hom}(V_1, V_2) \simeq \mathrm{Hom}(\pi_*(V_1), \pi_*(V_2));$$

in particular $V_1 \simeq V_2 \iff \pi_*(V_1) \simeq \pi_*(V_2)$.

Let us assume (iii). Let E_i ($i = 1, 2$) be as in (i) above. Then by $(*)$ if $\pi_*(E_1) \simeq \pi_*(E_2)$, we see that

$$I_{p_1,p_2}E_1|_X \simeq I_{p_1,p_2}E_2|_X,$$

multiplying (i.e. tensoring) by the line bundle $I^{-1}_{p_1,p_2}$, we deduce that $E_1|_X \simeq E_2|_X$. This proves the claim (i) above.

Now if $F = E \otimes \pi^*(\mathcal{O}_{X_0}(\ell))$ as in (ii) above, we see that

$$\pi_*(F) = \pi_*(E)(\ell),$$

so that $\pi_*(F)$ is torsion free and $F|_R \simeq E|_R$ is strictly positive. Then we have as in $(*)$

$$I_{X_0,p}(\pi_*(F)) = \pi_*(I_{p_1,p_2}F|_X).$$

If $\{\pi_*(E)\}$ is a bounded family, we can suppose that for $\ell \gg 0$ the LHS is generated by global sections, its H^1 is zero and without loss of generality that its dimension is independent of F. Hence if we set $W = I_{p_1,p_2}F|_X$, then for the family of vector bundles $\{W\}$ on X, we can suppose that $H^0(\pi_*(W))$ generates $\pi_*(W)$ (here $\pi : X \to X_0$ is the normalisation map), $\dim H^0(\pi_*(W))$ is independent of W and $H^1(\pi_*(W)) = 0$.

Now $H^0(\pi_*(W)) \simeq H^0(W)$ ($\pi : X \to X_0$ is an affine morphism) and we see that

$$H^0(\pi_*(W)) \text{ generates } \pi_*(W) \Rightarrow H^0(W) \text{ generates } W$$

(for

$$\pi_*(W)/m_{X_0,p}\pi_*(W) \simeq W_{p_1} \oplus W_{p_2},$$

W_{p_i} are the fibers of the vector bundle W at p_i). Hence for the family $\{W\}$, every W is the quotient of a trivial bundle, whose rank is independent of W and our hypothesis implies that $H^1(W) = 0$, so that the degree of W is independent of W. Hence one knows that $\{W\}$ is a bounded family (by the theory of Quot schemes). Thus we see that $\{I_{p_1,p_2}F|_X\}$ is a bounded family which implies that $\{F|_X\}$ and hence $\{E|_X\}$ is a bounded family. Then by the same arguments as in Proposition 4, we see that the claim (ii) follows.

Thus it remains to prove the above claim (iii). We see this claim is local with respect to X_0. If $A = \mathcal{O}_{X_0,p}$ and B is the semi-local ring of X at p_1, p_2, then B is the integral closure of A. The V_i in (iii) corresponds to the free B module B^n and the assertion (iii) reduces to showing that an A module isomorphism $B^n \to B^n$ is in fact a B module isomorphism. Such an A module isomorphism is an $(n \times n)$ matrix with entries in $\mathrm{Hom}_A(B,B)$ and thus it suffices to show that $\mathrm{Hom}_A(B,B) \simeq B$ (multiplication by elements of B). This is easy and well-known. This proves (iii).

3. The moduli space

We shall hereafter assume that the arithmetic genus g of X_0 is ≥ 2.

DEFINITION 4

(i) Let E be a vector bundle on X_k such that $E|_R$ is strictly positive. It is said to be stable if $\pi_*(E)$ ($\pi : X_k \to X_0$) is a stable (torsion free) sheaf on X_0 (cf. [10]). Note that it has then all the nice properties stated in Proposition 5, in particular $E|_R$ is standard, $k \leq n$, etc.

(ii) We call two vector bundles E_1, E_2 on X_k to be equivalent if $E_1 \simeq g^*(E_2)$, where g is an automorphism of X_k, which is identity on the component X (g could move points on R).

(iii) We set

$$G(n,d)_k = \begin{cases} \text{equivalence classes of stable vector bundles} \\ \text{(in the sense of (i) and (ii)) on } X_k \text{ of rank } n \\ \text{and degree } d \end{cases}$$

$$G(n,d) = \coprod_{k \leq n} G(n,d)_k \text{ (disjoint sum).}$$

Note that $G(n,d)_0$ is the set of isomorphism classes of stable vector bundles of rank n and degree d on X_0.

We shall see that if n and d are coprime, $G(n,d)$ is a projective variety with a birational morphism onto the projective variety $U(n,d)$ of stable torsion free sheaves of rank n and

degree d on X_0 (cf. Theorems 1 and 2) and it has all the good properties like specialisation stated in the introduction. One can also define semi-stability and its moduli space but it has to be done in a more subtle manner.

Remark 5. Let L be a line bundle on X_0. If E is a vector bundle on X_k, then since

$$\pi_*(E \otimes \pi^*(L)) = \pi_*(E) \otimes L$$

we note that $\pi_*(E)$ is torsion free (resp. stable) if and only if $\pi_*(E \otimes \pi^*(L))$ is torsion free (resp. stable). Then by (ii) of Remark 3 (since stable torsion free sheaves on X_0 of rank n and degree d form a bounded family), we can find $\ell \gg 0$ such that for any $E \in G(n,d)$ (or rather a vector bundle represented by an element of $G(n,d)$), $F = \pi_*(E \otimes \pi^*(\mathcal{O}_{X_0}(\ell))$ is generated by its global sections, the canonical morphism

$$\phi_F : X_k \to \mathrm{Gr}(H^0(F), n)$$

is a closed immersion and the properties (ii) (a), (b) of Remark 3 are also satisfied. We see that $m = \dim (H^0(F))$ and $e = \deg (F)$ are independent of F. We obtain a bijection

$$G(n,d)_k \xrightarrow{\sim} G(n,e)_k,$$
$$E \mapsto F = \pi_*(E \otimes \pi^*(\mathcal{O}_{X_0}(\ell)).$$

Thus without loss of generality, we can suppose that if E is a vector bundle representing an element of $G(n,d)_k$ i.e. a stable vector bundle on X_k, $H^0(E)$ generates E, the canonical morphism

$$\phi_E : X_k \to \mathrm{Gr}(H^0(E), n)$$

is a closed immersion and properties (ii) (a), (b) of Remark 4 are satisfied.

If we choose a basis of $H^0(E)$, $\mathrm{Gr}(H^0(E), n)$ can be identified with the standard Grassmannian $\mathrm{Gr}(m,n)$ ($m = \dim H^0(E)$), and ϕ_E identified with a morphism (we denote it again by ϕ_E)

$$\phi_E : X_k \to \mathrm{Gr}(m,n).$$

Now $PGL(m)$ operates canonically on $\mathrm{Gr}(m,n)$ and also on $X_0 \times \mathrm{Gr}(m,n)$ by taking the identity action on X_0. Now ϕ_E gives rise to a closed immersion

$$\psi_E : X_k \hookrightarrow X_0 \times \mathrm{Gr}(m,n), \quad \psi_E = (\pi, \phi_E).$$

Let E_1, E_2 be stable vector bundles of rank n and degree d on X_k and ψ_{E_1}, ψ_{E_2} the imbeddings into $X_0 \times \mathrm{Gr}(m,n)$ (choosing basis of $H^0(E_1)$, $H^0(E_2)$ respectively). Then the important remark is the easily seen observation:

$$E_1 \sim E_2 \text{ (equivalence relation defining } G(n,d))$$
$$\iff g(\mathrm{Im}\, \psi_{E_1}) = (\mathrm{Im}\, \psi_{E_2}), \quad g \in PGL(m).$$

We observe also that the Hilbert polynomial P_1 of $\mathrm{Im}\psi_E$ remains the same for all stable vector bundles E of rank n and degree d on X_k. Thus $\mathrm{Im}\psi_E \in \mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$ (we choose some polarisation on $X_0 \times \mathrm{Gr}(m,n)$). Note that the action of $PGL(m)$ on $X_0 \times \mathrm{Gr}(m,n)$ induces a canonical action of $PGL(m)$ on $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. The foregoing discussion thus shows that $G(n,d)$ can be identified (set theoretically) as the set of $PGL(m)$ orbits of a certain $PGL(m)$ stable subset of $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$.

We observe that given ψ_E, E is expressed canonically as a quotient of the trivial rank m vector bundle:

$$\mathcal{O}^m_{X_k} \rightarrow E, \quad H^0(\mathcal{O}^m_{X_k}) \simeq H^0(E), \quad H^1(E) = 0.$$

This representation is the pull-back by ϕ_E of the tautological rank n bundle on $\mathrm{Gr}(m,n)$ expressed as a quotient of the trivial rank m bundle. Then by (ii) (a) of Remark 4, $\pi_*(E)$ is a quotient of the trivial vector bundle of rank m on X_0

$$\mathcal{O}^m_{X_0} \rightarrow \pi_*(E) \quad \text{and} \quad H^0(\mathcal{O}^m_{X_0}) \simeq H^0(\pi_*(E)). \tag{$*$}$$

Let P_2 be the Hilbert polynomial of the stable torsion free sheaf on X_0 of rank n and degree d. Let $Q(m,P_2)$ be the Quot scheme of quotients with Hilbert polynomial P_2 of the trivial vector bundle of rank m on X_0. Let R be the $PGL(m)$ stable open subset of $Q(m, P_2)$ of quotients $\mathcal{O}^m_{X_0} \rightarrow F$ such that $H^0(\mathcal{O}^m_{X_0}) \rightarrow H^0(F)$ is an isomorphism; moreover let R^s be the $PGL(m)$ stable open subset of R such that F is stable (which is of course torsion free). Then we see that $(*)$ gives a point of R^s. Let $U(n,d)_s$ be the moduli space of stable torsion free sheaves of rank n and degree d on X_0. Recall that (cf. [7], [9], [10])

$$R^s \bmod PGL(m) \simeq U(n,d)_s.$$

In fact R^s is a principal $PGL(m)$ bundle over $U(n,d)_s$.

We shall now give the main steps in giving a canonical structure of a quasi-projective scheme on $G(n,d)$.

I The subset $Y^s = Y(n,d)^s \subset \mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$, $Y^s = \{\mathrm{Im}\,\psi_E\}$ (E representing elements of $G(n,d)$) is $PGL(m)$ stable and has a natural structure of an (irreducible) variety whose singularities are (analytic) normal crossings.

II The map $\theta : Y^s \rightarrow R^s$ defined by y (represented by ψ_E or $\mathrm{Im}\,\psi_E$) $\mapsto$ the element of R^s represented by $(*)$ above, is a $PGL(m)$ equivariant morphism.

III The morphism θ is proper.

We shall now see how admitting I, II and III, we would get a canonical structure of a quasi-projective variety on $G(n,d)$ with a proper birational morphism onto $U(n,d)_s$, the moduli space of stable torsion free sheaves on X_0.

Let R^s_v denote the $PGL(m)$ stable open subset of R^s such that the torsion free sheaves on X_0 represented by its points are locally free i.e. vector bundles.

Then a point of $\theta^{-1}(R^s_v)$ is represented by ψ_E such that the equivalence class of E is in $G(n,d)_0$, i.e. a closed immersion

$$\psi_E : X_0 \hookrightarrow X_0 \times \mathrm{Gr}(m,n).$$

Then it is rather easy to see that the morphism,

$$\theta : \theta^{-1}(R^s_v) \rightarrow R^s_v$$

is an isomorphism. Hence it follows that θ is a proper birational morphism. Since

$$\theta : Y^s \rightarrow R^s$$

is a $PGL(m)$ morphism and $R^s \rightarrow U(n,d)_s$ is a principal $PGL(m)$ bundle, it is seen easily that the quotient Y mod $PGL(m)$ exists and in fact that $Y^s \rightarrow Y^s$ mod $PGL(m)$ is a principal $PGL(m)$ bundle (since $R^s \rightarrow U(n,d)_s$ is locally isotrivial, choose a $PGL(m)$ stable open subset W in R^s such that W has an isotrivial section s over W mod $PGL(m)$,

then $\theta^{-1}(s)$ provides an isotrivial section of $\theta^{-1}(W)$ over $\theta^{-1}(W)$ mod $PGL(m)$ etc.). As we saw in Remark 4, $G(n,d) = Y^s$ mod $PGL(m)$, set theoretically. Thus we get a canonical scheme theoretic structure on $G(n,d)$ which is a variety. Further, since $Y^s \to Y^s$ mod $PGL(m)$, is a principal fibre space and Y has normal crossings as singularities, we see that the singularities of $G(n,d)$ are normal crossings. Besides θ gives rise to a proper birational morphism.

$$\begin{array}{ccc} \delta : G(n,d) & \to & U(n,d)_s \\ \| & & \| \\ Y^s \operatorname{mod} PGL(m) & & R^s \operatorname{mod} PGL(m). \end{array}$$

To prove that $G(n,d)$ is quasi-projective, we make use of GIT (cf. [7], [13]). This will also achieve, at the same time, giving the scheme theoretic structure on $G(n,d)$, which we did above.

Recall the basic fact in the construction of the moduli spaces of vector bundles (or torsion free sheaves), namely that there is a projective variety W with an action of $PGL(m)$ which lifts to an action of an ample line bundle $\mathcal{O}_W(1)$ on W, such that if W^{ss} (resp. W^s) represents the open subscheme of semi-stable (resp. stable or more precisely properly stable) points of W for this action, we have

(a) $W^s = R^s$
(b) $W^{ss} = R^{ss}$

where R^{ss} denotes the open subscheme of R represented by semi-stable torsion free sheaves F. These can be found in (cf. [12]). In the recent work of Simpson (cf. [14]), it is in fact proved that W can be taken to be the closure of R^{ss} in the Quot scheme $Q(m, P_2)$.

We claim now that we can find a $PGL(m)$ equivariant factorisation

$$\begin{array}{ccc} Y^s & \hookrightarrow & Z \\ \theta \downarrow & & \downarrow \lambda, \\ R^s & \hookrightarrow & W \end{array}$$

where Z is a projective variety with an action of $PGL(m)$ lifting to an ample line bundle $\mathcal{O}_Z(1)$. To see this take Z_1 to be the closure of Y^s in $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. Then we get a rational map $Z_1 \to W$ and we take Z to be the graph of this rational map. Note that Z is an (irreducible) variety since Y^s is a variety. The choice of $\mathcal{O}_Z(1)$ is obvious.

Consider now the polarisation $L = \lambda^*(\mathcal{O}_W(a)) \otimes \mathcal{O}_Z(1)$ on Z. Then with the usual notations one knows that if a is sufficiently large, we have ([7], [13])

(i) $\lambda^{-1}(R^s)(= \lambda^{-1}(W^s)) \subset Z(L)^s$
(ii) λ maps $Z(L)^{ss}$ onto R^{ss}.

It follows then that $Z(L)^s$ mod $PGL(m)$ exists as a quasi-projective scheme and $Z(L)^s \to Z(L)^s$ mod $PGL(m)$ is a principal fibre space. Note that we have an open immersion $Y^s \hookrightarrow \lambda^{-1}(R^s)$ and hence the following commutative diagram

$$\begin{array}{ccc} Y^s & \stackrel{i}{\hookrightarrow} & \lambda^{-1}(R^s) \\ \theta \searrow & & \swarrow \lambda \\ & R^s & \end{array}$$

where θ, λ are proper (birational). This implies that i is proper. But since i is also an open immersion and Z is a variety, it follows that i is an isomorphism. Hence $Y^s = \lambda^{-1}(R^s)$. Thus

we conclude Y^s mod $PGL(m) = G(n,d)$ is quasi-projective. If moreover $(n,d)=1$, then $U(n,d)=U(n,d)_s$ is projective, so that in this case $G(n,d)$ is also projective.

We shall now indicate how I and II are proved. The assertion III will be proved in the next section.

DEFINITION 5

Let $\mathcal{G} = \mathcal{G}(n,d)$ be the functor (called the Gieseker functor) defined as follows:

$$\mathcal{G} : (k\text{-schemes}) \to (\text{sets})$$

$\mathcal{G}(T) =$ set of closed subschemes $\Delta \hookrightarrow X_0 \times T \times \mathrm{Gr}(m,n)$ such that

(i) the induced projection map $p_{23} : \Delta \to T \times \mathrm{Gr}(m,n)$ is a closed immersion. We denote by E the rank n vector bundle on Δ which is the pull-back of the tautological rank n quotient bundle on $\mathrm{Gr}(m,n)$,

(ii) the projection $p_2 : \Delta \to T$ is a flat family of curves $\Delta_t (t \in T)$ such that Δ_t is a curve of the form X_k. Besides, the canonical map $\Delta_t \to X_0$ is the map $\pi : X_k (= \Delta_t) \to X_0$ that we have been considering,

(iii) the vector bundle E_t on Δ_t ($E_t = E|_{\Delta_t}$) is of degree d (and rank n) with $d = m + n(g-1)$.

(iv) By the definition of E, we get a quotient representation

$$\mathcal{O}^m_{\Delta_t} \to E_t$$

and we assume that this induces an isomorphism

$$H^0(\mathcal{O}^m_{\Delta_t}) \xrightarrow{\sim} H^0(E_t).$$

In particular, dim $H^0(E_t) = m$. It follows that

$$H^1(E_t) = 0.$$

PROPOSITION 6

The Gieseker functor $\mathcal{G}$ is represented by a $PGL(m)$ stable open subscheme Y of $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. *($P_1$ being the Hilbert polynomial of the closed subscheme Δ_t of $X_0 \times \mathrm{Gr}(m,n)$, choosing of course a polarisation). Further Y is an (irreducible) variety with singularities as (analytic) normal crossings.*

Proof. See Proposition 8 (of this paper) where a more general result is stated.

PROPOSITION 7

Let Δ be the universal object representing the Gieseker functor $\mathcal{G}$ above. Consider the "universal" closed immersion

$$\Delta \hookrightarrow X_0 \times Y \times \mathrm{Gr}(m,n)$$

defined by $\mathcal{G}$. This defines a flat family of curves $\Delta \to Y$. We have also a vector bundle E on Δ obtained as the pull-back of the tautological quotient bundle of rank n on $\mathrm{Gr}(m,n)$. *If Δ_y denotes the fibre of $\Delta \to Y$ over $y \in Y$, E defines a family $\{E_y\}$ of vector bundles on $\{\Delta_y\}, y \in Y$. We have the map $\Delta_y \to X_0$, defined by the first projection p_1, which we*

denote by π_y to be consistent with our earlier notation. We observe that $(\pi_y)_*(E_y)$ comes with a quotient representation

$$\mathcal{O}^m_{X_0} \to (\pi_y)_*(E_y) \text{ with } H^0(\mathcal{O}^m_{X_0}) \xrightarrow{\sim} H^0((\pi_y)_*(E_y)). \qquad (*)$$

Hence (*) defines a point of the open subscheme R of $Q(m, p_2)$ (see Remark 4). Then

$$y \mapsto \text{the point of } R \text{ defined by } (*)$$

defines a morphism $\theta : Y \to R$.

Proof. We have a commutative diagram.

$$\begin{array}{ccc} \Delta & \xrightarrow{\pi} & X_0 \times Y \\ p \searrow & & \swarrow q \\ & Y & \end{array}$$

where π is the projection p_{12}, $p =$ projection p_2, and $q =$ canonical projection onto Y. We observe that

$$\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{X_0 \times Y}. \qquad \text{(a)}$$

To see this since the fibres of π are connected (either a point or a chain of projective lines) and π is proper, we have $\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_Z$, where $Z \to X_0 \times Y$ is a proper bijective morphism. Note also that $Z \to X_0 \times Y$ is birational since π is birational (if Y_v denotes the subset defined by $y \in Y$ such that $\Delta_y \simeq X_0$, then π is an isomorphism over $X_0 \times Y_v$). Since X_0 and Y have normal crossing singularities, the proper bijective map $Z \to X_0 \times Y$ becomes an isomorphism (since all the "analytic branches" of $X_0 \times Y$ are again normal). This proves (a).

Now to prove the proposition, we claim that it suffices to prove

$$\begin{cases} \pi_*(E)|_{q^{-1}(y)} \simeq (\pi_y)_*(E_y) \\ q^{-1}(y) \simeq X_0 \times y \simeq X_0. \end{cases} \qquad \text{(b)}$$

To prove this claim suppose then that (b) holds.

We have the quotient representation

$$\mathcal{O}^m_\Delta \to E \text{ on } \Delta. \qquad \text{(c)}$$

Using (a), we get applying $(\pi)_*$

$$\mathcal{O}^m_{X_0 \times Y} \to \pi_*(E). \qquad \text{(d)}$$

We have then the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}^m_{X_0} & \to & (\pi_*(E))|_{q^{-1}(y)} \\ \| & & \downarrow \\ \mathcal{O}^m_{X_0} & \to & (\pi_y)_*(E_y), \end{array} \qquad \text{(e)}$$

where the first horizontal row is the restriction of (d) to $q^{-1}(y) \simeq X_0$ and the second horizontal row is obtained by applying $(\pi_y)_*$ to the quotient

$$\mathcal{O}^m_{q^{-1}(y)} \to E_y.$$

One knows that the second horizontal map of (e) is surjective by (ii) (a) of Remark 3. By (b) this implies that the first horizontal map of (e) is also surjective (for all $y \in Y$).

This implies that (d) is surjective i.e. $\pi_*(E)$ is a quotient of $\mathcal{O}^m_{X_0\times Y}$. Besides by (b) again the Hilbert polynomial of $\pi_*(E)|_{q^{-1}(y)}$ is P_2 and since Y is reduced, we see that $\pi_*(E)$ is flat over Y. This shows that (d) defines a morphism of Y into $Q(m, P_2)$ and the above claim is proved.

Thus to prove the proposition, it suffices to prove the following:

Lemma 4. *Suppose that we have a commutative diagram*

$$\begin{tikzcd} Z \arrow[rr, "\pi"] \arrow[dr, "p"'] & & W \arrow[dl, "q"] \\ & T & \end{tikzcd}$$

such that p and q are projective morphisms (which implies π is proper), $\pi_*(\mathcal{O}_Z) = \mathcal{O}_W$ and p is flat. Let E be a vector bundle on Z such that

$$R^1(\pi_t)_*(E|_{Z_t}) = 0, \quad i \geq 1.$$

Then $\pi_*(E)$ behaves well for restriction to fibres over T i.e.

$$\pi_*(E)|_{W_t} = (\pi_t)_*(E|_{Z_t}) \quad \textit{for all } t \in T.$$

Further $H^0(Z_t, E|_{Z_t}) \simeq H^0(W_t, (\pi_t)_*(E|_{Y_t}))$.

Proof. The required property is local with respect to T so that we can suppose that $T = \operatorname{Spec} A$. Let $\mathcal{O}_W(1)$ be a relatively ample sheaf with respect to q. Then by Serre's theorem, the coherent sheaf $\pi_*(E)$ on W is the sheaf associated to the graded module

$$\bigoplus_{n\geq 0} H^0(q_*(\pi_*(E)(n))). \tag{i}$$

If we denote by $E[n]$ the sheaf $E \otimes \pi^*(\mathcal{O}_W(n))$, we have

$$\pi_*(E)(n) = \pi_*(E[n]), \quad \text{since } \pi_*(\mathcal{O}_Z) = \mathcal{O}_W.$$

Further, since $q_*(\pi_*(E[n])) = p_*(E[n])$, we see that the graded module in (i) can be identified with

$$\bigoplus_{n\geq 0} H^0(p_*(E[n])) = \bigoplus_{n\geq 0} H^0(Z, E[n]). \tag{ii}$$

Similarly the sheaf $(\pi_t)_*(E|_{Z_t})$ on Z_t is the one associated to the graded k vector space ($k = k(t)$ residue at the closed point $t \in T$)

$$\bigoplus_{n\geq 0} (p_t)_*((E|_{Z_t})[n]) = \bigoplus_{n\geq 0} H^0(Z_t, (E|_{Z_t})[n]). \tag{iii}$$

Note that our hypothesis also implies that

$$R^i(\pi_t)_*((E|_{Y_t})[n]) = 0, \quad i \geq 1.$$

This implies the last assertion of the lemma. Further, we have

$$H^1(Z_t, (E|_{Z_t})[n]) = H^1(W_t, ((\pi_t)_*(E|_{Z_t}))(n)).$$

But the RHS is zero for $n \geq n_0$ (for some n_0 and for all $t \in T$) since $\mathcal{O}_W(1)$ is relatively ample with respect to q. Thus we see that

$$H^1(Z_t, (E|_{Z_t})[n]) = 0 \quad \text{for } n \geq n_0.$$

Since p is flat, we deduce that

$$\begin{cases} (p_*(E[n]) \otimes k(t) = (p_t)_*((E|_{Z_t})[n]) \\ \text{for } n \geq n_0 \ (k(t) = k \text{ residue field at } t). \end{cases}$$

Hence the graded module in (ii) tensored by $k(t)$ coincides with that in (iii) if we neglect terms of degree $\leq n_0$. But for determining the corresponding sheaves this suffices. We see that this proves the lemma. Consequently Proposition 7 also follows.

COROLLARY 1

(Proof of I and II). The subset $Y^s = Y(n,d)^s$ is open in Y and represents the subfunctor $\mathcal{G}^s$ of $\mathcal{G}$, for which in the definition of $\mathcal{G}$ (cf. Def. 4), we add moreover the condition

$$E_t \text{ is stable on } \Delta_t \text{ i.e. its equivalence class is in } G(n,d). \qquad \text{(v)}$$

Besides, $\theta : Y^s \longrightarrow R^s$ is a morphism.

Proof. Since R^s is open in R (being torsion free and stable give open conditions) and $\theta : Y \longrightarrow R$ is a morphism, the corollary follows immediately.

Thus admiting the properness property III, we have shown the following (a more general version will be given in the next section).

Theorem 1. *There exists a canonical structure of a quasi-projective variety on $G(n,d)$ and a canonical proper birational morphism*

$$\pi_* : G(n,d) \longrightarrow U(n,d)_s$$

onto the moduli space of stable torsion free sheaves on X_0. The singularities of $G(n,d)$ are (analytic) normal crossings. If $(n,d)=1$, $G(n,d)$ is projective, since $U(n,d)_s = U(n,d)$ is projective.

Remark 6 (semi-stable moduli). Consider the morphism $\theta : Y \longrightarrow R$ of Proposition 7. Set $Y^0 = \theta^{-1}(R^{ss})$. As we shall see in the next section, $\theta : Y^0 \longrightarrow R^{ss}$ is also proper. As we did in the discussions before Def. 4 for constructing a quasi-projective scheme structure on $G(n,d) = Y^s \bmod PGL(m)$, we can find a $PGL(m)$ equivariant factorisation

$$\begin{array}{ccc} Y^0 & \hookrightarrow & Z \\ \theta\downarrow & & \downarrow\lambda \\ R^{ss} & \hookrightarrow & W \end{array}$$

where Z and W are projective varieties with the actions of $PGL(m)$ lifting to ample line bundles etc. As we saw, $W^{ss} = R^{ss}$. Then by the same results and arguments which we used there, we see λ maps $Z(L)^{ss}$ onto W^{ss} and we have

$$Y^s \subset Z(L)^{ss} \subset Y^0.$$

However, $Z(L)^{ss}$ may not be equal to Y^0. Now the GIT quotient $Z(L)^{ss}//PGL(m)$ exists as a projective variety and we have a morphism of this GIT quotient onto $U(n,d) = R^{ss}//PGL(m)$, the moduli space of semi-stable torsion free sheaves of rank n and degree d on X_0. We could call $Z(L)^{ss}//PGL(m)$ the *generalized Gieseker semi-stable moduli space* of rank n and degree d on X_0. One has to show that it is intrinsically defined (there has been some choice of polarisations).

4. Properness and specialisation

To prove the specialisation property of $G(n,d)$, we have to take a family, say of smooth projective curves specialising to X_0 and show that the corresponding moduli spaces vary nicely. For simplicity, we work in the following context:

DEFINITION-NOTATION 6

Notation 6. Let $S = \operatorname{Spec} A$, where A is a discrete valuation ring with residue field the ground field k (which has been assumed algebraically closed). Let $\mathcal{X} \longrightarrow S$ be a flat family of projective curves such that the closed fibre $\mathcal{X}_{s_0} \simeq X_0$ and the generic fibre $\mathcal{X}_\xi$ (s_0-closed point of S and ξ the generic point of S) is smooth of genus g (recall $g =$ arithmetic genus of X_0 with $g \geq 2$). We assume that as a scheme over k, $\mathcal{X}$ is regular. Let $\mathcal{O}_{\mathcal{X}}(1)$ be an S-ample line bundle on $\mathcal{X}$ (we could assume that it is of degree one on the fibres over S).

One can formulate the definition of the Gieseker functor (cf. Def. 4) over the base S and Prop. 6, Prop. 7 go through easily in this generality. Further the construction of $G(n,d)$ over the base S also goes through, since we are taking quotients for free actions of the projective group and also the fact that *GIT* works over an arbitrary base (cf. [11]). Of course one has to add that all these go through, provided the required property of properness holds.

We shall now go through the generalisations rapidly and then take up properness.

Let $\mathrm{Gr}(m,n)$ or rather $\mathrm{Gr}(m,n)_S$ be the Grassmannian over S of n dimensional quotient spaces of the standard m dimensional space (we denote it by the same letter as we did for the case when the base field is k. Our object is simply the base change by S by the one we considered over k).

DEFINITION 7

Let $\mathcal{G}_S = \mathcal{G}(n,d)_S$ be the functor (called the Gieseker functor) defined as follows:

$$\mathcal{G}_S : (S\text{-schemes}) \longrightarrow (\text{sets}).$$

$\mathcal{G}_S(T) =$ set of closed subschemes $\Delta \hookrightarrow \mathcal{X} \times_S T \times_S \mathrm{Gr}(m,n)$ such that

(i) the induced projection map $p_{23} : \Delta \longrightarrow T \times_S \mathrm{Gr}(m,n)$ is a closed immersion. We denote by E the rank n vector bundle on Δ which is induced by the tautological rank n quotient bundle on $\mathrm{Gr}(m,n)$.

(ii) the projection map $p_1 : \Delta \longrightarrow T$ is a flat family of curves Δ_t ($t \in T$) such that Δ_t is a curve of the form X_k if t is over s_0 and is the curve $\mathcal{X}_\xi$ if t is over ξ.

Further consider the canonical commutative diagram

$$\begin{tikzcd} \Delta \arrow[rr] \arrow[dr] & & (\mathcal{X} \times_S T) \arrow[dl] \\ & T & \end{tikzcd}$$

This induces a canonical morphism $\Delta_t \longrightarrow (\mathcal{X} \times_S T)_t$. This morphism should be an isomorphism when $(\mathcal{X} \times_S T)_t$ is smooth (i.e. when t does not map to the closed point s_0 of S); further when t maps to s_0, in which case $(\mathcal{X} \times_S T)_t \simeq X_0$, this morphism should reduce to the canonical morphism $\Delta_t(\simeq X_k) \longrightarrow X_0$. We observe that Δ_t is a closed subscheme of the $k(t)$ scheme $(\mathcal{X} \times_S \mathrm{Gr}(m,n)) \times_S \mathrm{Spec}\, k(t)$ ($k(t)$ residue field at $t \in T$) and its Hilbert polynomial is P_1.

(iii) the vector bundle E_t on Δ_t ($E_t = E|_{\Delta_t}$) is of degree d (and rank n) with $d = m + n(g-1)$.

(iv) By the definition of E, we get a quotient representation

$$\mathcal{O}^m_{\Delta_t} \longrightarrow E_t$$

and we assume that this induces an isomorphism

$$H^0(\mathcal{O}^m_{\Delta_t}) \xrightarrow{\sim} H^0(E_t).$$

In particular, $\dim H^0(E_t) = m$.

PROPOSITION 8

The functor $\mathcal{G}_S$ is represented by a $PGL(m)$ stable open subscheme $\mathcal{Y}$ of the S-scheme $\mathrm{Hilb}^{P_1}(\mathcal{X} \times_S \mathrm{Gr}(m,n))$. *Further the S-scheme $\mathcal{Y}$ has the following properties:*

(i) *the closed fibre $\mathcal{Y}_{s_0}$ (of $\mathcal{Y}$ over S) is the variety Y with normal crossings (in Proposition 6)*

(ii) *the generic fibre $\mathcal{Y}_\xi$ is smooth,*

(iii) *as a scheme over k, $\mathcal{Y}$ is regular.*

Proof. The proof is essentially to be found in Gieseker's work (cf. Prop. 4.1, [5]). In the appendix to this paper we give a brief outline of the proof.

Let V be a vector bundle on $\mathcal{X}$ of rank n and degree d on the fibres over S. Let P_2 be the Hilbert polynomial of V. Let $Q_S(m, P_2)$ be the Quot scheme of quotients with Hilbert polynomial P_2 of the trivial vector bundle of rank m on $\mathcal{X}$. Then recall that $Q_S(m, P_2)$ is projective over S. Recall that an element of $Q_S(m, P_2)(T)$ (T being an S-scheme) is the following:

$$\left\{ \begin{array}{l} \text{A quotient } \mathcal{O}^m_{\mathcal{X} \times_S T} \xrightarrow{j} F, \text{ where } F \text{ is coherent on} \\ \mathcal{X} \times_S T, \text{ flat over } T \text{ with Hilbert polynomial } P_2. \end{array} \right.$$

In particular, we can take $T = Q_S(m, P_2)$ and we get the universal quotient:

$$\mathcal{O}^m_{\mathcal{X} \times_S Q_S(m, P_2)} \longrightarrow \mathcal{F}.$$

Now if $q \in Q_S(m, P_2)$, we denote by $\overline{q}$ the image of q in S ($Q_S(m, P_2)$ is an S-scheme). Let $\mathcal{X}_{\overline{q}}$ denote the fibre of $\mathcal{X} \longrightarrow S$ over q. With this notation, for $q \in Q(m, P_2)$, $\mathcal{F}_q$ is canonically a quotient of $\mathcal{O}^m_{\mathcal{X}_{\overline{q}}}$.

Let $\mathcal{R}$ be the $PGL(m)$ stable open subscheme of $Q_S(m, P_2)$ formed of $q \in Q_S(m, P_2)$ such that the canonical map $H^0(\mathcal{O}_{\mathcal{X}_{\bar{q}}}) \longrightarrow H^0(\mathcal{F}_q)$ is an isomorphism. Then we see that $H^1(\mathcal{F}_q) = 0$. Let $\mathcal{R}^s$ be the $PGL(m)$ stable open subscheme of $\mathcal{R}$ formed of q such that $\mathcal{F}_q$ is a stable (torsion free) sheaf on $\mathcal{X}_{\bar{q}}$. Let $\mathcal{U}(n, d)$ be the moduli space of semi-stable torsion free sheaves along the fibres of $\mathcal{X} \to S$. This is an S-projective scheme whose construction is based on the usual considerations when the base is a field and the fact that *GIT* works over an arbitrary base (cf. [7], [11]). Let $\mathcal{U}(n,d)_s$ be the open subscheme of $\mathcal{U}(n,d)$ corresponding to stable torsion free sheaves. Then we have $\mathcal{R}^s$ mod $PGL(m) \simeq \mathcal{U}(n,d)_s$. In fact $\mathcal{R}^s$ is a principal $PGL(m)$ fibre bundle over $\mathcal{U}(n,d)_s$.

PROPOSITION 9

Let Δ be the universal object defining $\mathcal{G}_S$ so that we have the universal closed immersion

$$\Delta \hookrightarrow \mathcal{X} \times_S \mathcal{Y} \times_S \mathrm{Gr}(m, n).$$

This gives a commutative diagram

$$\begin{tikzcd} \Delta \arrow[rr, "\pi"] \arrow[dr, "p"'] & & \mathcal{X} \times_S \mathcal{Y} \arrow[dl, "q"] \\ & \mathcal{Y} & \end{tikzcd}$$

where $\pi =$ projection p_{12}, $p =$ projection p_2, $q =$ canonical projection. We observe that for $y \in \mathcal{Y}$, $(\pi_y)_*(E_y)$ is a $k(y)$ valued point of $\mathcal{R}$. Then

$$y \mapsto (\pi_y)_*(E_y)$$

defines a morphism θ of $\mathcal{Y}$ into $\mathcal{R}$ (to be very formal we have to work with T valued points of $\mathcal{R}$, rather than $k(y)$ valued points).

Proof. The proof is on the same lines as that of Proposition 7. There is a mild difference in proving

$$\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{\mathcal{X} \times_S \mathcal{Y}}.$$

As in the case of Prop. 7, again π is birational as we note that the base change of the S-morphism π by $k(\xi)$ (ξ generic point of S) is an isomorphism (easy consequence of the fact that if $y \in \mathcal{Y}$ maps to ξ by the structure morphism $\mathcal{Y} \longrightarrow S$, then $\Delta_y \longrightarrow (\mathcal{X} \times_S \mathcal{Y}) \times_S k(y)$ is an isomorphism. Note we have assumed $\mathcal{X}_\xi$ is smooth). We claim that $\mathcal{X} \times_S \mathcal{Y}$ is normal. This can be seen as follows. We see that the closed subscheme $\mathcal{X} \times_S \mathcal{Y}_{s_0}$ ($\mathcal{Y}_{s_0}$-closed fibre of $\mathcal{Y} \longrightarrow S$) is of codimension one and defined by one equation (since the closed point s_0 of S is defined by one equation). We see that

$$\mathcal{X} \times_S \mathcal{Y}_{s_0} \simeq X_0 \times \mathcal{Y}_{s_0}$$

is of course Cohen–Macaulay (since X_0 and $\mathcal{Y}_{s_0}$ have normal crossing singularities). It follows that $\mathcal{X} \times_S \mathcal{Y}$ is Cohen–Macaulay. Further, it is easily seen that its singularity is of codimension ≥ 2. It follows that $\mathcal{X} \times_S \mathcal{Y}$ is normal. Then since π is proper birational and $\mathcal{X} \times_S \mathcal{Y}$ is normal, we see that $\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{\mathcal{X} \times_S \mathcal{Y}}$.

COROLLARY 2

Let $\mathcal{Y}^s = \Theta^{-1}(\mathcal{R}^s)$. Then $\mathcal{Y}^s$ is the open subset of $\mathcal{Y}$ such that $(\pi_y)_*(E_y)$ is stable, further it represents the subfunctor $\mathcal{G}^s_S$ defined in a way similar to that of $\mathcal{G}^s$ (cf. Corollary 1). Let $\mathcal{R}^f$ denote the open subset of $\mathcal{R}$ defined by $q \in Q_S(m, P_2)$ such that $\mathcal{F}_q$ is torsion free. We set $\mathcal{Y}^f = \Theta^{-1}(\mathcal{R}^f)$ and $\mathcal{Y}^0 = \Theta^{-1}(\mathcal{R}^{ss})$. Then we have open immersions

$$\mathcal{Y}^s \subset \mathcal{Y}^0 \subset \mathcal{Y}^{ss}$$

and Θ induces morphisms

$$\mathcal{Y}^s \longrightarrow \mathcal{R}^s, \quad \mathcal{Y}^0 \longrightarrow \mathcal{R}^{ss}, \quad \mathcal{Y}^f \longrightarrow \mathcal{R}^f.$$

Remark 7. Consider the above S-morphisms Θ (e.g. $\Theta : \mathcal{Y}^s \longrightarrow \mathcal{R}^s$). They are all isomorphisms over $S - \{s_0\}$ or equivalently Θ induces an isomorphism of their generic fibres over S (e.g. $\Theta_\xi : \mathcal{Y}^s_\xi \xrightarrow{\sim} \mathcal{R}^s_\xi$). In fact Θ is an isomorphism over the bigger open subset $\mathcal{R}_v$ defined by $q \in Q_s(m, P_2)$ such that $\mathcal{F}_q$ is locally free.

PROPOSITION 10

The morphism

$$\Theta : \mathcal{Y}^s \longrightarrow \mathcal{R}^s \quad (\textit{resp. } \mathcal{Y}^0 \longrightarrow \mathcal{R}^{ss}, \quad \mathcal{Y}^f \longrightarrow \mathcal{R}^f)$$

is proper.

Proof. It suffices to prove that $\Theta : \mathcal{Y}^f \longrightarrow \mathcal{R}^f$ is proper, as $\mathcal{Y}^s = \Theta^{-1}(\mathcal{R}^s)$ and $\mathcal{Y}^0 = \Theta^{-1}(\mathcal{R}^{ss})$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{Y}^f \arrow[r, hook, "i"] \arrow[dr, "\Theta"'] & Z \arrow[d, "q"] \\ & \mathcal{R}^f \end{tikzcd}$$

where i is an open immersion and $q : Z \longrightarrow \mathcal{R}^f$ is a projective morphism (since $\mathcal{Y}^f \longrightarrow \mathcal{R}^f$ is a quasi-projective morphism). As we saw in Remark 6, $\Theta^{-1}(\mathcal{R}^f_\xi) = \mathcal{Y}^f_s \longrightarrow \mathcal{R}^f_\xi$ is an isomorphism, in particular it is a proper morphism (note that $\mathcal{R}^f_\xi$ is open in $\mathcal{R}^f$). From this it follows that we can assume without loss of generality that Z is the closure of $\mathcal{Y}^f_\xi$. Then to prove that Θ is proper, we have only to show that $Z = \mathcal{Y}^f$ or equivalently $\mathcal{Y}^f_{s_0} = Z_{s_0}$ (these represent the closed fibres, s_0 being the closed point of S). Suppose that $z_0 \in Z_{s_0}$. Then we can find an S-morphism $T \longrightarrow Z$

$$T \xrightarrow{\mu} Z$$

such that

(i) $T = \operatorname{Spec} B$, B a d.v.r with k as residue field.
(ii) μ (closed point of T) $= z_0$.
(iii) $T \longrightarrow S$ is surjective, which implies that μ (generic point of T) $\in \mathcal{Y}^f_\xi \simeq \mathcal{R}^f_\xi$.

Consider the morphism $q \circ \mu : T \longrightarrow \mathcal{R}^f$. Then this gives a coherent sheaf $\mathcal{F}$ on $\mathcal{X}_T = (\mathcal{X} \times_S T)$ which is flat over T and torsion free over the fibres of $\mathcal{X}_T \longrightarrow T$. Further $\mathcal{F}$ has a quotient representation $\mathcal{O}^m_{\mathcal{X}_T} \longrightarrow \mathcal{F}$. Note that the closed fibre of $\mathcal{X}_T \simeq$ closed fibre of $\mathcal{X} \simeq X_0$ and $\mathcal{X}_T$ is regular outside the singular point p of X_0. Further, if η is the generic point of T, we denote the generic fibre of $\mathcal{X}_T \longrightarrow T$ by $\mathcal{X}_\eta$, which is a base change of the generic fibre $\mathcal{X}_\xi$ of $\mathcal{X} \longrightarrow S$. Now $\mathcal{F}$ is locally free on $\mathcal{X}_T$ outside p and the quotient representation defines a T-morphism

$$g : \mathcal{X}_T \setminus \{p\} \longrightarrow \operatorname{Gr}(m, n) \ \text{(rather } \operatorname{Gr}(m, n)_T\text{)}.$$

We can assume that g is an immersion (by suitable tensorisation by a power of $\mathcal{O}_{\mathcal{X}}(1)$, a similar property can be supposed to hold for the defining torsion free sheaf on $\mathcal{X} \times_S Q$ (m, P_2), so that this property follows by base change). Let Γ_g be the graph of g, considered as a rational morphism of $\mathcal{X}_T$, so that we have a closed immersion of T-schemes

$$\Gamma_g \hookrightarrow \mathcal{X}_T \times_T \operatorname{Gr}(m, n). \tag{$*$}$$

Let π_g denote the canonical projection (a T-morphism)

$$\pi_g : \Gamma_g \longrightarrow \mathcal{X}_T.$$

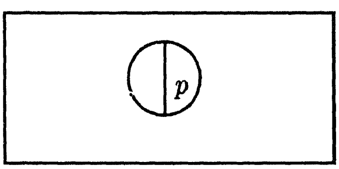
Obviously, π_g induces an isomorphism of the generic fibres over T, in fact it is an isomorphism over $\mathcal{X}_T \setminus \{p\}$. Let E be the vector bundle of rank n on Γ_g induced by the tautological quotient vector bundle of rank n on $\operatorname{Gr}(m, n)$ (through $(*)$). Consider the following:

$$\left\{\begin{array}{l} \text{(i) The closed fibre of } \Gamma_g \text{ over } T \text{ is a curve of the form } X_k \\ \text{(this implies that the morphism induced by } \pi_g \text{ on the closed} \\ \text{fibres is the canonical morphism } X_k \longrightarrow X_0) \\ \text{(ii) } (\pi_g)_*(E) = \mathcal{F}. \end{array}\right. \tag{A}$$

We claim that (A) $\Longrightarrow$ properness of $\Theta : \mathcal{Y}^f \longrightarrow \mathcal{R}^f$ i.e. $z_0 \in \mathcal{Y}^f_{s_0}$. To see this observe first that $\Gamma_g \longrightarrow T$ is a flat family of curves. To prove the claim, we have only to show that $(*)$ defines a T valued point of the Gieseker functor $\mathcal{G}_s$. We see that all the properties in the definition of the Gieseker functor are satisfied (Def. 6). This follows by Prop. 3 (especially the property (iii) of this proposition) and the fact that $(\pi_g)_*(E)$ behaves well for restriction to fibres over T (Lemma 4, to apply this Lemma we require the property $(\pi_g)_*(\mathcal{O}_{\Gamma_g}) = \mathcal{O}_{\mathcal{X}_T}$, which follows using the fact that $\mathcal{X}_T$ is normal and π_g is birational).

We see that to check the assertions in (A) it suffices to check them over a neighbourhood of the point p in $\mathcal{X}_T$. More precisely, let C be the local ring of $\mathcal{X}_T$ at p. Let C_0 be the local ring at p of the closed fibre ($\approx X_0$) of $\mathcal{X}_T \longrightarrow T$.

$\mathcal{X}_T$



Ball $\leftrightarrow$ Spec C

Vertical line $\leftrightarrow$ Spec C_0

Let Γ_C (resp. Γ_{C_0}) be the base change of $\pi_g : \Gamma_g \longrightarrow \mathcal{X}_T$ by $\operatorname{Spec} C \longrightarrow \mathcal{X}_T$ (resp. $\operatorname{Spec} C_0 \longrightarrow \mathcal{X}_T$). Let $\mathcal{F}_C$ denote the stalk of $\mathcal{F}$ at p and π_C (resp. π_{C_0}) the canonical morphism

$$\Gamma_C \longrightarrow \operatorname{Spec} C \text{ (resp. } \Gamma_{C_0} \longrightarrow \operatorname{Spec} C_0).$$

For a curve of the form X_k with its canonical morphism $X_k \longrightarrow X_0$ we denote by $(X_k)_{C_0}$, its base change by $\operatorname{Spec} C_0 \longrightarrow X_0$. We denote by E_C the restriction of E to Γ_C. Note that we have a commutative diagram

$$\begin{tikzcd} \Gamma_C \arrow[r] \arrow[dr] & \operatorname{Spec} C \arrow[d] \\ & \operatorname{Spec} B = T \end{tikzcd}$$

Then (A) is equivalent to

$$\left\{\begin{array}{l} \text{(i) The closed fibre of } \Gamma_C \longrightarrow T \text{ is of the form } (X_k)_{C_0} \text{ and} \\ \Gamma_{C_0} \longrightarrow \operatorname{Spec} C_0 \text{ identifies with the canonical morphism} \\ (X_k)_{C_0} \longrightarrow \operatorname{Spec} C_0 \\ \text{(ii) } (\pi_C)_*(E_C) = \mathcal{F}_C. \end{array}\right. \tag{A$_1$}$$

We have a quotient representation of the C-module $\mathcal{F}_C$

$$C^m \longrightarrow \mathcal{F}_C$$

induced by the quotient representation $\mathcal{O}^m_{\mathcal{X}_T} \longrightarrow \mathcal{F}$. Then if m_1 denotes the minimal number of generators of the C-module $\mathcal{F}_C$, we have a commutative diagram

$$\begin{tikzcd} C^m \arrow[rr] \arrow[dr] & & \mathcal{F}_C \\ & C^{m_1} \arrow[ur] & \end{tikzcd}$$

where $C^m \longrightarrow C^{m_1}$ is surjective. We have a canonical closed immersion $\operatorname{Gr}(m_1, n) \hookrightarrow \operatorname{Gr}(m, n)$ and the closed immersion

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times_T \operatorname{Gr}(m, n) \tag{$*$}'$$

induced by $(*)$ factorises as follows:

$$\begin{tikzcd} \Gamma_C \arrow[r, hook] \arrow[dr] & \operatorname{Spec} C \times_T \operatorname{Gr}(m, n) \\ & \operatorname{Spec} C \times_T \operatorname{Gr}(m_1, n) \arrow[u] \end{tikzcd}$$

i.e. Γ_C is the graph of the rational map

$$\operatorname{Spec} C \longrightarrow \operatorname{Gr}(m_1, n)$$

defined by the canonical morphism

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \operatorname{Gr}(m_1, n)$$

which is defined by a minimal set of generators of the C-module $\mathcal{F}_C$ which is locally free on $\operatorname{Spec} C \setminus \{p\}$.

We observe that in the foregoing discussion about the local nature of the assertion (A) over a neighbourhood of p, we could have supposed that A ($S = \operatorname{Spec} A$) and B are complete and C is the completion of the local ring of $\mathcal{X}_T$ at p. We assume this in the sequel.

We shall now give a concrete determination of $\mathcal{F}_C$, which would facilitate the checking of (A_1). For this we need the claim

$$\mathcal{F}^{**} = \mathcal{F} (\Longleftrightarrow \mathcal{F}_C^{**} = \mathcal{F}_C),$$

where $\mathcal{F}^{**}$ denotes the double dual of $\mathcal{F}$. To prove this claim note that we have a canonical inclusion $\mathcal{F} \hookrightarrow \mathcal{F}^{**}$ and $\mathcal{F}^{**}$ is flat over T. Consider the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{**} \longrightarrow N \longrightarrow 0.$$

Now the support of N is at p, in particular it is a torsion C-module. Let us use the notation, say $\mathcal{F}_t$ for the restriction $\mathcal{F}$ to the fibre of $\mathcal{X}_T \longrightarrow T$ over t. Let t_0 denote the closed point of T.

Since $\mathcal{F}_{t_0}$ is torsion free, the above exact sequence restricted to the fibre over t_0, remains exact i.e. we have the exact sequence

$$0 \longrightarrow \mathcal{F}_{t_0} \longrightarrow \mathcal{F}_{t_0}^{**} \longrightarrow N_{t_0} \longrightarrow 0.$$

Since $\mathcal{F}^{**}$ is flat over T and $\mathcal{F}_\eta = \mathcal{F}_\eta^{**}$ (η generic point of T), we conclude that Hilbert polynomial of $\mathcal{F}_{t_0}$ = Hilbert polynomial of $\mathcal{F}_{t_0}^{**}$. Since N_{t_0} is of finite length, it follows that $N_{t_0} = 0$, which in turn implies $N = 0$. This shows that $\mathcal{F} = \mathcal{F}^{**}$.

For simplicity we shall assume that the base field k is of characteristic zero, say $k = \mathbb{C}$.

Let $\widehat{\mathcal{O}}_p$ be the completion of the local ring of $\mathcal{X}$ at p. Then since $\mathcal{X}$ has been supposed to be regular, $\widehat{\mathcal{O}}_p = k[[x, y]]$. Besides, it is not difficult to see that x, y can be chosen so that the canonical homomorphism $A \longrightarrow \widehat{\mathcal{O}}_p$ is given by $t \mapsto xy$ ($A \simeq k[[t]]$). The canonical homomorphism $A \longrightarrow B$ ($B \simeq k[[t]]$) is defined by $t \mapsto t^r$ (r a positive integer, for a choice of the uniformising parameter t). Then we see easily that the completion C of the local ring of $\mathcal{X}_T$ at p is of the form

$$C \simeq k[[x, y, t]]/(xy = t^r) \qquad (**)$$

and the canonical homomorphism $B \longrightarrow C$ is defined by $t \mapsto$ the image of t in C. Let $D = k[[u, v]]$. Consider the action of the cyclic group Γ_r operating on D by the action

$$\begin{cases} (u, v) \mapsto (\zeta u, \bar{\zeta} v), \ \zeta \in \Gamma_r \text{ represented by an} \\ r\text{th root of unity, } \bar{\zeta} \text{ being the complex conjugate of } \zeta. \end{cases}$$

Then we have

$$C = D^{\Gamma_r} \ (\Gamma_r \text{ invariants in } D), \ \text{taking } x = u^r, \ y = v^r \ t = uv.$$

Now C is normal with an isolated singularity at p and the representation ($**$) means that this singularity is an ordinary double point of type A (cf. [1]). Let f denote the canonical morphism

$$f : \operatorname{Spec} D \longrightarrow \operatorname{Spec} C \ (C \hookrightarrow D)$$

which induces an etale covering:

$$f_0 : \operatorname{Spec} D \setminus (0) \longrightarrow \operatorname{Spec} C \setminus \{p\}.$$

Consider $f_0^*(\mathcal{F}_C)$ (here $\mathcal{F}_C$ denotes the restriction of $\mathcal{F}_C$ to $\operatorname{Spec} C \setminus \{p\}$). Then it is locally free and extends to a locally free coherent sheaf $\mathcal{F}'$ on $\operatorname{Spec} D$ i.e. $\mathcal{F}'$ is represented by a free D-module of rank n, which we denote again by $\mathcal{F}'$. In fact we have an action of Γ_r on $\mathcal{F}'$ consistent with its action on D (we call $\mathcal{F}'$ a $D - \Gamma_r$ module). It is easy to see that $(\mathcal{F}')^{\Gamma_r}$ is a reflexive C-module and that the restriction of $(\mathcal{F}')^{\Gamma_r}$ to $\operatorname{Spec} C \setminus \{p\}$ can be canonically identified with the restriction of $\mathcal{F}_C$ to $\operatorname{Spec} C \setminus \{p\}$. Now since $\mathcal{F}_C$ is reflexive, it follows that

$$\mathcal{F}_C \simeq (\mathcal{F}')^{\Gamma_r}.$$

It is known (cf. [6]) that a free $(D - \Gamma_r)$ module is associated to a representation of Γ_r i.e. the space of sections over $\operatorname{Spec} D$ of the trivial vector bundle $\operatorname{Spec} D \times V$, where V is a finite dimensional representation of Γ_r. We take the diagonal representation of Γ_r on $\operatorname{Spec} D \times V$ and through this action, the above space of sections acquires a canonical $(D - \Gamma_r)$ module structure. Now a finite dimensional Γ_r representation is a direct sum of 1-dimensional representations. Then if $\dim V = 1$ so that $V \simeq \mathbb{C}$, we see that a Γ_r action on $(\operatorname{Spec} D \times \mathbb{C}) = L$ is given as follows:

$$\zeta \cdot \{(u, v) \times \mathbb{C}\} = (\zeta u, \bar{\zeta} v, \zeta^s \theta), \quad \theta \in \mathbb{C}$$

where we take $\operatorname{Spec} D$ as a 2-dimensional disc with the origin as centre and Γ_r is identified with rth roots of unity. A Γ_r-invariant section of this line bundle L is easily identified with a function F on the disc satisfying the following condition:

$$F(\zeta u, \bar{\zeta} v) = \zeta^s F(u, v).$$

We see easily that the Γ_r-invariant sections of L are generated by u^s and v^{r-s} as a C-module. We have

$$u^{r-s}(u^s, v^{r-s}) = (u^r, (uv)^{r-s}) = (x, t^{r-s})$$

i.e. the C-module $(L)^{\Gamma_r}$ is isomorphic to an ideal in C of the form (x, t^{r-s}). Thus the C-module $\mathcal{F}_C$ is of the following form:

$$\mathcal{F}_C \simeq \bigoplus_{i=1}^{n} (t^{a_i}, x), \quad 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n.$$

We see that (t^{a_i}, x) is principal if and only if $a_i = 0$. Hence we can write

$$\begin{array}{rcl} \mathcal{F}_C = (\mathcal{O}_C^b) & \bigoplus_{i=1}^{n-b} & (t^{a_i}, x), \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_n \\ \| & & \| \\ = \mathcal{F}_1 & \oplus & \mathcal{F}_2. \end{array}$$

Now $\mathcal{F}_1$ is free of rank b. We saw above that Γ_C is the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \operatorname{Gr}(m_1, n)$$

efined by a minimal set of generators of the C-module $\mathcal{F}_C$. Choosing $2(n-b)$ generators of $\mathcal{F}_2$ (and adding the canonical generators of $\mathcal{F}_1$), we see easily that the above

map factorises as follows:

$$
\begin{tikzcd}
\operatorname{Spec} C \setminus \{b\} \arrow[rr] \arrow[drr] & & \mathrm{Gr}\,(m_1, n) \\
& & \mathrm{Gr}\,(2(n-b), (n-b)) \arrow[u, hook]
\end{tikzcd}
$$

Thus for our purpose (studying Γ_C and checking (A_1) above), we can assume without loss of generality that $b = 0$ or equivalently $\mathcal{F}_1 = 0$ i.e.

$$\mathcal{F}_C = \bigoplus_{i=1}^{n} (t^{a_i}, x), \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_n.$$

Then we have $m_1 = 2n$ (minimal number of generators for the C-module $\mathcal{F}_C$) and Γ_C is the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \mathrm{Gr}(2n, n)$$

defined by a minimal number of generators of $\mathcal{F}_C$. Now

$$\mathcal{F}_C = \bigoplus_{k=1}^{n} I_k, \quad I_k = (t^{a_k}, x).$$

Picking up the two generators t^{a_k} and x from each I_k, we see easily that the above rational map factorises as follows:

$$
\begin{tikzcd}
\operatorname{Spec} C - \{p\} \arrow[rr] \arrow[drr] & & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^n \arrow[d, hook] \\
& & \mathrm{Gr}\,(2n, n)
\end{tikzcd}
$$

Let $\alpha_1, \ldots, \alpha_l$ be the distinct ones among the above $\{a_i\}$ occurring with multiplicity $\{\beta_i\}$, $1 \leq i \leq l$. Then the above rational map of $\operatorname{Spec} C \setminus \{p\}$ into $(\mathbb{P}^1)^n$ factorises as follows:

$$
\begin{tikzcd}
\operatorname{Spec} C \setminus \{p\} \arrow[rr] \arrow[drr] & & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ (l \text{ times}) \arrow[d, hook, "\Delta_1"', "\Delta_l"] \\
& & (\mathbb{P}^1)^{\beta_1} \times \cdots \times (\mathbb{P}^1)^{\beta_l} = (\mathbb{P}^1)^n \arrow[d] \\
& & \mathrm{Gr}\,(2n, n)
\end{tikzcd}
$$

where $\{\Delta_i\}$ are the diagonal morphisms of $\mathbb{P}^1$ into $(\mathbb{P}^1)^{\beta_i}$, $1 \leq i \leq l$. Thus we have

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l \tag{$***$}$$

and it is obtained as the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow (\mathbb{P}^1)^l$$

choosing the generators t^{α_k}, x of $I_k = (t^{\alpha_k}, x)$ with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$. We observe that the pull-back of the tautological quotient bundle on $\mathrm{Gr}(2n, n)$ to $(\mathbb{P}^1)^l$ (by the canonical map given above) identifies with

$$\begin{cases} \mathcal{O}(1)^{\beta_1} \oplus \cdots \oplus \mathcal{O}(1)^{\beta_l}, \ \mathcal{O}(1)^{\beta_i} \text{ denotes the line bundle} \\ \text{coming from } i\text{th } \mathbb{P}^1\text{-factor.} \end{cases} \tag{a}$$

We shall now determine the fibre of the morphism $\Gamma_C \longrightarrow \operatorname{Spec} C$ in $(***)$ over the closed point of $\operatorname{Spec} C$ (which is the point p of $\mathcal{X}_T$).

A curve in $\operatorname{Spec} C$ (passing through p) is given by a morphism $\operatorname{Spec} E \longrightarrow \operatorname{Spec} C$, where E is a d.v.r (with residue field the base field), such that if π is a uniformising parameter of E, we have

$$\begin{cases} x = \pi^{\lambda} u, \ y = \pi^{\mu} v, \ t = \pi^{(\lambda+\mu)/r}(uv)^{1/r} \\ u, v \text{ units in } E. \text{ Set } \delta = (\lambda + \mu)/r \text{ (positive integer).} \end{cases}$$

Then through the canonical map $\operatorname{Spec} C \setminus \{p\} \longrightarrow (\mathbb{P}^1)^l$ and the k-th projection of $(\mathbb{P}^1)^l$ onto $\mathbb{P}^1$, we get a map of $\operatorname{Spec} E$ into $\mathbb{P}^1$ given by

$$(\pi^{\delta\alpha_k} *, \pi^{\lambda} u).$$

If $\lambda < \delta\alpha_k$, the image is (1,0). If $\lambda = \delta\alpha_k$, varying the units i.e. the curves, we would get all the points of $\mathbb{P}^1$ in the image. From these considerations it follows easily that the fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$ identifies with the union of $\mathbb{P}^1$'s in $(\mathbb{P}^1)^l$ of the form:

$$\begin{gathered} \{\mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \{(1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \\ \cup \{(1,0) \times (1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \cdots \cup \{(1,0) \times \cdots \times (1,0) \times \mathbb{P}^1\}. \end{gathered} \tag{b}$$

Thus if we denote by F the fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$, we see easily that

$$F_{\mathrm{red}} = \text{a chain } R \text{ of } \mathbb{P}^1\text{'s in } (\mathbb{P}^1)^l \text{ of length } l \text{ of the above form.}$$

Recall that Γ_{C_0} is the base change of $\Gamma_C \longrightarrow \operatorname{Spec} C$ by $\operatorname{Spec} C_0 \longrightarrow \operatorname{Spec} C$ (or the inverse of $\operatorname{Spec} C_0$), where (recall that) C_0 denotes the completion of the local ring at p of the closed fibre of $\mathcal{X}_T \longrightarrow T$. We have

$$\begin{aligned} &\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l \\ &\Gamma_{C_0} \hookrightarrow \operatorname{Spec} C_0 \times (\mathbb{P}^1)^l. \end{aligned}$$

Now the analytic local ring $C_0 \simeq k[[x, y]]/(xy)$ so that $\operatorname{Spec} C_0$ has two smooth components $\operatorname{Spec} E_1$ and $\operatorname{Spec} E_2$, where $\operatorname{Spec} E_1$ (resp. $\operatorname{Spec} E_2$) is defined by $y = 0$ (resp. $x = 0$). Of course as closed subschemes of $\operatorname{Spec} C$, we have $t = 0$ on $\operatorname{Spec} E_1$ and $\operatorname{Spec} E_2$. To find the image of the closed point of $\operatorname{Spec} E_1$ (resp. $\operatorname{Spec} E_2$) in Γ_{C_0}, we observe that

$$\begin{aligned} &(t^{\alpha_i}, x) = (0, x) \text{ on } \operatorname{Spec} E_1 \longrightarrow (0, 1) \\ &(t^{\alpha_i}, x) = ((xy)^{\alpha_i/r}, x) \sim (y^{\alpha_i/r}, x^{(r-\alpha_i)/r}) \longrightarrow (1, 0) \ (r - \alpha_i > 0). \end{aligned}$$

From these we conclude easily that the image $\operatorname{Spec} E_1$ in Γ_{C_0} is represented by

$$\operatorname{Spec} E_1 \times (0,1) \times \cdots \times (0,1)$$

and the image of $\operatorname{Spec} E_2$ in Γ_{C_0} is represented by

$$\operatorname{Spec} E_2 \times (1,0) \times \cdots \times (1,0).$$

This shows that

$$(\Gamma_{C_0})_{\mathrm{red}} \simeq (X_l)_{C_0} \tag{c}$$

and the canonical morphism $(\Gamma_{C_0})_{\mathrm{red}} \longrightarrow \operatorname{Spec} C_0$ identifies with the canonical morphism $(X_l)_{C_0} \longrightarrow \operatorname{Spec} C_0$.

We now see easily that

$$(\Gamma_{C_0})_{\mathrm{red}} = \Gamma_{C_0} \Longleftrightarrow F_{\mathrm{red}} = F.$$

Then if $F = F_{\mathrm{red}}$, by the arguments leading to (A_1), we see that if $\overline{\Gamma}_g$ represents the closed fibre of $\Gamma_g \longrightarrow T$, we have

$$\overline{\Gamma}_g = (\overline{\Gamma}_g)_{\mathrm{red}} \simeq X_l. \tag{d}$$

We will now show that $F_{\mathrm{red}} = F$, which would prove (d). If $l = 1$, this is rather immediate. Suppose then that $l \geq 2$.

Consider the closed immersion (see $(***)$ above)

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l.$$

Let us take the homogeneous coordinates

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_l, Y_l)$$

for $(\mathbb{P}^1)^l$, We see that the following equations hold on Γ_C:

$$\begin{aligned} Y_1 X_2 &= t^{\alpha_2 - \alpha_1} X_1 Y_2 \\ Y_2 X_3 &= t^{\alpha_3 - \alpha_2} X_2 Y_3 \\ &\vdots \\ Y_{l-1} X_l &= t^{\alpha_l - \alpha_{l-1}} X_{\alpha-1} Y_l. \end{aligned}$$

This implies that F (fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$, $F \hookrightarrow (\mathbb{P}^1)^l$) satisfies the following equations:

$$Y_1 X_2 = 0, \quad Y_2 X_3 = 0, \ldots, Y_{l-1} X_l = 0.$$

It is easy to see that F is defined by these equations and that F is reduced. This proves (d).

Let E be the pull-back on Γ_g of the tautological quotient vector bundle of rank n on $\mathrm{Gr}(m,n)$. Then by (a) above, it follows that the restriction of E to $\overline{\Gamma}_g \simeq X_l$ is a standard vector bundle. Besides, by the representation (b) it follows easily that this restriction satisfies the properties of Lemma 2. Hence if $\overline{\pi}_g$ denotes the canonical morphism

$$\overline{\pi}_g : \overline{\Gamma}_g \simeq X_l \longrightarrow X_0 (\simeq \text{ closed fibre of } \mathcal{X}_T \longrightarrow T)$$

we see that $(\overline{\pi}_g)_*(E)$ is torsion free. Then by the arguments of Lemma 4, it follows that $(\pi_g)_*(E)$ is a family of torsion free sheaves on $\mathcal{X}_T$, parametrized by T. Then by an argument similar to proving that $\mathcal{F}$ is reflexive, we see that $(\pi_g)_*(E)$ is reflexive. But $(\pi_g)_*(E)$ and $\mathcal{F}$ coincide outside p. Hence it follows that

$$(\pi_g)_*(E) = \mathcal{F}.$$

Thus all the properties of (A_1) or (A) follow. This completes the proof of the required properness and Proposition 10 follows.

Remark 7. We have the well-known property that there is a desingularisation of $\mathcal{X}_T$ (whose singularity is an ordinary double point of type A) such that it is an isomorphism over $\mathcal{X}_T \setminus \{p\}$ and the fibre over p is a chain of $\mathbb{P}^1$'s (cf. [1]). We have not made use of this fact but it was a motivation for this proof. There is also the plausibility of another proof of properness along the following lines. One sees that to prove the properness of $\mathcal{G}_S$, it suffices to prove the properness of $\mathcal{G}$ i.e. in the case when $S = \operatorname{Spec} k$. For this let F be a family of torsion free sheaves on $X_0 \times T$ (T advr.) such that F is locally free of rank n outside $p \times t_0$ (t_0 closed point of T). It should be possible to write down a fairly explicit form of F in a neighbourhood of $(x_0 \times p)$, since one knows the versal deformation of the stalk of F at $(x_0 \times p)$, considered as a module over the local ring at $(x_0 \times p)$ (cf. [4], [10]). Then choosing a minimal set m_1 of generators of the stalk of F at $(x_0 \times p)$, we consider the canonical rational map of a neighbourhood of $(x_0 \times p)$ (which is a morphism outside $(x_0 \times p)$). Then the graph of this rational map should be proved to have the same properties as was done in the proof of the above proposition.

Now we have the main result of this paper.

Theorem 2. *Let $\mathcal{X} \longrightarrow S$ be a flat family of projective curves as in Definition–Notation 6. Then the Gieseker functor $\mathcal{G}(n,d)_S = \mathcal{G}_S$ (cf. Def. 7) is represented by a scheme $G(n,d)_S$ which is quasi-projective (and flat) over S. The closed fibre of $G(n,d)_S$ over S is a variety with analytic normal crossings. Besides, we have a canonical proper morphism*

$$\pi_* : G(n,d)_S \longrightarrow U(n,d)_S$$

where $U(n,d)_S$ is the relative moduli space of stable torsion free sheaves of $\mathcal{X} \longrightarrow S$ of rank n and deg *d. If $(n,d) = 1$, $G(n,d)_S$ is projective over S (of course it is known in this case that $U(n,d)_S$ is projective over S). Further (since we have assumed that $\mathcal{X}$ as a scheme is regular), $G(n,d)_S$ is regular as a scheme over k.*

Proof. Except the last assertion all the statements in the theorem have been proved above. The construction of the moduli space $G(n,d)_S$ follows on the same lines the proof given for the case $S = \operatorname{Spec} k$, given in the previous section before Def. 4. The proof of the last assertion follows on the same lines as in the work of Gieseker ([4]).

Remark 8. We have made simplifying hypotheses in the construction of the generalized moduli spaces. It should not be difficult to generalize it in the context of a general family $\mathcal{X} \longrightarrow S$ of stable curves and also work out the generalized Gieseker moduli in the semi-stable case (cf. Remark 5).

Remark 9. Let say $(n,d) = 1$. Consider the canonical morphism

$$\pi_* : G(n,d) \longrightarrow U(n,d).$$

Then it can be shown that if the torsion free sheaf $F \in U(n,d)$ on X_0 is of type r at p, then the fibre $(\pi_*)^{-1}(F)$ over F is isomorphic to the wonderful compactification of $PGL(r)$ (in the sense of De Concini and Procesi, cf. [2]). A crucial remark for determining this fibre is (1) of Remark 3 which states that for all vector bundles E on X_l such that $\pi_*(E) = F$ the

restriction of E to X (normalisation of X_0) remains the same. Hence one has to investigate the patching data which extend $E|_X$ to X_l. These will be taken up in a later work.

Remark 10. It seems possible to construct $G(n,d)$ in a rather explicit manner, from the moduli space on X. Gieseker does this when $n=2$.

Appendix: Local theory

We shall now outline a proof of Proposition 8, essentially as in Gieseker (cf. [5]).

I. Let $\mathcal{G}'_S$ be the functor obtained from $\mathcal{G}_S$ by forgetting the imbeddings into Grassmannians, so that an element of $\mathcal{G}'_S(T)$ is represented by a family of curves $\Delta \longrightarrow T$ and a morphism $\Delta \longrightarrow \mathcal{X} \times_S T$ satisfying the condition (ii) of Def. 7. We have a canonical morphism

$$\mathcal{G}_S \longrightarrow \mathcal{G}'_S.$$

We claim that this functor is formally smooth. The proof of this is quite standard. Given an element θ of $\mathcal{G}'_S(T)$ where T is the spectrum of an artin local ring, such that if $\theta_0 \in \mathcal{G}'_S(T_0)$ (T_0 closed subscheme of T defined by an ideal of dimension one) obtained by restricting θ to T_0, can be lifted to an element of $\mathcal{G}_S(T_0)$, then we have to show that θ can be lifted to an element of $\mathcal{G}_S(T)$. Let θ be defined by $\Delta \longrightarrow T$ and $\Delta \longrightarrow \mathcal{X} \times_S T$. Then the lifting of θ_0 to an element of $\mathcal{G}_S(T_0)$ defines a vector bundle E_0 on Δ_0 (Δ_0 – the restriction of Δ to T_0), obtained as the pull-back of the tautological quotient bundle on $\mathrm{Gr}(m,n)$. It is easily seen that the problem is to extend E_0 to a vector bundle E on Δ and then the sections of E_0 to those of E. Let E_1 be the restriction of E_0 to the fibre of $\Delta \longrightarrow T$ over the closed point of T. The obstruction to extending E_0 lies in $H^2(\mathrm{End}\, E_1)$, which is zero. Extending sections is possible, since $H^1(E_1)=0$ (cf. (iv) of Def. 5).

II. We shall hereafter take $S = \mathrm{Spec}\, A$, $A = k[[t]]$. Let $W = \mathrm{Spec}\, k[[t_0, \ldots, t_r]]$ endowed with an S-scheme structure $W \longrightarrow S$, defined by $t \mapsto t_0 \ldots t_r$. A crucial point is the construction of an element θ of $\mathcal{G}'_S(W)$ (cf. Lemma 4.2 [5]) defined by a family of curves $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$, having the following properties:

(a) the closed fibre of $Z \longrightarrow W$ is X_r
(b) the closed subscheme of W corresponding to the singular fibres of $Z \longrightarrow W$ is the union of $t_i = 0$ so that it has normal crossing singularities and is the inverse image of the closed point of S (by the morphism $W \longrightarrow S$).
(c) Let V be an affine open subset of the closed fibre of $Z \longrightarrow W$ containing its singular points (or we can take the semi-local ring at the singular points). Then $Z \longrightarrow W$ defines a deformation $Z_V \longrightarrow W$ of V. The property is that this is an (effective) miniversal deformation of V.
(d) $\psi^*(M) = L$, where L (resp. M) is the dualizing sheaf of Z (resp. $\mathcal{X} \times_S W$) relative to W.

Roughly speaking Z is obtained by taking the base change of $\mathcal{X}$ by $W \longrightarrow S$ and performing certain blow ups.

III. Let $T = \mathrm{Spec}\, B$, B an artin local ring (k = algebra) such that T is also an S-scheme. We take elements of $\mathcal{G}'_S(T)$ represented by $Z' \longrightarrow T$ and $\psi' : Z' \longrightarrow \mathcal{X} \times_S T$ such that the fibre of $Z' \longrightarrow T$ over the closed point of T is the curve X_r. In this way we obtain a functor

$$\mathcal{G}'_r : (\text{Artin } S\text{-schemes}) \longrightarrow \text{sets}.$$

Through the element $\theta \in \mathcal{G}'_S(W)$ in II above, we get a canonical morphism

$$\lambda : W \longrightarrow \mathcal{G}'_r. \tag{i}$$

Gieseker shows that this morphism is formally smooth (cf. Proposition 4.5 and its proof, [5]). In other words suppose that we are given an element $\delta \in \mathcal{G}'_r(T)$ defined by $Z' \longrightarrow T$ and $\psi' : Z' \longrightarrow \mathcal{X} \times_S T$. We suppose further that there is a morphism $\phi_0 : T_0 \longrightarrow W$ such that the pull-backs by ϕ_0 of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ coincide with the restriction $\delta_0 \in \mathcal{G}'_r(T_0)$ of δ to T_0 (T_0 closed subscheme of T as above). Then it is shown that ϕ_0 can be extended to a morphism $\phi : T \longrightarrow W$ such that the pull-backs of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ are isomorphic to $Z' \longrightarrow T$ and ψ'.

The proof can be sketched as follows. Given Z', ψ' and ϕ_0, we can find a morphism $\phi : T \longrightarrow W$ such that the pull-back (Z_1, ψ_1) of (Z, ψ) by ϕ is isomorphic to (Z', ψ') over T_0 (i.e. the restrictions to T_0 are isomorphic); besides (Z_1, ψ_1) and (Z', ψ') are locally isomorphic over T. This latter fact is a consequence of (c) of II. Given these local isomorphisms (whose restrictions to T_0 define the given isomorphism of (Z_1, ψ_1) with (Z', ψ') over T_0), we find the obstruction to extending these local isomorphisms to a global one over T is an element μ

$$\mu \in H^1(X_r, \mathrm{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})),$$

where $\Omega^1_{X_r}$ denotes the sheaf of differentials and Hom denotes the "sheaf Hom". Similarly, the obstruction to extending

$$\psi'_0 : Z'_0 \longrightarrow \mathcal{X} \times_S T_0$$

to a morphism $Z' \longrightarrow \mathcal{X} \times_S T$ is an element μ'

$$\mu' \in H^1(X_r, \mathrm{Hom}(\pi * \Omega^1_{X_r}, \mathcal{O}_{X_r}))$$

and we see that μ maps to μ' under the canonical homomorphism

$$H^1(X_r, \mathrm{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow H^1(X_r, \mathrm{Hom}(\pi^* \Omega^1_{X_0}, \mathcal{O}_{X_r})), \tag{ii}$$

where π is the canonical morphism $X_r \longrightarrow X_0$. But since ψ' extends ψ'_0 we see that $\mu' = 0$. It is shown that (ii) is injective (Cor. 4.4, [5]) so that $\mu = 0$ and we get the required isomorphism of (Z', ψ') with (Z_1, ψ_1). Thus we see that λ is formally smooth.

One can view $\mathcal{G}'_r$ as a functor

$$\mathcal{G}'_r : (\text{Artin}\, k\text{-schemes}) \longrightarrow \text{sets},$$

i.e. if T is an artin k-scheme, in the definition of an element of $\mathcal{G}'_r(T)$ we take also an S-scheme structure for T. Then we see that an element of $\mathcal{G}'_r(T)$ gives a deformation of the singularities of X_r; to be more precise we get a deformation of V (V as in (c) of II above). Thus we get a functor

$$\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V). \tag{iii}$$

Then by (c) of II above, we see that the dimension d of the k-linear space of first order deformations of $\mathcal{G}'_r$ is $\geq (r+1)$. On the other hand by the formal smoothness of λ in (i) above, we see that $d \leq (r+1)$. Hence $d = (r+1)$ and we conclude that the functor λ (as well as the functor in (iii) above) is an isomorphism. Note that we have

$$\begin{cases} \text{the pull-back by } \psi' \text{ of the dualizing sheaf of } \mathcal{X} \times_S T \\ \text{is the dualizing sheaf of } Z' \text{ (all relative to } T). \end{cases} \tag{iv}$$

IV. Let h be a closed point of $H = \mathrm{Hilb}^{P_1}(\mathcal{X} \times_S \mathrm{Gr}(m,n)$ such that $h \in \mathcal{G}_S(k)$. We have a curve X_r associated to h. Let $T = \mathrm{Spec}\, B$, B an artin local ring (with residue field k) such that T is also an S-scheme. We take elements of $\mathcal{G}_S(T)$ (resp. $H(T)$), represented by $\Delta \hookrightarrow \mathcal{X} \times_S T \times_S \mathrm{Gr}(m,n)$ such that the canonical image $\mathcal{G}_S(k)$ in $\mathcal{G}_S(T)$ is h. Then the fibre of $\Delta \longrightarrow T$ over the closed point of T is the curve X_r. In this way we obtain functors

$$\mathcal{G}_h(\text{resp. } H_h) : (\text{Artin } S\text{-scheme}) \longrightarrow \text{sets}.$$

We observe also that

$$\mathcal{G}_h = H_h \,(\text{since } \mathcal{G}_h(T) = H_h(T)). \tag{i}$$

Then we get a canonical functor $\mathcal{G}_h \longrightarrow \mathcal{G}'_r$, which is formally smooth by I. By (III), we have

$$\mathcal{G}'_r \xrightarrow{\sim} \mathrm{Def}(V) \simeq W. \tag{ii}$$

If we assume now that $\mathcal{G}_S$ is represented by an open subscheme $\mathcal{Y}$ of H, then by all the above considerations, we see that $\mathcal{Y}$ is smooth over k and $\mathcal{Y}_{s_0}$ (fibre over the closed point s_0 of S) has normal crossing singularities.

We denote by

$$\begin{cases} p_H : \Delta_H \longrightarrow H, \text{ the universal family over } H, \text{ and } \psi_H \\ \text{the canonical morphism } \psi_H : \Delta_H \longrightarrow \mathcal{X} \times_S H = \mathcal{X}_H. \end{cases} \tag{iii}$$

Let $\mathcal{O}$ be the local ring of H at h. We write $C = \mathrm{Spec}\, \mathcal{O}$ and $C_n = \mathrm{Spec}\, \mathcal{O}/m^n$ (m maximal ideal of $\mathcal{O}$).

Let

$$\begin{cases} p_C : \Delta_C \longrightarrow C, \; \psi : \Delta_C \longrightarrow \mathcal{X}_C. \\ p_n : \Delta_{C_n} \longrightarrow C_n, \; \psi : \Delta_{C_n} \longrightarrow \mathcal{X}_{C_n} \end{cases} \tag{iv}$$

denote the base changes of (iii) by the canonical morphism $C \longrightarrow H$, $C_n \longrightarrow H$. Since the fibre of ψ_H over the closed point of C is X_r, by the deformation theory of ordinary double points, we see that the fibres of p_C have only ordinary double point singularities. Let L (resp. M) denote the dualizing sheaf (in our case a line bundle) of Δ_C (resp. $\mathcal{X}_C$) relative to C. We denote the base changes of L and M by the morphism $C_n \longrightarrow C$ by L_n and M_n respectively. Note that L_n (resp. M_n) is the dualizing sheaf of Δ_{C_n} (resp. $\mathcal{X}_{C_n}$) relative to C_n (cf. [3]). Then by (iv) of (III), we have

$$\psi_n^*(M_n) = L_n \quad \text{for all } n. \tag{v}$$

We claim that

$$\psi_C^*(M) \simeq L. \tag{vi}$$

To prove (vi), set $N = L^{-1} \otimes \psi_C^*(M)$ and $N_n = L_n^{-1} \otimes \psi_n^*(M_n)$. Now N_n are trivial line bundles. Then by Grothendeick's comparison theorems, we have

$$\widehat{(p_C)_*(N)} = \varprojlim_n (p_n)_*(N_n) \tag{vii}$$

where the LHS denotes the completion of the direct image $(p_C)_*(N)$ considered as an $\mathcal{O}$-module. Since N_n are trivial, we see that the RHS of (vii) is $\simeq \hat{\mathcal{O}}$. Then the restriction of N to the generic fibre of p_C has a non-trivial section. Applying the semi-continuity theorem, the space of sections of N restricted to any fibre of p_C is 1-dimensional, which

implies that "$(p_C)_*$ commutes with base change". From this one concludes easily that there is a section s of N which does not vanish identically on the closed fibre of p_C. Since the restriction of N to the closed fibre of p_C is trivial, we see that s is, in fact, everywhere non-zero on Δ_C. This shows that N is trivial and proves the claim (vi).

From the above considerations, we see easily that there is a neighbourhood U of h in H such that for the morphisms

$$p_U : \Delta_U \longrightarrow U, \psi_U : \Delta_U \longrightarrow \mathcal{X}_U \tag{viii}$$

obtained as base change of p_H and ψ_H by $U \longrightarrow H$, we have the following properties:

$$\left\{\begin{array}{l}\text{The fibres of } p_U \text{ have only ordinary double point singularities.}\\ \text{Besides } \psi_U^*(M_U) = L_U, \text{ where } L_U \text{ (resp.) is the dualizing}\\ \text{sheaf of } \Delta_U \text{ (resp. } \mathcal{X}_U) \text{ relative to } U.\end{array}\right. \tag{ix}$$

We claim that (ix) implies that the fibres of p_U are smooth or curves of the form X_n and ψ_U induces the canonical morphism on fibres over U i.e. the property (ii) of Def. 7 is satisfied. The claim is a consequence of the following:

Lemma. Let Y be a connected projective curve of arithmetic genus g with only ordinary double point singularities. Let $f : Y \longrightarrow D$ be a morphism, where either D is a smooth projective curve of genus g or $D \simeq X_0$. Suppose that the pull-back by f of the dualizing sheaf of D is isomorphic to the dualizing sheaf of Y. Then f is an isomorphism if D is smooth; otherwise Y is of the form X_n and f is the canonical morphism $X_n \longrightarrow X_0$.

The above lemma is an easy consequence of the characterization of the dualizing sheaf of Y by a sheaf of meromorphic differentials on the normalization of Y (cf. [3]).

Thus it follows that

$$\Delta_U \longrightarrow U \quad \text{and} \quad \psi_U : \Delta_U \longrightarrow \mathcal{X}_U \tag{x}$$

satisfy the property (ii) of Def. 7 define open subsets of H so that we can suppose that the morphisms in (x) satisfy all the conditions of Def. 7. Thus it follows that $\mathcal{G}_S$ is represented by an open subscheme $\mathcal{Y}$ of H.

Thus to conclude the proof of Proposition 8, it remains to prove that $\mathcal{Y}_{s_0}$ is irreducible. Since $\mathcal{Y}_{s_0}$ has normal crossing singularities, it follows that the open subset $\mathcal{Y}_1$ of smooth points of $\mathcal{Y}_{s_0}$ is dense in $\mathcal{Y}_{s_0}$. It suffices to prove that $\mathcal{Y}_1$ is irreducible. Then $\mathcal{Y}_1$ can be identified with an open subset of the open subscheme R of the quot scheme $Q(m, P_2)$. One knows that R is irreducible and the required irreducibility follows.

Remark. Let Def (X_r) denote the functor of deformations of X_r. Then we have morphisms

$$\mathcal{G}'_r \xrightarrow{i_1} \text{Def}\,(X_r) \xrightarrow{j_1} \text{Def}\,(V) \tag{i}$$

where i defines a subfunctor. Recall that the first order deformations of X_r can be identified with $\text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})$ and we have an exact sequence (cf. [3]).

$$\begin{aligned}&0 \longrightarrow H^1(X_r, \text{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow \text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})\\ &\xrightarrow{j_2} H^0(X_r, \text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow 0.\end{aligned} \tag{ii}$$

Now j_2 can be identified with the canonical map of the first order deformations induced by j_1. The above considerations show that the first order deformations of $\mathcal{G}'_r$ can be identified with a supplement of $\ker j_2$.

One can arrange the above proof of Proposition 8 slightly differently as follows. The argument in IV above shows that if N denotes the subspace of $\mathrm{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})$ corresponding to the first order deformations of $\mathcal{G}'_r$, we have $N \cap \ker j_2 = (0)$. This shows that the linear map on first order deformations induced from the canonical morphism $\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V)$ is injective, in particular $\dim N \leq (r+1)$. On the other hand, as we saw before, by (c) of II above, $\dim N \geq (r+1)$. We then easily conclude that $\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V)$ is an isomorphism i.e. $\mathcal{G}'_r \longrightarrow W$ is an isomorphism, which would prove Proposition 8.

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On a new mean and functions of bounded deviation

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Abstract. The main purpose of the present paper is to introduce a new type of mean of a function and to study some of its special properties. In the last section we make use of this mean to show that a function of bounded deviation is not necessarily a function of bounded variation.

Keywords. Fractional integral mean; bounded deviation; bounded variation; bounded variation in the mean.

1. Preliminaries

First we recall some standard notations for several of the function classes of interest.

$L^1 = L^1\,[0, 2\pi]$ is the space of 2π-periodic and Lebesgue integrable functions over $[0, 2\pi]$,
$L^\infty = L^\infty\,[0, 2\pi]$ consists of these L^1 functions which are essentially bounded,
$\mathrm{BV} = \mathrm{BV}\,[0, 2\pi]$ consists of those L^1 functions which are of bounded variation on $[0, 2\pi]$.

Bounded deviation (*BD*) (see [7], p. 229; [3]). The class BD of functions of bounded deviation is the subset of L^1 consisting of those functions f such that a relation

$$\left| n \int_a^b f(t)\mathrm{e}^{-int}\mathrm{d}t \right| \leq C \tag{1.1}$$

holds, where C is a constant independent of the sub-interval (a, b) of $[0, 2\pi]$. If in addition f is even, the interval $[0, 2\pi]$ in the above definition can be replaced by $[0, \pi]$.

Bounded variation in the mean (V_1). V_1 denotes the subset of L^1 consisting of those functions f for which there exists a constant Q such that if $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)$ is any set of non-overlapping intervals in $[0, 2\pi]$ then

$$\int_0^{2\pi} \left| \sum_{k=1}^{m} [f(\beta_k - t) - f(\alpha_k - t)] \right| \mathrm{d}t \leq Q. \tag{1.2}$$

In [3] it is shown that a function belongs to V_1, if and only if, its sequence of Fourier co-efficients is $0(\frac{1}{n})$. Obviously if a function belongs to BD then its Fourier co-efficients satisfy this order relation and hence $\mathrm{BD} \subseteq V_1$. Also it is easy to see that $\mathrm{BV} \subseteq \mathrm{BD}$. It has been observed by Zygmund ([7], p. 229) that a function of bounded deviation is not necessarily a function of bounded variation.

Introducing a new mean. Let $f \in L^1$. Let the Fourier series of f at $t = x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.3}$$

We write

$$\phi(t) = \phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$
$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\mathrm{d}u, \quad \alpha > 0$$
$$\Phi_0(t) = \phi(t),$$
$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t).$$

$\Phi_\alpha(t)$ and $\phi_\alpha(t)$ are respectively known as the α-th integral and α-th integral mean of $\phi(t)$.

As far as the authors are aware, writing $P(t) = \phi(t) - \phi_1(t)$, it was Chandra [4] who first studied some properties of $P(t)$. He made extensive use of this mean (see [4], [5], [6]) for studying the summability and convergence problems of Fourier series. Chandra's work on $P(t)$ motivated us to introduce a new mean as follows:

We define

$$p(k,t) = \sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}\phi_\nu(t), \quad k \in N$$
$$p(0,t) = \phi(t),$$
$$p(1,t) = P(t), \text{ the mean studied by Chandra.}$$

At this stage we wish to acknowledge that our present work has been greatly influenced by the works of Prem Chandra (see [4], [5], [6]).

Let $\alpha \geq 0$, we write $G_\alpha(k,t)$ and $g_\alpha(k,t)$ respectively for the α-th integral and α-th integral mean of $p(k,t)$.

Purpose of the present work. In § 2 of the present paper we first study some special properties of the mean $p(k,t)$. In § 3 we apply Chandra's mean (that is $p(k,t)$ with $k = 1$) to give another demonstration of the known fact ([7], p. 229) that BD(a,b) is not contained in BV(a,b).

2. Properties of $p(k,t)$

We prove the following

Theorem 1. *For each $k \in N$*

(a) $\phi(t) \in \mathrm{BV}[0,\pi] \Leftrightarrow p(k,t) \in \mathrm{BV}[0,\pi]$ *and* $t^{-1}p(k,t) \in L(0,\pi)$
(b) $p(k,t) \in \mathrm{BV}[0,\pi] \Rightarrow p(k+1,t) \in \mathrm{BV}[0,\pi]$
(c) $t^{-1}p(k,t) \in L[0,\pi] \Rightarrow t^{-1}p(k+1,t) \in L[0,\pi]$.

Remarks. The case $k = 1$ of part (a) is due to Chandra [4]. For $k = 0$, the results at (b) and (c) are special cases of known results (see Lemma 1).

We need the following.

Lemma 1. *For* $\beta > \alpha \geq 0$

(i) [1] $\phi_\alpha(t) \in \mathrm{BV}[0, \pi] \Rightarrow \phi_\beta(t) \in \mathrm{BV}[0, \pi]$

(ii) [2] $\phi_\alpha(t)/t \in L[0, \pi] \Rightarrow \phi_\beta(t)/t \in L[0, \pi]$.

Proof of Theorem 1. Part (a) By a simple computation, we get

$$\mathrm{d}(t^k p(k, t)) = t^k \mathrm{d}\phi(t) \tag{2.1}$$

which further ensures that

$$\mathrm{d}\phi(t) = kt^{-1}p(k, t) + \mathrm{d}p(k, t). \tag{2.2}$$

By Lemma 1(i), $p(k, t) \in \mathrm{BV}[0, \pi]$ for every $k \in N$ whenever $\phi(t) \in \mathrm{BV}[0, \pi]$ and hence part (a) follows from (2.2).

Part (b) and (c) For any $k \in N$, we have

$$\begin{aligned} p(k+1, t) &= \sum_{\nu=0}^{k+1} (-1)^\nu \left(\binom{k}{\nu} + \binom{k}{\nu - 1} \right) \phi_\nu(t), \\ &= p(k, t) - \sum_{s=0}^{k} (-1)^s \binom{k}{s} \phi_{s+1}(t). \end{aligned} \tag{2.3}$$

Integrating by parts k times in succession, we get

$$\begin{aligned} \int_0^t u^k p(k, u)\mathrm{d}u &= \sum_{s=0}^{k} (-1)^s \binom{k}{s} s!\, t^{k-s} G_{s+1}(k, t), \\ &= t^{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} (s+1)^{-1} g_{s+1}(k, t). \end{aligned} \tag{2.4}$$

In the like manner, we have

$$\begin{aligned} \int_0^t u^{k-\nu} \Phi_\nu(u)\mathrm{d}u &= \sum_{s=0}^{k-\nu} (-1)^s \binom{k-\nu}{s} s!\, t^{k-\nu-s} \Phi_{\nu+s+1}(t) \\ &= t^{k+1} \sum_{s=\nu}^{k} (-1)^{s-\nu} \frac{(k-\nu)!}{(k-s)!(s+1)!} \phi_{s+1}(t). \end{aligned} \tag{2.5}$$

On the other hand using (2.5) in the left side of (2.4), we obtain

$$\begin{aligned} \int_0^t u^k p(k, u)\mathrm{d}u &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \nu! \int_0^t u^{k-\nu} \Phi_\nu(u)\mathrm{d}u \\ &= t^{k+1} \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \nu! \sum_{n=\nu}^{k} (-1)^{s-\nu} \frac{(k-\nu)!}{(k-s)!(s+1)!} \phi_{s+1}(t) \\ &= t^{k+1} \sum_{\nu=0}^{k} \sum_{s=\nu}^{k} (-1)^s \binom{k}{s} (s+1)^{-1} \phi_{s+1}(t) \end{aligned}$$

$$= t^{k+1}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\phi_{s+1}(t)$$

$$= t^{k+1}[p(k,t)-p(k+1,t)], \tag{2.6}$$

using (2.3).

Now, combining the results of (2.4) and (2.6), we get

$$p(k+1,t)=p(k,t)-\sum_{s=0}^{k}(-1)^s\binom{k}{s}(s+1)^{-1}g_{s+1}(k,t). \tag{2.7}$$

As $g_{s+1}(k,t)$ is the $(s+1)$-th integral mean of $p(k,t)$ by Lemma 1, we obtain for $s=0,1,2,\ldots$

(A) $p(k,t)\in \mathrm{BV}[0,\pi]\Rightarrow g_{s+1}(k,t)\in \mathrm{BV}[0,\pi]$
(B) $t^{-1}p(k,t)\in L[0,\pi]\Rightarrow t^{-1}g_{s+1}(k,t)\in L[0,\pi]$.

Hence parts (b) and (c) follow at once from (2.7).

COROLLARY

(I) *Let r and s be any pair of positive integers. Then* $\phi(t)\in \mathrm{BV}[0,\pi]\Leftrightarrow p(r,t)\in \mathrm{BV}[0,\pi]$ *and* $p(s,t)/t\in L[0,\pi]$
(II) *Let* $p(k,t)\in \mathrm{BV}[0,\pi]$ *for some* $k\in N$. *Then for* $r,s\in N\, t^{-1}p(r,t)\in L[0,\pi]\Leftrightarrow t^{-1}p(s,t)\in L[0,\pi]$
(III) *Let* $t^{-1}p(k,t)\in L[0,\pi]$ *for some* $k\in N$. *Then for* $r,s\in N\, p(r,t)\in \mathrm{BV}[0,\pi]\Leftrightarrow p(s,t)\in \mathrm{BV}[0,\pi]$.

Proof of (I). By Theorem 1(a), 1(b) and 1(c) for any $r,s\in N\,\phi(t)\in \mathrm{BV}[0,\pi]\Rightarrow p(r,t)\in \mathrm{BV}[0,\pi]$ and $t^{-1}p(s,t)\in L[0,\pi]$. Let $m=\max(r,s)$. By Theorem 1(b) and (c), we have

$$p(r,t)\in \mathrm{BV}[0,\pi]\Rightarrow p(m,t)\in \mathrm{BV}[0,\pi]$$

and

$$t^{-1}p(s,t)\in L[0,\pi]\Rightarrow t^{-1}p(m,t)\in L[0,\pi].$$

Hence by Theorem 1(a)

$$p(r,t)\in \mathrm{BV}[0,\pi] \quad\text{and}\quad t^{-1}p(s,t)\in L[0,\pi]\Rightarrow \phi(t)\in \mathrm{BV}[0,\pi]$$

and this completes the proof of (I).

Clearly (II) and (III) follow from (I) and Theorem 1(a).

The conditions imposed on $p(k,t)$ in Theorem 1(a) are mutually exclusive one being of the Jordan type and other one being of the Dini type. Hence the bounded variation of $p(k,t)$ need not ensure the bounded variation of $\phi(t)$.

3. BD class strictly includes BV class

We prove

Theorem 2. *There exists a function in BD class which is not necessarily in BV class.*

We need the following.

Lemma 2. *There exists a function f such that*

(i) $f \in L^{\infty}$,
(ii) $p(1,t) \in \mathrm{BV}[0,\pi]$,
(iii) *for $n \in N$ and $0 < 1/n < c \leq \pi$ there is a positive constant B independent of n and c so that*

$$\left| \int_{1/n}^{c} \frac{p(1,t)}{t} \mathrm{d}t \right| \leq B,$$

(iv) $\lim \int_{1/n}^{c} (p(1,u)/u)\,\mathrm{d}u$ *exists an $n \to \infty$ for every c with $0 < c \leq \pi$, and*
(v) $p(1,t)/t \notin L(0,\pi)$.

Proof. If f is an even function and $x = 0$, then $\phi(t) = f(t)$. Let $0 < \epsilon < 1/2$ and let us write

$$h(t) = \frac{\cos\sqrt{\log\frac{2\pi}{t}}}{t(\log\frac{2\pi}{t})^{(1/2)+\epsilon}}, \quad 0 < t \leq \pi, \text{ and}$$

$$H(t) = \int_0^t h(u)\mathrm{d}u, \quad 0 < t \leq \pi.$$

We define

$$f(t) = \phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}[tH(t)], \quad 0 < t \leq \pi$$

and elsewhere by periodicity, as an even function.

Then $\Phi_1(t) = tH(t)$, $\phi_1(t) = H(t)$ and

$$p(1,t) = t\,h(t) = \frac{\cos\sqrt{\log\frac{2\pi}{t}}}{(\log\frac{2\pi}{t})^{(1/2)+\epsilon}}, \quad 0 < t \leq \pi.$$

It is easy to verify that f satisfies conditions (i)–(v) of the lemma.

Proof of Theorem 2. Let f be an even 2π-periodic function and $x = 0$. Then $f(t) = \phi(t)$. Suppose that f satisfies (i)–(v) of Lemma 2.

By Theorem 1(a) conditions (i) and (v) taken together ensure that $\phi(t) \notin \mathrm{BV}[0,\pi]$. Now it remains to prove that

$$\left| \int_a^b \phi(t)\mathrm{e}^{-int}\mathrm{d}t \right| \leq C/n \tag{3.1}$$

for every subinterval (a,b) in $[0,\pi]$ where C is some positive constant independent of a and b.

For every c with $0 < c \leq \pi$, we have

$$\int_0^c \phi(t)\frac{1-\mathrm{e}^{-int}}{t}\mathrm{d}t = \frac{\Phi_1(c)}{c}(1-\mathrm{e}^{-inc}) - \int_0^c \Phi_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1-\mathrm{e}^{-int}}{t}\right]\mathrm{d}t$$

$$
\begin{aligned}
&= \phi_1(c)(1-\mathrm{e}^{-inc}) - \int_0^c t\phi_1(t)\left[\frac{\mathrm{in}\,\mathrm{e}^{-int}}{t} - \frac{1-\mathrm{e}^{-int}}{t^2}\right]\mathrm{d}t \\
&= \phi_1(c)(1-\mathrm{e}^{-inc}) - \mathrm{in}\int_0^c \phi_1(t)\mathrm{e}^{-int}\mathrm{d}t \\
&\quad + \int_0^c \phi_1(t)\left(\frac{1-\mathrm{e}^{-int}}{t}\right)\mathrm{d}t
\end{aligned}
$$

which ensures that

$$
\begin{aligned}
\int_0^c \phi(t)\mathrm{e}^{-int}\mathrm{d}t &= \int_0^c p(1,t)\mathrm{e}^{-int}\mathrm{d}t - \int_0^c p(1,t)\frac{1-\mathrm{e}^{-int}}{\mathrm{in}\,t}\mathrm{d}t + \phi_1(c)\left(\frac{1-\mathrm{e}^{-inc}}{\mathrm{in}}\right) \\
&= I - \frac{1}{\mathrm{in}}J + \frac{1}{\mathrm{in}}K, \quad \text{say.}
\end{aligned}
\tag{3.2}
$$

We write

$$
\begin{aligned}
J &= \int_0^{1/n} p(1,t)\frac{1-\mathrm{e}^{-int}}{t}\mathrm{d}t + \int_{1/n}^c \frac{p(1,t)}{t}\mathrm{d}t - \int_{1/n}^c p(1,t)\frac{\mathrm{e}^{-int}}{t}\mathrm{d}t \\
&= J_1 + J_2 - J_3, \quad \text{say.}
\end{aligned}
\tag{3.3}
$$

As $p(1,t)\in \mathrm{BV}[0,\pi]$ there exists a positive constant C_1 so that $|p(1,t)|\le \frac{1}{2}C_1$ for all $0<t\le\pi$ and hence

$$
\begin{aligned}
|J_1| &\le \frac{1}{2}C_1\int_0^{1/n}\frac{|1-\mathrm{e}^{-int}|}{t}\mathrm{d}t \\
&\le C_1 n\int_0^{1/n}\mathrm{d}t = C_1.
\end{aligned}
\tag{3.4}
$$

As $p(1,t)$ satisfies condition (iii) there exists a positive constant C_2 independent of c so that

$$
|J_2|\le C_2 \text{ for all } n\in N \text{ and } 0<c\le\pi. \tag{3.5}
$$

Now using the familiar technique employed in proving the convergence tests ([7], p. 5 and p. 59) for Fourier series (Jordan's test) and conjugate series (analogue of Jordan's test) it can be proved that

$$
|J_3|\le C_3, \tag{3.6}
$$

where C_3 is a constant independent of c.

By routine argument $p(1,t)\in\mathrm{BV}[0,\pi]$ implies that

$$
|I|\le C_4/n, \tag{3.7}
$$

where C_4 is a constant independent of c. Lastly since $f\in L^1$ there exists a positive constant C_5 independent of c such that

$$
|K|\le C_5. \tag{3.8}
$$

Collecting the results of (3.2)–(3.8), we have

$$
\left|\int_0^c \phi(t)\mathrm{e}^{-int}\mathrm{d}t\right|\le A/n, \tag{3.9}
$$

where A is a constant independent of c. As we can write

$$\int_a^b \phi(t)e^{-int}dt = \int_0^b \phi(t)e^{-int}dt - \int_0^a \phi(t)e^{-int}dt$$

the inequality (3.1) follows by an appeal to (3.9) and this completes the proof of Theorem 2.

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Nonlinear boundary value problems

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Abstract. In this paper we consider two quasilinear boundary value problems. The first is vector valued and has periodic boundary conditions. The second is scalar valued with nonlinear boundary conditions determined by multivalued maximal monotone maps. Using the theory of maximal monotone operators for reflexive Banach spaces and the Leray–Schauder principle we establish the existence of solutions for both problems.

Keywords. Monotone operator; maximal monotone operator; demicontinuous operator; weakly coercive operator; surjective operator; periodic problem; Leray–Schauder principle; Sobolev space; compact embedding.

1. Introduction

In this paper we study the following two problems:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) = x(b),\ x'(0) = x'(b),\ 2 \le p < \infty \end{aligned} \tag{1}$$

and

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x'(0) \in \xi_1(x(0)),\ -x'(b) \in \xi_2(x(b)),\ 2 \le p < \infty. \end{aligned} \tag{2}$$

We shall study problem (1) in R^N and problem (2) in R. In problem (2) ξ_1, ξ_2 are two maximal monotone graphs in R^2. Recently quasilinear ordinary differential equations were studied by Boccardo–Drabek–Giachetti–Kucera [2], Drabek [4], DelPino–Elqueta–Manasevich [3], Guo [6] and Zhang [16]. With the exception of Zhang, these works deal with the scalar equation. Boccardo *et al*, Drabek, DelPino–Elqueta–Manasevich and Zhang study the Dirichlet problem and the first three assume that the function f is independent of the derivative x'. In this case the one dimensional p–Laplacian $A(x) = -(|x'|^{p-2}x')'$ is invertible and its inverse is compact from $C(T)$ into itself. So the Leray–Schauder degree theory can be applied. Guo considers the periodic and Neumann problems and in the Neumann problem f is independent of the derivative x'. For these two problems the quasilinear differential operator is not invertible and so some generalized degree theory, such as coincidence theory of Mawhin is needed. Note that these works (except Zhang [16]) assume that the vector field f is continuous in all variables (including the time variable). The semilinear case (i.e. $p = 2$) is covered in the well written books of Gaines–Mawhin [5] and Mawhin [11] where the interested reader can find a

comprehensive account of the coincidence degree theory and in the papers of Knobloch [9] and Mawhin [10]. In the same vein we should also mention the relevant works of Mawhin–Ward [12], [13] which deal with the Lienard and Duffing equations using nonresonance hypotheses. Our approach here is different and relies heavily on the theory of maximal monotone operators.

2. Preliminaries

Since our approach is based on the theory of maximal monotone operators, in this section we recall some basic definitions and facts from this theory. Details can be found in Zeidler [15].

Let X be a reflexive Banach space and X^* its topological dual. A possibly multivalued map $A : D \subseteq X \to 2^{X^*}$ is said to be "monotone", if for any $x, y \in D$ $(x^* - y^*, x - y) \geq 0$ holds for all $x^* \in A(x)$ and $y^* \in A(y)$. Here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X^*, X). When $(x^* - y^*, x - y) = 0$ implies that $x = y$, then we say that A is "strictly monotone". A monotone map for which the inequalities $(x^* - y^*, x - y) \geq 0$ for all $[y, y^*] \in \mathrm{Gr}A$, imply $[x, x^*] \in \mathrm{Gr}A$ is said to be "maximal monotone". Here by $\mathrm{Gr}A$ we denote the graph of A, i.e. $\mathrm{Gr}A = \{[y, y^*] \in X \times X^* : y^* \in A(y)\}$. From this definition it follows that A is maximal monotone if its graph is maximal with respect to inclusion among the graphs of monotone maps (i.e. $\mathrm{Gr}A$ is not properly included in the graph of another monotone map). When $A(\cdot)$ is maximal monotone, then for every $x \in D$, $A(x)$ is closed and convex in X^* and the set $\mathrm{Gr}A$ is demiclosed, i.e. if $x_n \to x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* as $n \to \infty$ (or $x_n \xrightarrow{w} x$ in X, $x_n^* \to x^*$ in X^* as $n \to \infty$) and $x_n^* \in A(x_n)$, $n \geq 1$, then $x^* \in A(x)$. Now let $D = X$ and $A : X \to X^*$ be single valued. We say that $A(\cdot)$ is "demicontinuous" if it is sequentially continuous from X into X_w^*, where X_w^* denotes the dual space X^* furnished with the weak topology (i.e. if $x_n \to x$ in X, then $A(x_n) \xrightarrow{w} A(x)$ in X^* as $n \to \infty$). A monotone, demicontinuous map is maximal monotone. A map $A : D \subseteq X \to 2^{X^*}$ is said to be "weakly coercive", if D is bounded in X or D is unbounded and $\inf[\|x^*\|_* : x^* \in A(x)] \to \infty$ as $\|x\| \to \infty$, $x \in D$ (here by $\|\cdot\|$ and $\|\cdot\|_*$ we denote the norms of X and X^* respectively). A maximal monotone, weakly coercive $A : D \subseteq X \to 2^{X^*}$ is surjective, i.e. $R(A) = X^*$. In particular a monotone, demicontinuous and weakly coercive map $A : X \to X^*$ is surjective.

Our method of proof will produce the solution via a fixed point argument based on the Leray–Schauder theorem, which for the convenience of the reader, we recall here (see Mawhin [11]):

Theorem 1. *If Y is a Banach space, $K : Y \to Y$ is compact and there exists $r > 0$ such that if $y = \mu K(y)$ with $0 < \mu < 1$, we have $\|y\| \leq r$ (a priori bound), then K has a fixed point, i.e. there exists $y \in Y$ such that $y = K(y)$.*

Finally in the sequel we will use the following elementary inequality

$$2^{2-p}|a - c|^p \leq (|a|^{p-2}a - |c|^{p-2}c)(a - c)$$

for all $a, c \in R$ and all $2 \leq p < \infty$.

3. Periodic vector problem

In this section we prove an existence theorem for problem (1). For this purpose we will need the following hypotheses on $f(t, x, y)$.

$\mathrm{H(f)}_1$: $f : T \times R^N \times R^N \to R^N$ is a function such that

(i) for every $x, y \in R^N$, $t \to f(t,x,y)$ is measurable;
(ii) for almost all $t \in T$, $(x,y) \to f(t,x,y)$ is continuous;
(iii) for almost all $t \in T$ and all $x, y \in R^N$, we have

$$(f(t,x,y),x)_{R^N} \geq -a\|x\|^p - \beta\|x\|^r\|y\|^{p-r} - c(t)\|x\|^s$$

with $a, \beta \geq 0$, $1 \leq r, s < p$ and $c \in L^1(T)_+$;
(iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{R^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$ we have

$$\inf\left[(f(t,x,y),x)_{R^N} + \|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta\right] \geq c;$$

(v) for almost all $t \in T$ and all $x, y \in R^N$, we have

$$\|f(t,x,y)\| \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\|y\|^{p-1}$$

with $\sup_{0\leq r\leq k} \gamma_1(t,r) \leq \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k}(\cdot) \in L^q(T)$ and $\sup_{0\leq r\leq k}\gamma_2(t,r) \leq \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^\infty(T)$.

Remark. Hypothesis $\mathrm{H(f)}_1$ (iv) is a suitable extension of the classical Nagumo–Hartman condition for continuous vector fields (see Hartman [7], p. 433 and Knobloch [9], Mawhin [10]).

DEFINITION

By a solution of (1) we mean a function $x \in C^1(T, R^N)$ such that $(\|x'(\cdot)\|^{p-2}x'(\cdot))' \in W^{1,q}(T, R^N)$ and satisfies (1).

Given $g \in L^q(T, R^N)$ we consider the following auxiliary problem:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) = g(t) \quad \text{a.e. on } T \\ &x(0) = x(b),\ x'(0) = x'(b). \end{aligned} \tag{3}$$

PROPOSITION 2

For every $g \in L^q(T, R^N)$ problem (3) *has a unique solution.*

Proof. Let $a \in R^N$ and consider the following boundary value problem:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) = g(t) \text{ a.e. on } T \\ &x(0) = x(b) = a. \end{aligned} \tag{4}$$

Setting $y(\cdot) = x(\cdot) - a$, problem (4) becomes a homogeneous Dirichlet problem in y:

$$\begin{aligned} &-(\|y'(t)\|^{p-2}y'(t))' + \|y(t) + a\|^{p-2}(y(t) + a) = g(t) \quad \text{a.e. on } T \\ &y(0) = y(b) = 0. \end{aligned} \tag{5}$$

We solve problem (5). To this end consider the operator $A_1 : W_0^{1,p}(T, R^N) \to W^{-1,q}(T, R^N)$ defined by

$$\langle A_1(y), z\rangle = \int_0^b (\|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N} + \|y(t) + a\|^{p-2}(y(t) + a, z(t))_{R^N})\, dt.$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(T,R^N), W^{-1,q}(T,R^N))$. We claim that $A_1(\cdot)$ is strictly monotone and demicontinuous. We have:

$$\begin{aligned}
&\langle A_1(y)-A_1(z), y-z\rangle\\
&= \int_0^b (\|y'(t)\|^{p-2}(y'(t), y'(t)-z'(t))_{R^N} - \|z'(t)\|^{p-2}(z'(t), y'(t)-z'(t))_{R^N})\,\mathrm{d}t\\
&\quad + \int_0^b (\|y(t)+a\|^{p-2}(y(t)+a, y(t)-z(t))_{R^N}\\
&\quad - \|z(t)+a\|^{p-2}(z(t)+a, y(t)-z(t))_{R^N})\,\mathrm{d}t.
\end{aligned}$$

First we estimate the first integral of the rhs of the above equation. We have:

$$\begin{aligned}
&\int_0^b (\|y'(t)\|^{p-2}(y'(t), y'(t)-z'(t))_{R^N} - \|z'(t)\|^{p-2}(z'(t), y'(t)-z'(t))_{R^N})\,\mathrm{d}t\\
&\geq \int_0^b (\|y'(t)\|^p - \|y'(t)\|^{p-1}\|z'(t)\| - \|z'(t)\|^{p-1}\|y'(t)\| + \|z'(t)\|^p)\,\mathrm{d}t\\
&= \int_0^b (\|y'(t)\|^{p-1}(\|y'(t)\| - \|z'(t)\|) - \|z'(t)\|^{p-1}(\|y'(t)\| - \|z'(t)\|))\,\mathrm{d}t\\
&= \int_0^b (\|y'(t)\|^{p-1} - \|z'(t)\|^{p-1})(\|y'(t)\| - \|z'(t)\|)\,\mathrm{d}t\\
&\geq 2^{2-p}\int_0^b |\|y'(t)\| - \|z'(t)\||^p\,\mathrm{d}t \quad \text{(see § 2)}.
\end{aligned} \tag{6}$$

In a similar way, we obtain

$$\begin{aligned}
&\int_0^b (\|y(t)+a\|^{p-2}(y(t)+a, y(t)-z(t))_{R^N}\\
&\qquad - \|z(t)+a\|^{p-2}(z(t)+a, y(t)-z(t))_{R^N})\,\mathrm{d}t\\
&\geq 2^{2-p}\int_0^b |\|y(t)+a\| - \|z(t)+a\||^p\,\mathrm{d}t.
\end{aligned} \tag{7}$$

From (6) and (7) we have the monotonicity of $A_1(\cdot)$. Moreover, if $\langle A_1(y)-A_1(z), y-z\rangle = 0$, we have $\|y'(t)\| = \|z'(t)\|$ a.e. on T and $\|y(t)+a\| = \|z(t)+a\|$ for all $t\in T$. Then it follows that

$$\begin{aligned}
&\int_0^b \|y'(t)\|^{p-2}\|y'(t)-z'(t)\|^2\,\mathrm{d}t + \int_0^b \|y(t)+a\|^{p-2}\|y(t)-z(t)\|^2\,\mathrm{d}t = 0\\
&\Rightarrow \int_0^b \|y'(t)\|^{p-2}\|y'(t)-z'(t)\|^2\,\mathrm{d}t = 0 \text{ and}\\
&\quad \int_0^b \|y(t)+a\|^{p-2}\|y(t)-z(t)\|^2\,\mathrm{d}t = 0
\end{aligned}$$

from which we deduce easily that $y = z$. Therefore $A_1(\cdot)$ is strictly monotone.

Next we show that $A_1(\cdot)$ is demicontinuous. To this end let $y_n \to y$ in $W_0^{1,p}(T,R^N)$ as $n\to\infty$. For every $z\in W_0^{1,p}(T,R^N)$ we have

$$
\begin{aligned}
&|\langle A_1(y_n) - A_1(y), z\rangle| \\
&\leq \left| \int_0^b (\|y_n'(t)\|^{p-2}(y_n'(t), z'(t))_{R^N} - \|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N})\,\mathrm{d}t \right| \\
&\quad + \left| \int_0^b (\|y_n(t) + a\|^{p-2}(y_n(t)+a, z(t))_{R^N} - \|y(t)+a\|^{p-2}(y(t)+a, z(t))_{R^N})\mathrm{d}t \right|.
\end{aligned}
$$

By passing to a subsequence if necessary we may assume that $y_n'(t) \to y'(t)$ a.e. on T and $y_n(t) \to y(t)$ for all $t \in T$ as $n \to \infty$ (the latter convergence follows from the fact that $W_0^{1,p}(T, R^N)$ is embedded continuously in $C(T, R^N)$). So by the extended dominated convergence theorem (see Ash [1], theorem 7.5.2, p. 295), we have

$$
\left| \int_0^b (\|y_n'(t)\|^{p-2}(y_n'(t), z'(t))_{R^N} - \|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N})\,\mathrm{d}t \right| \to 0 \text{ and}
$$

$$
\left| \int_0^b (\|y_n(t) + a\|^{p-2}(y_n(t) + a, z(t))_{R^N} - \|y(t) + a\|^{p-2}(y(t) + a, z(t))_{R^N})\,\mathrm{d}t \right| \to 0 \text{ as } n \to \infty.
$$

Therefore $|\langle A_1(y_n) - A_1(y), z\rangle| \to 0$ as $n \to \infty$ and since $z \in W_0^{1,p}(T, R^N)$ was arbitrary we conclude that $A_1(y_n) \xrightarrow{w} A_1(y)$ in $W^{-1,q}(T, R^N)$ as $n \to \infty$, i.e. $A_1(\cdot)$ is demicontinuous. Since $A_1(\cdot)$ is monotone and demicontinuous, it is maximal monotone (see § 2).

Finally we will show that $A_1(\cdot)$ is weakly coercive. We have:

$$
\begin{aligned}
\langle A_1(y), y\rangle &= \|y'\|_p^p + \int_0^b \|y(t) + a\|^{p-2}(y(t) + a, y(t))_{R^N}\,\mathrm{d}t \\
&= \|y'\|_p^p + \|y + a\|_p^p - \int_0^b \|y(t) + a\|^{p-2}(y(t) + a, a)_{R^N}\,\mathrm{d}t \\
&\geq \|y'\|_p^p + \|y + a\|_p^p - \int_0^b \|y(t) + a\|^{p-1}\|a\|\,\mathrm{d}t \\
&\geq \|y + a\|_{1,p}^p - c\|y + a\|_p^{p-1} \quad \text{for some } c > 0,
\end{aligned}
$$

where $\|\cdot\|_{1,p}$ denotes the norm of the Sobolev space $W_0^{1,p}(T, R^N)$. So we proved that $A_1(\cdot)$ is weakly coercive. Hence $A_1(\cdot)$ is surjective. This means that we can find $y \in W_0^{1,p}(T, R^N)$ such that $A_1(y) = g$. Then for all $\phi \in C_0^\infty(T, R^N)$ we have

$$
\int_0^b \|y'(t)\|^{p-2}(y'(t), \phi'(t))_{R^N}\mathrm{d}t = \int_0^b (g(t) - \|y(t)+a\|^{p-2}(y(t)+a(g(t)+\alpha, \phi(t))_{R^N}\mathrm{d}t.
$$

From the definition of the distributional derivative, we infer that

$$
\begin{aligned}
&-(\|y'(t)\|^{p-2}y'(t))' = g(t) - \|y(t) + a\|^{p-2}(y(t) + a) \quad \text{a.e. on } T \\
&y(0) = y(b) = 0.
\end{aligned}
$$

Hence $y(\cdot)$ is a solution of (5) and so $x(\cdot) = y(\cdot) + a$ is a solution of (4). Moreover, by virtue of the strict monotonicity of A_1, this solution of (4) is unique. So we can define a map $\theta : R^N \to C^1(T, R^N)$ by setting $\theta(a)$ to be the unique solution of (4). Then introduce the map $\rho : R^N \to R^N$ defined by

$$\rho(a) = \|\theta'(a)(b)\|^{p-2}\theta'(a)(b) - \|\theta'(a)(0)\|^{p-2}\theta'(a)(0)$$
$$= \int_0^b (\|\theta'(a)(t)\|^{p-2}\theta'(a)(t))'\, dt.$$

Claim 1. $\rho(\cdot)$ is strictly monotone.

We need to show that for all $a, \beta \in R^N$ $(\rho(a) - \rho(\beta), a - \beta)_{R^N} \geq 0$ and if equality holds, then $a = \beta$. We have

$$\begin{aligned}
&(\rho(a) - \rho(\beta), a - \beta)_{R^N} \\
&= \int_0^b ((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))',\ a - \theta(a)(t) \\
&\quad - (\beta - \theta(\beta)(t)))_{R^N}\, dt + \int_0^b ((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) \\
&\quad - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))', \theta(a)(t) - \theta(\beta)(t))_{R^N}\, dt. \qquad (8)
\end{aligned}$$

Since by definition $\theta(a)(\cdot)$ and $\theta(\beta)(\cdot)$ are solutions of (4) with boundary values a and β respectively, we can write that

$$\begin{aligned}
&(\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))' \\
&= \|\theta(a)(t)\|^{p-2}\theta(a)(t) - \|\theta(\beta)(t)\|^{p-2}\theta(\beta)(t) \quad \text{a.e. on } T.
\end{aligned}$$

Using this fact in the second integral of the rhs in (8) and performing an integration by parts on the first integral, we obtain

$$\begin{aligned}
&(\rho(a) - \rho(\beta), a - \beta)_{R^N} \\
&= \int_0^b [\|\theta'(a)(t)\|^{p-2}(\theta'(a)(t), \theta'(a)(t) - \theta'(\beta)(t))_{R^N} \\
&\quad - \|\theta'(\beta)(t)\|^{p-2}(\theta'(\beta)(t), \theta'(a)(t) - \theta'(\beta)(t))_{R^N}]\, dt \\
&\quad + \int_0^b (\|\theta(a)(t)\|^{p-2}\theta(a)(t) - \|\theta(\beta)(t)\|^{p-2}\theta(\beta)(t), \theta(a)(t) - \theta(\beta)(t))_{R^N}\, dt \\
&\geq 2^{2-p}\left[\int_0^b |\|\theta'(a)(t)\| - \|\theta'(\beta)(t)\||^p\, dt + \int_0^b |\|\theta(a)(t)\| - \|\theta(\beta)(t)\||^p\, dt\right].
\end{aligned}$$

From this last inequality as before, we infer that $\rho(\cdot)$ is indeed strictly monotone.

Claim 2. $\rho : R^N \to R^N$ is continuous.

Let $a_n \to a$ in R^N as $n \to \infty$ and set $x_n = \theta(a_n)$, $y_n = x_n - a_n$, $n \geq 1$. Taking the inner product of (5) with $y_n(t)$, integrating over T and finally performing an integration by parts, we obtain

$$\begin{aligned}
&\|y_n'\|_p^p + \|y_n + a_n\|_p^p \leq \|g\|_q(\|y_n + a_n\|_p + b^{1/p}\|a_n\|) + \|y_n + a_n\|_p^{p-1} b^{1/p}\|a_n\| \\
&\Rightarrow \|y_n + a_n\|_{1,p}^p \leq \|g\|_q(\|y_n + a_n\|_p + b^{1/p}\|a_n\|) + \|y_n + a_n\|_p^{p-1} b^{1/p}\|a_n\|.
\end{aligned}$$

From this inequality it follows that $\{x_n = y_n + a_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T, R^N)$. Hence $\{\|x_n\|^{p-2}x_n\}_{n\geq 1}$ and $\{\|x_n'\|^{p-2}x_n'\}_{n\geq 1}$ are bounded in $L^q(T, R^N)$. From these facts and eq. (4) it follows that $\{\|x_n'\|^{p-2}x_n'\}_{n\geq 1}$ is bounded in $W^{1,q}(T, R^N)$. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{x} u$ in $W^{1,p}(T, R^N)$ and $\|x_n'\|^{p-2}\, x_n' \xrightarrow{w} v$

in $W^{1,q}(T, R^N)$ as $n \to \infty$. In particular we have $\|x_n'\|^{p-2}x_n' \xrightarrow{w} \nu'$ in $L^q(T, R^N)$ as $n \to \infty$. Recall that $W^{1,p}(T, R^N)$ is embedded compactly in $C(T, R^N)$. So we have $x_n \to u$ in $C(T, R^N)$ as $n \to \infty$. Note that

$$\begin{aligned}&\big\|\|x_n(t)\|^{p-2}x_n(t) - \|u(t)\|^{p-2}u(t)\big\|\\ &\quad \le \|x_n(t)\|^{p-2}\|x_n(t) - u(t)\| + \|u(t)\|\big|\|x_n(t)\|^{p-2} - \|u(t)\|^{p-2}\big|.\end{aligned}$$

From an elementary inequality (see Rudin [14], p. 78), we have

$$\begin{aligned}&\big\|\|x_n(t)\|^{p-2}x_n(t) - \|u(t)\|^{p-2}u(t)\big\|\\ &\quad \le \begin{cases}\|x_n\|_\infty^{p-2}\|x_n - u\|_\infty + \|u\|_\infty\|x_n - u\|_\infty^{p-2} & \text{if } 2 \le p < 3\\ \|x_n\|_\infty^{p-2}\|x_n - u\|_\infty + (p-2)\|x_n - u\|_\infty(\|x_n\|_\infty + \|u\|_\infty)^{p-3} & \text{if } 3 \le p < \infty\end{cases}\\ &\Rightarrow \|x_n\|^{p-2}x_n \to \|u\|^{p-2}u \ \text{ in } \ C(T, R^N) \text{ as } n \to \infty.\end{aligned}$$

Since for every $n \ge 1$ we have $-(\|x_n'(t)\|^{p-2}x_n'(t))' + \|x_n(t)\|^{p-2}x_n(t) = g(t)$ a.e. on T in the limit as $n \to \infty$ we obtain $-\nu'(t) + \|u(t)\|^{p-2}u(t) = g(t)$ a.e. on T. Also from the compact embedding of $W^{1,q}(T, R^N)$ in $C(T, R^N)$, we have that $\|x_n'\|^{p-2}x_n' \to \nu$ in $C(T, R^N)$ as $n \to \infty$. Consider the map $\sigma : R^N \to R^N$ defined by $\sigma(z) = \|z\|^{p-2}z$. Clearly this map is strictly monotone, continuous and weakly coercive, thus surjective. Hence $\sigma^{-1} : R^N \to R^N$ is well-defined and is easily seen to be continuous. Then we have $x_n'(t) \to \sigma^{-1}(\nu(t))$ for all $t \in T$ as $n \to \infty$. Hence $x_n' \to \sigma^{-1}(\nu)$ in $L^p(T, R^N)$ as $n \to \infty$ and so in the limit we have $-(\|u'(t)\|^{p-2}u'(t))' + \|u(t)\|^{p-2}u(t) = g(t)$ a.e. on T, $u(0) = u(b) = a$, therefore $u = \theta(a)$. Since $W^{1,q}(T, R^N)$ is compactly embedded in $C(T, R^N)$ we have $\|x_n'\|^{p-2}x_n' \to \|u'\|^{p-2}u' = \nu$ in $C(T, R^N)$ and so $\rho(a_n) \to \rho(a)$ as $n \to \infty$, which proves the continuity of $\rho(\cdot)$.

Claim 3. $\rho(\cdot)$ is weakly coercive.

To establish this claim we argue as follows:

$$\begin{aligned}\frac{(\rho(a), a)_{R^N}}{\|a\|} &= \frac{(\|\theta'(a)(b)\|^{p-2}\theta'(a)(b) - \|\theta'(a)(0)\|^{p-2}\theta'(a)(0), a)_{R^N}}{\|a\|}\\ &= \frac{\int_0^b ((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t))', \theta(a)(t))_{R^N}\,\mathrm{d}t + \|\theta'(a)\|_p^p}{\|a\|}\\ &\ge \frac{\|\theta(a)\|_p^p + \|\theta'(a)\|_p^p - \|h\|_q\|\theta(a)\|_p}{\|a\|} \quad \text{(see eq. (4)).}\end{aligned}$$

Using the mean value theorem for integrals (see Hewitt–Stromberg [8], Theorem 21.69, p. 420), we can find $0 < t_0 < b$ such that

$$b\|\theta(a)(t_0)\| = \int_0^b \|\theta(a)(t)\|\,\mathrm{d}t.$$

Hence for every $t \in T$, we have

$$\begin{aligned}&\theta(a)(t) = \theta(a)(t_0) + \int_{t_0}^t \theta'(a)(s)\,\mathrm{d}s\\ &\quad\Rightarrow \|\theta(a)(t)\| \le \|\theta(a)(t_0)\| + \|\theta'(a)\|_1 \le \frac{1}{b}\|\theta(a)\|_1 + \|\theta'(a)\|_1\\ &\quad\Rightarrow \|a\| \le \|\theta(a)\|_\infty \le \gamma_1\|\theta(a)\|_{1,p} \ \text{ for some } \gamma_1 > 0.\end{aligned}$$

So we can write

$$\frac{(\rho(a),a)_{R^N}}{\|a\|} \geq \frac{\|\theta(a)\|_p^p + \|\theta'(a)\|_p^p - \|h\|_q\|\theta(a)\|_p}{\gamma_1\|\theta(a)\|_{1,p}}$$
$$\geq \frac{\|\theta(a)\|_{1,p}(\|\theta(a)\|_{1,p}^{p-1} - \|h\|_q)}{\gamma_1\|\theta(a)\|_{1,p}}.$$

If $\|a\| \to \infty$, we have $\|\theta(a)\|_{1,p} \to \infty$. Indeed $\|a\| \leq \|\theta(a)\|_\infty \leq \gamma_2\|\theta(a)\|_{1,p}$ for some $\gamma_2 > 0$ (the last inequality being a consequence of the fact that $W^{1,p}(T,R^N)$ is embedded compactly in $C(T,R^N)$). So $\rho(\cdot)$ is weakly coercive as claimed.

Therefore $\rho(\cdot)$ is monotone, continuous, weakly coercive, hence surjective. So we can find $a \in R^N$ such that $\rho(a) = 0$, which implies that $\|\theta'(a)(0)\|^{p-2}\theta'(a)(0) = \|\theta'(a)(b)\|^{p-2}\theta'(a)(b)$. Acting with σ^{-1} on both sides of this equality, we obtain $\theta'(a)(0) = \theta'(a)(b)$ (recall $\sigma(z) = \|z\|^{p-2}z$). Thus $x = \theta(a)$ is a solution of (3).

Finally to show the uniqueness of this solution, let x, y be two solutions of (3). Via an integration by parts and using the periodic boundary conditions, we obtain

$$0 = \int_0^b (\|x'(t)\|^{p-2}x'(t) - \|y'(t)\|^{p-2}y'(t), x'(t) - y'(t))_{R^N}\, dt$$
$$+ \int_0^b (\|x(t)\|^{p-2}x(t) - \|y(t)\|^{p-2}y(t), x(t) - y(t))_{R^N}\, dt$$
$$\geq 2^{2-p}\left[\int_0^b \big|\|x'(t)\| - \|y'(t)\|\big|^p\, dt + \big|\|x(t)\| - \|y(t)\|\big|^p\, dt\right].$$

From this inequality, as before, we infer that $x = y$. Hence the solution of (3) is unique. Using proposition 2, we can prove an existence theorem for problem (1).

Theorem 3. *If hypotheses* $\mathrm{H(f)}_1$ *hold, then problem* (1) *has a solution.*

Proof. Let $D = \{x \in C^1(T,R^N) : \|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T,R^N),\ x(0) = x(b),\ x'(0) = x'(b)\}$ and let $A : D \subseteq L^p(T,R^N) \to L^q(T,R^N)$ be defined by $A(x) = -(\|x'\|^{p-2}x')'$.

Claim 1. $A(\cdot)$ is a maximal monotone operator.

Let $(\cdot,\cdot)_{pq}$ denote the duality brackets for $(L^p(T,R^N),\ L^q(T,R^N))$. For every $x, y \in D$ we have

$$(A(x) - A(y), x - y)_{pq}$$
$$= \int_0^b (-(\|x'(t)\|^{p-2}x'(t))' + (\|y'(t)\|^{p-2}y'(t))', x(t) - y(t))_{R^N}\, dt$$
$$= \int_0^b (\|x'(t)\|^{p-2}x(t)x'(t) - \|y'(t)\|^{p-2}y'(t), x'(t) - y'(t))_{R^N}\, dt$$

(integration by parts)

$$\geq \int_0^b (\|x'(t)\|^{p-1} - \|y'(t)\|^{p-1})(\|x'(t)\| - \|y'(t)\|)\, dt \geq 0.$$

So $A(\cdot)$ is monotone (in fact strictly monotone). To prove the maximality of $A(\cdot)$ it suffices to show that if $J : L^p(T,R^N) \to L^q(T,R^N)$ is defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$,

then $R(A+J)=L^q(T,R^N)$. Indeed suppose that $(A+J)(\cdot)$ is surjective and let $y \in L^p(T,R^N)$, $\nu \in L^q(T,R^N)$ be such that

$$(A(x)-\nu, x-y)_{pq} \geq 0 \quad \text{for all } x \in D. \tag{9}$$

Since we have assumed that $R(A+J)=L^q(T,R^N)$, we can find $x_1 \in D$ such that

$$A(x_1)+J(x_1)=\nu+J(y).$$

In (9) we set $x=x_1 \in D$. We obtain

$$0 \leq (A(x_1)-A(x_1)-J(x_1)+J(y), x_1-y)_{pq}=(J(y)-J(x_1), x_1-y)_{pq}. \tag{10}$$

But $J(\cdot)$ being the Frechet derivative of the strictly convex map $x \to 1/p\|x\|_p^p$ on $L^p(T,R^N)$ is maximal monotone and strictly monotone. So from (5) it follows that $(J(y)-J(x_1), x_1-y)_{pq}=0$ and so $x_1=y$. Hence $y \in D$ and $\nu=A(x_1)$. This proves the maximality of $A(\cdot)$.

Therefore we have to show that $R(A+J)=L^q(T,R^N)$. This is equivalent to saying that for every $g \in L^q(T,R^N)$ problem (3) has a solution. But this is exactly proposition 2. Moreover, since $J(\cdot)$ is strictly monotone, we see at once that $(A+J)^{-1}: L^q(T,R^N) \to D \subseteq W^{1,p}(T,R^N)$ is well-defined. In what follows we set $L=(A+J)^{-1}$.

Claim 2. $L: L^q(T,R^N) \to D \subseteq W^{1,p}(T,R^N)$ is compact.

Because of the reflexivity of $L^q(T,R^N)$ to prove the claim it suffices to show that if $\nu_n \xrightarrow{w} \nu$ in $L^q(T,R^N)$, then $L(\nu_n) \to L(\nu)$ in $W^{1,p}(T,R^N)$ as $n \to \infty$ (complete continuity of $L(\cdot)$). To this end let $x_n=L(\nu_n)$, $n \geq 1$. Then $x_n \in D$ and we have

$$\begin{aligned} A(x_n)+J(x_n)=\nu_n,\ n \geq 1 &\Rightarrow (A(x_n),x_n)_{pq}+(J(x_n),x_n)_{pq}=(\nu_n,x_n)_{pq} \\ &\Rightarrow \|x_n'\|_p^p+\|x_n\|_p^p \leq \|\nu_n\|_q\|x_n\|_p \\ &\Rightarrow \|x_n\|_{1,p}^{p-1} \leq \sup_{n\geq 1}\|\nu_n\|_q < \infty. \end{aligned}$$

So we see that $\{x_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T,R^N)$. By passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T,R^N)$. Since $W^{1,p}(T,R^N)$ is embedded compactly in $L^p(T,R^N)$, we also have that $x_n \to x$ in $L^p(T,R^N)$ as $n \to \infty$. So

$$\begin{aligned} &\lim(A(x_n)+J(x_n),x_n-x)_{pq}=\lim(\nu_n,x_n-x)_{pq}=0 \\ &\quad\Rightarrow \limsup(A(x_n),x_n-x)_{pq} \leq 0 \\ &\qquad (\text{since } J: L^p(T,R^N) \to L^q(T,R^N) \text{ is continuous}) \\ &\quad\Rightarrow \limsup\|x_n'\|_p^p \leq \limsup(A(x_n),x)_{pq}. \end{aligned}$$

Because $W^{1,p}(T,R^N)$ is compactly embedded in $C(T,R^N)$, we have that $x_n \to x$ in $C(T,R^N)$ as $n \to \infty$. So $x(0)=x(b)$. Using this and Green's formula (integration by parts), we have

$$\begin{aligned} (A(x_n),x)_{pq}=\int_0^b \|x_n'(t)\|^{p-2}(x_n'(t),x'(t))_{R^N}\,dt &\leq \|x_n'\|_p^{p-1}\|x'\|_p \\ \Rightarrow \limsup\|x_n'\|_p^p &\leq \limsup\|x_n'\|_p^{p-1}\|x'\| \\ \Rightarrow \limsup\|x_n'\|_p &\leq \|x'\|_p. \end{aligned} \tag{11}$$

On the other hand from the weak lower semicontinuity of the norm functional, we have that

$$\|x'\|_p \leq \liminf\|x_n'\|_p. \tag{12}$$

From (11) and (12) it follows that $\|x_n'\|_p \to \|x'\|_p$ as $n \to \infty$. Since $x_n' \xrightarrow{w} x'$ in $L^p(T,R^N)$ as $n \to \infty$ and $L^p(T,R^N)$ is uniformly convex, we deduce that $x_n' \to x'$ in $L^p(T,R^N)$ as $n \to \infty$ (Kadec–Klee property). Hence $x_n \to x$ in $W^{1,p}(T,R^N)$ as $n \to \infty$. Because $A(x_n) + J(x_n) = \nu_n$, we have $[x_n, \nu_n - J(x_n)] \in \operatorname{Gr} A$, $n \geq 1$. From the maximal monotonicity of $A(\cdot)$ (see claim 1), $\operatorname{Gr} A$ is demiclosed in $L^p(T,R^N) \times L^q(T,R^N)$ (see § 2) and so $[x, \nu - J(x)] \in \operatorname{Gr} A$, hence $A(x) + J(x) = \nu$, i.e. $x = L(\nu)$. This proves that $L(\cdot)$ is compact as claimed.

Now let $N : W^{1,p}(T,R^N) \to L^q(T,R^N)$ be the Nemitsky (superposition) operator corresponding to the function $f(t,x,y)$; i.e. $N(x)(\cdot) = f(\cdot, x(\cdot), x'(\cdot))$. Using hypothesis $\mathrm{H(f)}_1\,(v)$ it is easy to see that $N(\cdot)$ is continuous and bounded (i.e. maps bounded sets to bounded sets). Let $N_1(x) = -N(x) + J(x)$. This is a continuous and bounded map. Evidently the resolution of problem (1), is equivalent to solving the following abstract fixed point problem

$$x = LN_1(x). \tag{13}$$

Note that the continuity and boundedness of N_1 combined with claim 2, imply that $LN_1 : W^{1,p}(T,R^N) \to W^{1,p}(T,R^N)$ is a compact map. Then by virtue of theorem 1, we will be able to solve (13) if we show the validity of the following claim:

Claim 3. $\Gamma = \{x \in D \subseteq W^{1,p}(T,R^N) : x = \lambda LN_1(x), 0 < \lambda < 1\}$ is bounded in $W^{1,p}(T,R^N)$.

So let $x \in \Gamma$. By definition we have

$$\begin{aligned} x &= \lambda LN_1(x) \quad \text{for some} \quad 0 < \lambda < 1 \\ &\Rightarrow (A + J)\left(\frac{1}{\lambda}x\right) = N_1(x) \\ &\Rightarrow -(\|x'(t)\|^{p-2}x'(t))' = -\lambda^{p-1}f(t, x(t)), x'(t)) \\ &\quad + (\lambda^{p-1} - 1)\|x(t)\|^{p-2}x(t) \quad \text{a.e. on } T \\ x(0) &= x(b), x'(0) = x'(b). \end{aligned}$$

Take the inner product with $x(t)$ and then integrate over T. We have

$$\begin{aligned} &\int_0^b (-(\|x'(t)\|^{p-2}x'(t))', x(t))_{R^N}\, \mathrm{d}t \\ &= \lambda^{p-1} \int_0^b (-f(t, x(t), x'(t)), x(t))_{R^N}\, \mathrm{d}t + (\lambda^{p-1} - 1)\|x\|_p^p. \end{aligned} \tag{14}$$

Using Green's identity on the lhs, we have

$$\int_0^b (-(\|x'(t)\|^{p-2}x'(t))', x(t))_{R^N}\, \mathrm{d}t = \|x'\|_p^p. \tag{15}$$

Also from hypothesis $\mathrm{H(f)}_1$ (iii) we have

$$\begin{aligned} &\lambda^{p-1} \int_0^b (-f(t, x(t), x'(t)), x(t))_{R^N}\, \mathrm{d}t \\ &\leq \lambda^{p-1} a\|x\|_p^p + \lambda^{p-1}\beta \int_0^b \|x(t)\|^r \|x'(t)\|^{p-r}\, \mathrm{d}t + \lambda^{p-1}\|c\|_1\|x\|_\infty^s. \end{aligned}$$

Let $\tau = p - r$ and set $\eta = p/r$, $\eta' = p/\tau$. Using Hölder's inequality, we have

$$\int_0^b \|x(t)\|^r \|x'(t)\|^\tau \, dt \le \left(\int_0^b \|x(t)\|^{r\eta} \, dt \right)^{1/\eta} \left(\int_0^b \|x'(t)\|^{\tau\eta'} \right)^{1/\eta'} = \|x\|_p^r \|x'\|_p^\tau.$$

So we have

$$\lambda^{p-1} \int_0^b (-f(t, x(t), x'(t)), x(t))_{R^N} \, dt \le \lambda^{p-1} a \|x\|_p^p + \lambda^{p-1} \beta \|x\|_p^r \|x'\|_p^\tau + \lambda^{p-1} \|c\|_1 \|x\|_\infty^s. \tag{16}$$

We will also show that for all $x \in \Gamma$, we have $\|x\|_\infty \le M$ where $M > 0$ is as in hypothesis $\mathrm{H(f)}_1$ (iv). To this end let $r(t) = \|x(t)\|^p$ and let $t_0 \in T$ be the point where $r(\cdot)$ attains its maximum and suppose that $r(t_0) > M^p$. First assume that $0 < t_0 < b$. We have $r'(t_0) = p\|x(t_0)\|^{p-2}(x'(t_0), x(t_0))_{R^N} = 0 \Rightarrow (x'(t_0), x(t_0))_{R^N} = 0$. So by virtue of hypothesis $\mathrm{H(f)}_1$ (iv), we can find $c, \delta > 0$ such that for almost all $t \in T$,

$$\inf[(f(t,x,y),x) + \|y\|^p : \|x(t_0) - x\| + \|x'(t_0) - y\| < \delta] \ge c.$$

Note that $x \in \Gamma \subseteq D$ and so $\|x'(\cdot)\|^{p-2} x'(\cdot) \in W^{1,q}(T, R^N)$. Since $W^{1,q}(T, R^N)$ is embedded continuously (in fact compactly) in $C(T, R^N)$, we have that $t \to \|x'(t)\|^{p-2} x'(t) \in C(T, R^N)$ and so $t \to \sigma^{-1}(\|x'(t)\|^{p-2} x'(t)) = x'(t) \in C(T, R^N)$ (recall that $\sigma(z) = \|z\|^{p-2} z$; see the proof of proposition 2). Hence $\Gamma \subseteq C^1(T, R^N)$ and so we can find $\delta_1 > 0$ such that if $t_0 < t \le t_0 + \delta_1$ we have

$$\|x(t_0) - x(t)\| + \|x'(t_0) - x'(t)\| < \delta.$$

So for almost all $t \in (t_0, t_0 + \delta_1]$, we have

$$\begin{aligned}
&\lambda^{p-1}(f(t, x(t), x'(t)), x(t))_{R^N} + \lambda^{p-1}\|x'(t)\|^p \ge \lambda^{p-1} c \\
&\Rightarrow ((\|x'(t)\|^{p-2} x'(t))', x(t))_{R^N} + (\lambda^{p-1} - 1)\|x(t)\|^p + \lambda^{p-1}\|x'(t)\|^p \ge \lambda^{p-1} c \\
&\Rightarrow \int_{t_0}^t ((\|x'(s)\|^{p-2} x'(s))', x(s))_{R^N} \, ds + (\lambda^{p-1} - 1) \int_{t_0}^t \|x(s)\|^p \, ds \\
&\quad + \lambda^{p-1} \int_{t_0}^t \|x'(s)\|^p \, ds \ge \lambda^{p-1} c (t - t_0).
\end{aligned} \tag{17}$$

Using Green's identity (integration by parts) on the first integral of the lhs, of the last inequality, we obtain

$$\begin{aligned}
\int_{t_0}^t ((\|x'(s)\|^{p-2} x'(s))', x(s))_{R^N} \, ds &= \|x'(t)\|^{p-2}(x'(t), x(t))_{R^N} \\
&\quad - \|x'(t_0)\|^{p-2}(x'(t_0), x(t_0))_{R^N} - \int_{t_0}^t \|x'(s)\|^p \, ds.
\end{aligned} \tag{18}$$

Using (18) in (17) and since $0 < \lambda < 1$ and $(x'(t_0), x(t_0))_{R^N} = 0$, we have

$$\begin{aligned}
&\|x'(t)\|^{p-2}(x'(t), x(t))_{R^N} \ge \lambda^{p-1} c (t - t_0) > 0 \quad \text{for all} \quad t_0 < t \le t_0 + \delta_1 \\
&\Rightarrow r'(t) > 0, \quad \text{i.e.} \quad r(t) > r(t_0).
\end{aligned}$$

This contradicts the choice of $t_0 \in T$. So $\|x(t_0)\| \le M$. Next assume that $t_0 = 0$. Then we have $r(0) = r(b)$ and $r(0) \le 0$ and $r'(b) \ge 0$. Hence $0 \ge r'(0) = \|x(0)\|^{p-2}(x'(0),$

$x(0))_{R^N} = \|x(b)\|^{p-2}(x'(b), x(b))_{R^N} = r'(b) \geq 0 \Rightarrow r'(0) = 0$ and so the previous argument applies. Similarly if $t_0 = b$. So in all cases we conclude that $\|x\|_\infty \leq M$ for all $x \in \Gamma$. Using this fact together with estimates (15) and (16) in (14), we obtain

$$\begin{aligned}\|x'\|_p^p &\leq \lambda^{p-1}a\|x\|_p^p + \lambda^{p-1}\beta\|x\|_p^r\|x'\|_p^\tau + \lambda^{p-1}M^s\|c\|_1 \\ &\leq aM^b b + \beta M^r b^{r/p}\|x'\|_p^\tau + M^s\|c\|_1 \quad (\tau < p).\end{aligned}$$

From this inequality, we deduce that there exists $M_1 > 0$ such that for all $x \in \Gamma$ we have

$$\begin{aligned}&\|x'\|_p \leq M_1 \\ &\Rightarrow \Gamma \text{ is bounded in } W^{1,p}(T, R^N).\end{aligned}$$

Apply theorem 1 to conclude that there exists $x \in D$ which solves (13). Evidently this is a solution of problem (1).

Remark. Theorem 3 partially extends theorem 1 of Knobloch [9], where $p = 2$ and f is continuous (see also theorem 6.1 and corollary 6.2 of Mawhin [10]). Also partially improves theorems 4.1 and 4.5 of Guo [6] where $N = 1$ and f is continuous. Finally it partially extends the existence result of Zhang [16] (who deals with the Dirichlet problem) to the periodic problem.

4. Nonlinear scalar bvp problems

In this section we deal with the scalar nonlinear boundary value problem (2). We introduce the following hypotheses on the data of (2).

$\mathrm{H(f)}_2$: $f : T \times R \times R \to R$ satisfies hypotheses $\mathrm{H(f)}_2$ with $N = 1$.
$\mathrm{H}(\xi)$: $\xi_1 : R \to 2^R$ and $\xi_2 : R \to 2^R$ are maximal monotone maps with $0 \in \xi_1(0)$ and $0 \in \xi_2(0)$.

Remark. It is well-known from convex analysis that $\xi_1 = \partial\phi_1$ and $\xi_2 = \partial\phi_2$ with ϕ_1, ϕ_2: $R \to \overline{R} = R \cup \{+\infty\}$ being proper (i.e. $\phi_i \not\equiv +\infty$, $i = 1, 2$), convex and lower semicontinuous. In fact $\phi_i(r) = \int_0^r \xi_i^0(s)\,ds$ where $\xi_i^0(s) = \text{proj}\,(0; \xi_i(s))$ (i.e. $\xi_i^0(s)$ is the element of $\xi_i(s)$ of minimal absolute value), $i = 1, 2$. The map $s \to \xi_i^0(s)$ is increasing and $\xi_i(s) = [\xi_i^0(s^-), \xi_i^0(s^+)]$, $i = 1, 2$.

Let $D = \{x \in C^1(T) : |x'|^{p-2}x' \in W^{1,q}(T),\ x'(0) \in \xi_1(x(0)),\ -x'(b) \in \xi_2(x(b))\}$ and consider the operator $L : D \subseteq L^p(T) \to L^q(T)$ defined by

$$L(x) = -(|x'|^{p-2}x'), \quad x \in D.$$

PROPOSITION 4. *If hypotheses* $\mathrm{H(f)}_2$ *and* $\mathrm{H}(\xi)$ hold, *then* L *is maximal monotone.*

Proof. From the proof of theorem 3 we know that it suffices to show that $R(L + J) = L^q(T)$ with $J : L^p(T) \to L^q(T)$ defined by $J(x) = |x|^{p-2}x$. This surjective property of $L + J$ is equivalent to saying that for every $g \in L^q(T)$ the problem

$$\left.\begin{aligned}&-(|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \text{ a.e. on } T \\ &x'(0) \in \xi_1(x(0)),\ -x'(b) \in \xi_2(x(b))\end{aligned}\right\} \quad (19)$$

has a solution, where the notion of solution is defined as in § 3. To solve (19) we proceed as in the proof of proposition 2. So let $\nu, w \in R$ and consider the following auxiliary problem

$$-(|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \quad \text{a.e. on } T \tag{20}$$
$$x(0) = \nu,\ x(b) = w.$$

Let $\gamma(t) = (1-\frac{t}{b})\nu + \frac{t}{b}w$, $t \in T$ and set $y(t) = x(t) - \gamma(t)$. Then if we rewrite (20) in terms of the unknown function $y(\cdot)$, we have

$$\begin{aligned} &-\left(\left|y'(t)+\frac{w-\nu}{b}\right|^{p-2}\left(y'(t)+\frac{w-\nu}{b}\right)\right)' \\ &\qquad + |y(t)+\gamma(t)|^{p-2}(y(t)+\gamma(t)) = g(t) \quad \text{a.e. on } T \\ &y(0) = y(b) = 0. \end{aligned} \tag{21}$$

We will solve the homogeneous Dirichlet problem (21). To this end consider the operator $A : W_0^{1,p}(T) \to W^{-1,q}(T)$ defined by

$$\begin{aligned} \langle A(y), z\rangle &= \int_0^b \left|y'(t)+\frac{w-\nu}{b}\right|^{p-2}\left(y'(t)+\frac{w-\nu}{b}\right)z'(t)\,dt \\ &\quad + \int_0^b |y(t)+\gamma(t)|^{p-2}(y(t)+\gamma(t))z(t)\,dt, \end{aligned}$$

whereas before $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$.

Claim 1. $A(\cdot)$ is monotone, demicontinuous (hence maximal monotone) and weakly coercive.

To establish the monotonicity of $A(\cdot)$, let $y, z \in W_0^{1,p}(T)$. We have

$$\begin{aligned} &\langle A(y) - A(z), y - z\rangle \\ &= \int_0^b \Big[\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)(y'-z') \\ &\qquad - \left|z'+\frac{w-\nu}{b}\right|^{p-2}\left(z'+\frac{w-\nu}{b}\right)(y'-z')\Big]\,dt \\ &\quad + \int_0^b [|y+\gamma|^{p-2}(y+\gamma)(y-z) - |z+\gamma|^{p-2}(z+\gamma)(y-z)]\,dt \\ &= \int_0^b \Big[\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)\left(y'+\frac{w-\nu}{b}-z'-\frac{w-\nu}{b}\right) \\ &\qquad - \left|z'+\frac{w-\nu}{b}\right|^{p-2}\left(z'+\frac{w-\nu}{b}\right)\left(y'+\frac{w-\nu}{b}-z'-\frac{w-\nu}{b}\right)\Big]\,dt \\ &\quad + \int_0^b [|y+\gamma|^{p-2}(y+\gamma)(y+\gamma-z-\gamma) \\ &\qquad - |z+\gamma|^{p-2}(z+\gamma)(y+\gamma-z-\gamma)]\,dt \\ &\geq \int_0^b \Big[\left|y'+\frac{w-\nu}{b}\right|^{p} - \left|y'+\frac{w-\nu}{b}\right|^{p-1}\left|z'+\frac{w-\nu}{b}\right| - \left|z'+\frac{w-\nu}{b}\right|^{p-1}\left|y'+\frac{w-\nu}{b}\right| \end{aligned}$$

$$+\left|z'+\frac{w-\nu}{b}\right|^p\Bigg]\,dt+\int_0^b[|y+\gamma|^p-|y+\gamma|^{p-1}|z+\gamma|$$
$$-|z+\gamma|^{p-1}|y+\gamma|+|z+\gamma|^p]\,dt$$
$$=\int_0^b\left(\left|y'+\frac{w-\nu}{b}\right|^{p-1}-\left|z'+\frac{w-\nu}{b}\right|^{p-1}\right)\left(\left|y'+\frac{w-\nu}{b}\right|-\left|z'+\frac{w-\nu}{b}\right|\right)dt$$
$$+\int_0^b(|y+\gamma|^{p-1}-|z+\gamma|^{p-1})(|y+\gamma|-|z+\gamma|)\,dt\geq 0\quad(\text{see } \S\, 2).$$

This proves the monotonicity of $A(\cdot)$. For the demicontinuity, let $y_n\to y$ in $W^{1,p}(T)$ as $n\to\infty$. For every $z\in W_0^{1,p}(T)$, we have

$$|\langle A(y_n)-A(y),z\rangle|$$
$$\leq\left|\int_0^b\left(\left|y_n'+\frac{w-\nu}{b}\right|^{p-2}\left(y_n'+\frac{w-\nu}{b}\right)z'-\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)z'\right)dt\right|$$
$$+\left|\int_0^b(|y_n-\gamma|^{p-2}(y_n+\gamma)z-|y+\gamma|^{p-2}(y+\gamma)z)\,dt\right|.$$

By passing to a subsequence if necessary, we may assume that $y_n'(t)\to y'(t)$ a.e. on T and $y_n(t)\to y(t)$ for all $t\in T$. So via the extended dominated convergence theorem (see Ash [1], theorem 7.5.2, p. 295), we have that

$$|\langle A(y_n)-A(y),z\rangle|\to 0\quad\text{as } n\to\infty.$$

Since $z\in W_0^{1,p}(T)$ was arbitrary, we conclude that $A(y_n)\xrightarrow{w}A(y)$ in $W^{-1,q}(T)$ as $n\to\infty$. So $A(\cdot)$ is demicontinuous.

Finally we will show the weak coercivity of $A(\cdot)$. Indeed we have

$$\langle A(y),y\rangle=\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)y'\,dt+\int_0^b|y+\gamma|^{p-2}(y+\gamma)y\,dt.$$

Note that

$$\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)y'\,dt$$
$$=\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)\left(y'+\frac{w-\nu}{b}\right)dt$$
$$-\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)\left(\frac{w-\nu}{b}\right)dt$$
$$\geq\left\|y'+\frac{w-\nu}{b}\right\|_p^p-c_1\left\|y'+\frac{w-\nu}{b}\right\|_p^{p-1}\quad\text{for some } c_1>0. \tag{22}$$

Similarly we have

$$\int_0^b|y+\gamma|^{p-2}(y+\gamma)y\,dt\geq\|y+\gamma\|_p^p-c_2\|y+\gamma\|_p^{p-1}\quad\text{for some } c_2>0. \tag{23}$$

From (22) and (23) it follows that

$$\langle A(y),y\rangle\geq\|y+\gamma\|_{1,p}^p-c_3\|y+\gamma\|_{1,p}^{p-1}\quad\text{with } c_3=\max\{c_1,c_2\}.$$

This then gives us the desired weak coercivity of $A(\cdot)$. So we have proved the claim. Recall that a monotone, demicontinuous and weakly coercive operator is surjective. So we can find $y \in W_0^{1,p}(T)$ such that $A(y) = g$. For every $\psi \in C_0^\infty(T)$ we have

$$\int_0^b \left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)\psi'\, dt = \int_0^b (g(t)\psi(t) - |y+\gamma|^{p-2}(y+\gamma)\psi)\, dt.$$

From the definition of the distributional derivative it follows that

$$-\left(\left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)\right)'(t) = g(t) - (|y+\gamma|^{p-2}(y+\gamma))(t) \quad \text{a.e. on } T$$

$\Rightarrow y(\cdot)$ is a solution of (21).

Moreover, if x, y are two solutions of (21), we have

$$0 = \int_0^b (|x'|^{p-2}x' - |y'|^{p-2}y')(x'-y')\, dt + \int_0^b (|x|^{p-2}x - |y|^{p-2}y)(x-y)\, dt$$
$$\geq 2^{2-p}\int_0^b \big|\,|x'| - |y'|\,\big|^p\, dt + 2^{2-p}\int_0^b \big|\,|x| - |y|\,\big|^p\, dt.$$

From this inequality as in the proof of proposition 2, we obtain $x = y$. So the solution of (21) is unique, hence so is that of (20). Denote the unique solution of (20) by $\theta(\nu, w)$. By definition $\theta(\nu, w)(\cdot) \in C^1(T)$. So we can define $\rho : R \times R \to R \times R$ by

$$\rho(\nu, w) = (-|\theta(\nu,w)'(0)|^{p-2}\theta(\nu,w)'(0),\ |\theta(\nu,w)'(b)|^{p-2}\theta(\nu,w)'(b)).$$

Claim 2. $\rho(\cdot,\cdot)$ is monotone.

Let $x = \theta(a, \beta)$ and $x_1 = \theta(a_1, \beta_1)$. Using Green's formula, we have

$$\left(\rho(a,\beta) - \rho(a_1,\beta_1), \begin{pmatrix} a - a_1 \\ \beta - \beta_1 \end{pmatrix}\right)_{R^2} = -(|x'(0)|^{p-2}x'(0)$$
$$- |x_1'(0)|^{p-2}x_1'(0))(a - a_1) + (|x'(b)|^{p-2}x'(b) - |x_1'(b)|^{p-2}x_1'(b))(\beta - \beta_1)$$
$$= \int_0^b |x'(t)|^{p-2}x'(t)(x'(t) - x_1'(t))\, dt - \int_0^b |x_1'(t)|^{p-2}x_1'(t)(x'(t) - x_1'(t))\, dt$$
$$+ \int_0^b ((|x'(t)|^{p-2}x'(t))' - (|x_1'(t)|^{p-2}x_1'(t))')(x(t) - x_1(t))\, dt.$$

Note that by the elementary inequality mentioned in § 2, we have

$$\int_0^b |x'(t)|^{p-2}x'(t)(x'(t) - x_1'(t))\, dt - \int_0^b |x_1'(t)|^{p-2}x_1'(t)(x'(t) - x_1'(t))\, dt \geq 0.$$

Also since x, x_1 are solutions of (20), we have

$$\int_0^b ((|x'(t)|^{p-2}x'(t))' - (|x_1'(t)|^{p-2}x_1'(t))')(x(t) - x_1(t))\, dt$$
$$= \int_0^b (|x(t)|^{p-2}x(t) - |x_1(t)|^{p-2}x_1(t))(x(t) - x_1(t))\, dt \geq 0.$$

So finally we obtain

$$\left(\rho(a,\beta)-\rho(a_1,\beta_1),\begin{pmatrix}a-a_1\\ \beta-\beta_1\end{pmatrix}\right)_{R^2}\geq 0$$

$\Rightarrow \rho(\cdot,\cdot)$ is monotone.

Claim 3. $\rho(\cdot,\cdot)$ is continuous.

Assume $\nu_n \to \nu$ and $w_n \to w$ in R as $n\to\infty$. Set $x_n = \theta(\nu_n, w_n)$, $n\geq 1$, and $x=\theta(\nu,w)$. As before we introduce $\gamma_n(t)=(1-\frac{t}{b})\nu_n+\frac{t}{b}w_n$, $n\geq 1$ and $\gamma(t)=(1-\frac{t}{b})\nu+\frac{t}{b}w$ and define $y_n = x_n - \gamma_n$, $n\geq 1$, and $y=x-\gamma$. We have

$$-\left(\left|y_n'(t)+\frac{w-\nu}{b}\right|^{p-2}\left(y_n'(t)+\frac{w-\nu}{b}\right)\right)'+|y_n+\gamma_n|^{p-2}(y_n+\gamma_n)=g(t)\quad \text{a.e. on } T$$
$$y_n(0)=y_n(b)=0.$$

Multiplying with $y_n(t)$ and then integrating over T, we obtain

$$\int_0^b -(|(y_n+\gamma_n)'|^{p-2}(y_n+\gamma_n)')'y_n\,dt + \int_0^b |y_n+\gamma_n|^{p-2}(y_n+\gamma_n)y_n\,dt = \int_0^b g(t)y_n(t)\,dt.$$

Using Green's formula, we have

$$\begin{aligned}\int_0^b -(|(y_n+\gamma_n)'|^{p-2}(y_n+\gamma_n)')'y_n\,dt &= \int_0^b |y_n'+\gamma_n'|^{p-2}(y_n'+\gamma_n')y_n'\,dt\\ &= \|y_n'+\gamma_n'\|_p^p - \int_0^b |y_n'+\gamma_n'|(y_n'+\gamma_n')\gamma_n'\,dt\\ &\geq \|y_n'+\gamma_n'\|_p^p - c_4\|y_n'+\gamma_n'\|_p^{p-1}\quad \text{for some } c_4>0.\end{aligned}$$

Also we have that

$$\int_0^b |y_n+\gamma_n|^{p-2}(y_n+\gamma_n)\gamma_n\,dt \geq \|y_n+\gamma_n\|_p^p - \|y_n+\gamma_n\|_p^{p-1}\|\gamma_n\|_p.$$

Thus finally we have

$$\begin{aligned}\|y_n'+\gamma_n'\|_p^p+\|y_n+\gamma_n\|_p^p &\leq \|g\|_q(\|y_n+\gamma_n\|_p+\|\gamma_n\|_p)\\ &\quad+\|y_n+\gamma_n\|_p^{p-1}\|\gamma_n\|_p+c_4\|y_n'+\gamma_n'\|_p^{p-1}.\end{aligned}$$

Since $\sup_{n\geq 1}\|\gamma_n\|_p<\infty$, from this last inequality it follows that $\{x_n=y_n+\gamma_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T)$. Then $\{|x_n|^{p-2}x_n\}_{n\geq 1}$ and $\{|x_n'|^{p-2}x_n'\}_{n\geq 1}$ are both bounded in $L^q(T)$ and using eq. (20) we see that $\{|x_n'|^{p-2}x_n'\}_{n\geq 1}$ is bounded in $W^{1,q}(T)$. Thus we may assume that $x_n \xrightarrow{w} u$ in $W^{1,p}(T)$ and $|x_n'|^{p-2}x_n' \xrightarrow{w} \nu$ in $W^{1,q}(T)$ as $n\to\infty$. In particular $(|x_n'|^{p-2}x_n')' \xrightarrow{w} \nu'$ in $L^q(T)$ and $x_n\to u$ in $C(T)$ as $n\to\infty$. Moreover, as in the proof of proposition 2, we can check that $|x_n(\cdot)|^{p-2}x_n(\cdot)\to|u(\cdot)|^{p-2}u(\cdot)$ in $C(T)$ as $n\to\infty$. Hence in the limit as $n\to\infty$, we have

$$-\nu'(t)+|u(t)|^{p-2}u(t)=g(t)\quad \text{a.e. on } T$$
$$u(0)=\nu,\quad u(b)=\nu.$$

Since $|x_n'|^{p-2}x_n' \xrightarrow{w} \nu$ in $W^{1,q}(T)$, we have $|x_n'|^{p-2}x_n' \to \nu$ in $C(T)$ as $n \to \infty$ (recall that $W^{1,q}(T)$ is embedded compactly in $C(T)$). So $\sigma^{-1}(|x_n'|^{p-2}x_n') = x_n' \to \sigma^{-1}(\nu)$ in $C(T)$ as $n \to \infty$ (recall $\sigma(z) = |z|^{p-2}z$) and so $\sigma^{-1}(\nu) = u'$, hence $\nu = |u'|^{p-2}u'$. Thus in the limit as $n \to \infty$, we have

$$\begin{aligned}
&-(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t) = g(t) \quad \text{a.e. on } T\\
&u(0) = \nu, \quad u(b) = w\\
&\Rightarrow u = \theta(\nu, w) = x.
\end{aligned}$$

Also $|x_n'|^{p-2}x_n' \to |x'|^{p-2}x'$ in $C(T)$ as $n \to \infty$ and so finally $\theta(\nu_n, w_n)'(t) \to \theta(\nu, w)'(t)$ for all $t \in T$ as $n \to \infty$. Therefore we conclude that $\rho(\nu_n, w_n) \to \rho(\nu, w)$ which proves the continuity of $\rho(\cdot,\cdot)$.

Claim 4. $\rho(\cdot,\cdot)$ is weakly coercive.

We have

$$\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} = \frac{|x'(b)|^{p-2}x'(b)w - |x'(0)|^{p-2}x'(0)\nu}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|},$$

where $x = \theta(\nu, w)$. From Green's formula we have

$$\begin{aligned}
\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} &= \frac{\int_0^b (|x'(t)|^{p-2}x'(t))'x(t)\,dt + \|x'\|_p^p}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|}\\
&= \frac{\|x\|_p^p + \|x'\|_p^p - \int_0^b g(t)x(t)dt}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|}
\end{aligned}$$

$$(\text{since} -(|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \text{ a.e. on } T).$$

Using the mean value theorem for integrals (see Hewitt–Stromberg [8], theorem 21.69, p. 420), we can find $t_0 \in T$ such that $|x(t_0)|b = \int_0^b |x(t)|\,dt$. So for every $t \in T$ we have

$$\begin{aligned}
|x(t)| &\le |x(t_0)| + \int_{t_0}^t |x'(s)|\,ds \le \frac{1}{b}\|x\|_1 + b^{1/q}\|x'\|_p \le b^{-1/p}\|x\|_p\\
&\quad + b^{1/q}\|x'\|_p \le c_5\|x\|_{1,p} \quad \text{for some } c_5 > 0\\
&\Rightarrow \left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\| \le c_6\|x\|_{1,p} \quad \text{for some } c_6 > 0.
\end{aligned}$$

Thus finally we can write that

$$\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} \ge \frac{\|x\|_{1,p}^p - \|g\|_q\|x\|_{1,p}}{c_6\|x\|_{1,p}}$$

$$\Rightarrow \rho(\cdot,\cdot) \quad \text{is weakly coercive.}$$

Because $\rho(\cdot,\cdot)$ is monotone and continuous, is maximal monotone. Set $\widehat{\xi}_1 = \sigma \circ \xi_1$ and $\widehat{\xi}_2 = \sigma \circ \xi$ and define $\xi : R \times R \to 2^{R\times R}$ by $\xi(a,\beta) = [\widehat{\xi}_1(a), \widehat{\xi}_2(\beta)]$.

Claim 5. $\xi(\cdot,\cdot)$ is maximal monotone.

First we check the monotonicity of $\xi(\cdot,\cdot)$. It suffices to show the monotonicity of $\widehat{\xi}_i(\cdot)$, $i=1,2$. Let $a \le \beta$ if $a' \in \xi_1(a)$, $\beta' \in \xi_i(\beta)$, then $a' \le \beta'$ (since $\xi_i(\cdot)$ is monotone) and so $\sigma(a') \le \sigma(\beta')$ (since $\sigma(\cdot)$ is monotone). Hence $(\sigma(\beta') - \sigma(a'))(\beta - a) \ge 0$ which proves the monotonicity of $\widehat{\xi}_i(\cdot)$, hence the monotonicity of $\xi(\cdot,\cdot)$ too.

To check the maximality of $\xi(\cdot,\cdot)$ we proceed as follows. Suppose that for all $[a,\beta] \in \mathrm{dom}\,\xi = \mathrm{dom}\,\xi_1 \times \mathrm{dom}\,\xi_2$ and all $[a',\beta'] \in \xi(a,\beta) = [\widehat{\xi}_1(a), \widehat{\xi}_2(\beta)]$ we have

$$(\sigma(a') - \nu)(a - y) + (\sigma(\beta') - \nu_1)(\beta - y_1) \ge 0.$$

Let $u, u_1 \in R$ such that $\nu = \sigma(u)$, $\nu_1 = \sigma(u_2)$ (recall that $\sigma(\cdot)$ is surjective). Since $\xi_i + I$ ($I =$ identity map), $i = 1,2$, are surjective, we can find $\gamma, \delta \in R$ and $\gamma' \in \xi_1(\gamma)$, $\delta' \in \xi_2(\delta)$ such that

$$\gamma' + \gamma = u + y \ \text{ and } \ \delta' + \delta = u_1 + y_1.$$

Therefore we have

$$\sigma(\gamma') = \sigma(u + y - \gamma) \ \text{ and } \sigma(\delta') = \sigma(u_1 + y_1 - \delta). \tag{24}$$

So take $a = \gamma, \beta = \delta, a' = \gamma'$ and $\beta' = \delta'$. With such choices we have

$$(\sigma(u + y - \gamma) - \sigma(u))(\gamma - y) + (\sigma(u_1 + y_1 - \delta) - \sigma(u_1))(\delta - y_1) \ge 0.$$

But by virtue of the monotonicity of $\sigma(\cdot)$, each term in the lhs of the above inequality is nonpositive. So we infer that

$$\begin{aligned} &(\sigma(u + y - \gamma) - \sigma(u))(y - \gamma) = 0 \quad \text{and } (\sigma(u_1 + y_1 - \delta) - \sigma(u_1))(y_1 - \delta) = 0 \\ &\Rightarrow y = \gamma, y_1 = \delta \quad \text{(since } \sigma \text{ is strictly monotone)} \\ &\Rightarrow \gamma' = u, \delta' = u_1 \quad \text{(from (24) and the strict monotonicity of } \sigma). \end{aligned}$$

Thus $\nu \in \widehat{\xi}_1(\gamma)$ and $\nu_1 \in \widehat{\xi}_2(\delta)$, which prove the maximality of ξ.

Now set $\mu(\nu, w) = \xi(\nu, w) + \rho(\nu, w)$. Then from Zeidler [15] (theorem 21.I, p. 888), we have that μ is maximal monotone. Moreover, since $(0,0) \in \xi(0,0)$ (see hypothesis H(ξ)) and using claim 4, we can easily see that μ is weakly coercive. Therefore μ is surjective and so we can find $[\nu, w] \in R \times R$ such that $[0,0] \in \mu(\nu, w)$, hence $[\theta(\nu,w)'(0), -\theta(\nu,w)'(b)] = [\widehat{\xi}_1(\nu), \widehat{\xi}_2(w)]$. Then if $x = \theta(\nu, w)$, we can see that it solves problem (19). This proves the maximality of $L(\cdot)$.

This proposition leads us to the following existence theorem for problem (2).

Theorem 5. *If hypotheses* H(f)$_2$ *and* H(ξ) hold, *then problem* (2) *has a solution.*

Proof. By proposition 4, L is maximal monotone. Also $J : L^p(T) \to L^q(T)$ defined by $J(x) = \|x\|_p^{p-2} x$ is strictly monotone and continuous. So $L + J$ is maximal monotone (see Zeidler [15], theorem 32.I, p. 888) and strictly monotone. Therefore $K = (L+J)^{-1}$: $L^q(T) \to D \subseteq W^{1,p}(T)$ is a well-defined maximal monotone operator.

Arguing as in the proof of claim 2 of theorem 3 and using the fact that $0 \in \xi_i(0)$, $i = 1,2$ (see hypothesis H(ξ)) and since $\mathrm{Gr}\,\xi_i$ is closed, $i = 1,2$ (see § 2), we can check that K is compact. Having this we continue as in the proof of theorem 3 and via the Leray–Schauder principle we obtain a solution for (2).

Remark. A careful reading of the proof of proposition 4 reveals that in the vector case (i.e. $N > 1$) the proof of the maximality of ξ fails for $p > 2$. So it is an open problem whether theorem 5 also holds for vector equations with $p > 2$. It will be interesting to know the answer to this.

Some important special cases of theorem 5 are presented in the corollaries that follow:

(a) Let $K_1, K_2 \subseteq R$ be nonempty closed intervals containing the origin. Let $\xi_i = \partial\delta_{K_i}$, $i = 1, 2$, where $\delta_{K_i}(x) = \begin{cases} 0 \text{ if } x \in K_i \\ +\infty \text{ otherwise} \end{cases}$ and $\partial\delta_{K_i}(\cdot)$ its subdifferential. Then problem (2) takes the following form:

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) \in K_1, \ x(b) \in K_2 \\ &x'(0)x(0) = \sup_{v \in K_1} x'(0)v, \ -x'(b)x(b) = \sup_{w \in K_2} (-x'(b))w. \end{aligned} \tag{25}$$

COROLLARY 6

If hypotheses $\mathrm{H(f)}_2$ *hold and* ξ_1, ξ_2 *are as above, then problem* (25) *has a solution.*

(b) Let $K_1, K_2 = \{0\}$, in the above case. Then $\xi_i(x) = R$ for all $x \in R$ and so problem (2) becomes the Dirichlet problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) = x(b) = 0. \end{aligned} \tag{26}$$

COROLLARY 7

If hypotheses $\mathrm{H(f)}_2$ *hold, then problem* (26) *has a solution.*

Remark. This corollary partially extends the works of Boccardo *et al* [2], Drabek [4] and DelPino–Elgueta–Manasevich [3], where f is independent of x'. Also it is related to theorems 1 and 2 of Zhang [16] with the nonresonance conditions replaced by the Nagumo–Hartman condition $\mathrm{H(f)}_2$ (iv).

(c) Let $K_1, K_2 = R$. Then $\xi_i = \partial\delta_{K_i} = \{0\}$ for $i = 1, 2$. Then problem (2) becomes the homogeneous Neumann problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x'(0) = x'(b) = 0. \end{aligned} \tag{27}$$

COROLLARY 8

If hypotheses $\mathrm{H(f)}_2$ *hold, then problem* (27) *has a solution.*

Remark. This work partially extends theorem 5.2 of Guo [6], where f is independent of x'.

(d) Let $u_1, u_2 : R \to R$ be two contractions. Then $\xi_1 = u_1 - I, \xi_2 = u_2 - I$ are maximal monotone maps (I = identity map). Then problem (2) takes the following form:

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) + x'(0) = u_1(x(0)), \ \ x(b) - x'(b) = u_2(x(b)). \end{aligned} \tag{28}$$

COROLLARY 9

If hypotheses $\mathrm{H(f)}_2$ *hold and* $u_1, u_2 : R \to R$ *are contractions, then problem* (28) *has a solution.*

Remark. In Gaines–Mawhin [5] (p. 88) we can find semilinear problems (i.e. $p = 2$) with nonlinear boundary conditions.

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Equations of similitude

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Abstract. A general technique is developed to enlarge the Galois group of an equation from a subgroup of a finite classical isometry group towards the corresponding similitude group.

Keywords. Galois group; equation; isometry; similitude.

1. Introduction

In [Ab3, Ab4, Ab5], explicit equations were given whose Galois groups were the symplectic, orthogonal, and unitary groups $\mathrm{Sp}(2m,q)$, $\Omega^-(2m,q)$, and $\mathrm{SU}(2m-1,q')$, where $m>1$ is an integer and $q>1$ is a power of a prime p; in the unitary case $q=q'^2$ where $q'>1$ is a power of p. We shall now modify these equations so as to enlarge their Galois groups towards the corresponding similitude groups $\mathrm{GSp}(2m,q)$, $\mathrm{GO}^-(2m,q)$, and $\mathrm{GU}(2m-1,q')$. In § 2 we shall review the definitions of these finite classical groups. In § 3 we shall refine the composite polynomial lemma (2.4) of [Ab2] describing the Galois group of the composition of two polynomials. In § 4, the enlargement principle will be deduced as a consequence of this. In § 5, 6, and 7, the said principle will be applied to get the modified equations in the symplectic, orthogonal, and unitary cases respectively.

2. Review of classical groups

Recall that for any integer $n>0$, the general linear group $\mathrm{GL}(n,q)$ is the group of all nonsingular n by n matrices over the field $\mathrm{GF}(q)$ of cardinality q. Likewise, for an n-dimensional $\mathrm{GF}(q)$-vector-space V, the group of all $\mathrm{GF}(q)$-linear bijections $V\to V$ is denoted by $\mathrm{GL}(V)$. Moreover, for any basis $\beta=(\beta_1,\ldots,\beta_n)$ of V, we get a $\mathrm{GF}(q)$-linear bijection $\Lambda_\beta: V\to \mathrm{GF}(q)^n$ given by $\Lambda_\beta(\alpha_1\beta_1+\cdots+\alpha_n\beta_n)=(\alpha_1,\ldots,\alpha_n)$ for all $\alpha_1,\ldots,\alpha_n$ in $\mathrm{GF}(q)$, and this induces an isomorphism $\Lambda_\beta^*:\mathrm{GL}(V)\to\mathrm{GL}(n,q)$. Likewise, any $\mathrm{GF}(q)$-linear bijection $\Lambda: V\to V^\sharp$, where $V^\sharp$ is any other n-dimensional $\mathrm{GF}(q)$-vector-space, induces an isomorphism $\Lambda^*:\mathrm{GL}(V)\to\mathrm{GL}(V^\sharp)$. By $\mathrm{HL}(V)$ we denote the group of all homotheties of V, i.e., members of $\mathrm{GL}(V)$ of the form $v\mapsto\gamma v$ with $0\neq\gamma\in\mathrm{GF}(q)$, and by $\Theta_V:\mathrm{GL}(V)\to\mathrm{PGL}(V)$ we denote the canonical epimorphism where $\mathrm{PGL}(V)=\mathrm{GL}(V)/\mathrm{HL}(V)$. Likewise, by $\mathrm{HL}(n,q)$ we denote the group of all nonsingular n by n scalar matrices over $\mathrm{GF}(q)$, and by $\Theta_n:\mathrm{GL}(n,q)\to\mathrm{PGL}(n,q)$ we denote the canonical epimorphism where $\mathrm{PGL}(n,q)=\mathrm{GL}(n,q)/\mathrm{HL}(n,q)$. For any sub-

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group H of $\mathrm{GL}(V)$, by $\widehat{\Theta}_V(H)$ we denote the largest subgroup of $\mathrm{GL}(V)$ whose image under Θ_V coincides with the image of H, and we call $\widehat{\Theta}_V(H)$ the homothetization of H; note that then $\widehat{\Theta}_V(H) = \Theta_V^{-1}(\Theta_V(H))$ = the subgroup of $\mathrm{GL}(V)$ generated by H and $\mathrm{HL}(V)$. Likewise, for any subgroup H of $\mathrm{GL}(n,q)$, by $\widehat{\Theta}_n(H)$ we denote the largest subgroup of $\mathrm{GL}(n,q)$ whose image under Θ_n coincides with the image of H, and we call $\widehat{\Theta}_n(H)$ the homothetization of H; note that then $\widehat{\Theta}_n(H) = \Theta_n^{-1}(\Theta_n(H))$ = the subgroup of $\mathrm{GL}(n,q)$ generated by H and $\mathrm{HL}(n,q)$.

For any integer $m > 1$, the special linear group $\mathrm{SL}(m,q)$ is the subgroup of $\mathrm{GL}(m,q)$ consisting of all those matrices whose determinant is 1. The symplectic (isometry) group $\mathrm{Sp}(2m,q)$ is the group of all $\mu \in \mathrm{GL}(2m,q)$ which leave the symplectic form $\psi(x,y) = \sum_{i=1}^{m}(x_i y_{m+i} - y_i x_{m+i})$ unchanged, i.e., for which $\psi(x\mu, y\mu) = \psi(x,y)$. Note that $\mathrm{Sp}(2m,q) < \mathrm{SL}(2m,q)$ and $\mathrm{SL}(m,q) \lhd \mathrm{GL}(m,q)$ where $<$ and $\lhd$ indicate subgroup and normal subgroup, respectively. The symplectic similitude group $\mathrm{GSp}(2m,q)$ is the group of all $\mu \in \mathrm{GL}(2m,q)$ such that $\psi(x\mu, y\mu) = \gamma_\mu \psi(x,y)$ for some $0 \neq \gamma_\mu \in \mathrm{GF}(q)$. Note that $\mathrm{Sp}(2m,q) \lhd \mathrm{GSp}(2m,q)$ with $\mathrm{GSp}(2m,q)/\mathrm{Sp}(2m,q) = Z_{q-1}$ = the cyclic group of order $q-1$. By Chap. 2 of [KLi] we get the following:

Claim 2.1. If $p=2$ then $\widehat{\Theta}_{2m}(\mathrm{Sp}(2m,q)) = \mathrm{GSp}(2m,q)$. If $p>2$ then $\widehat{\Theta}_{2m}(\mathrm{Sp}(2m,q))$ is the unique group between $\mathrm{Sp}(2m,q)$ and $\mathrm{GSp}(2m,q)$ such that $[\mathrm{GSp}(2m,q): \widehat{\Theta}_{2m}(\mathrm{Sp}(2m,q))] = 2$.

To deal with the orthogonal (isometry) group $\mathrm{O}^-(2m,q)$, fix $\delta \in \mathrm{GF}(q)$ such that $T^2 + T + \delta$ is irreducible in $\mathrm{GF}(q)[T]$, and consider the quadratic form $\psi^-(x) = x_1 x_{m+1} + \cdots + x_{m-1}x_{2m-1} + x_m^2 + x_m x_{2m} + \delta x_{2m}^2$. Now $\mathrm{O}^-(2m,q)$ is the the group of all $\mu \in \mathrm{GL}(2m,q)$ which leave this form unchanged, i.e., for which $\psi^-(x\mu) = \psi^-(x)$. Moreover, the special orthogonal group $\mathrm{SO}^-(2m,q) = \mathrm{SL}(2m,q) \cap \mathrm{O}^-(2m,q)$, and the proper orthogonal group $\Omega^-(2m,q)$ = the commutator subgroup of $\mathrm{O}^-(2m,q)$. Note that $\Omega^-(2m,q) < \mathrm{SO}^-(2m,q) < \mathrm{O}^-(2m,q)$ with $[\mathrm{SO}^-(2m,q) : \Omega^-(2m,q)] = 2$ and $[\mathrm{O}^-(2m,q): \mathrm{SO}^-(2m,q)] = \mathrm{GCD}(2, q-1)$. The orthogonal similitude group $\mathrm{GO}^-(2m,q)$ is the group of all $\mu \in \mathrm{GL}(2m,q)$ such that $\psi^-(x\mu) = \gamma_\mu \psi^-(x)$ for some $0 \neq \gamma_\mu \in \mathrm{GF}(q)$. Note that $\mathrm{O}^-(2m,q) \lhd \mathrm{GO}^-(2m,q)$ with $\mathrm{GO}^-(2m,q)/\mathrm{O}^-(2m,q) = Z_{q-1}$. By Chap. 2 of [KLi] we get the following:

Claim 2.2. $\widehat{\Theta}_{2m}(\Omega^-(2m,q))$ is a group between $\Omega^-(2m,q)$ and $\mathrm{GO}^-(2m,q)$ such that $[\mathrm{GO}^-(2m,q) : \widehat{\Theta}_{2m}(\Omega^-(2m,q))] = 2$ or 4 or 8 according as $p=2$ or $p>2$ with $m(q-1)/4 \in \mathbb{Z}$ or $p>2$ with $m(q-1)/4 \notin \mathbb{Z}$.

Assuming $q = q'^2$ where $q' > 1$ is a power of p, the unitary (isometry) group $\mathrm{U}(2m-1,q')$ is the group of all $\nu \in \mathrm{GL}(2m-1,q)$ which leave the unitary form $\psi^\dagger(z) = z_1^{q'+1} + \cdots + z_{2m-1}^{q'+1}$ unchanged, i.e., for which $\psi^\dagger(z\nu) = \psi^\dagger(z)$. Moreover, the special unitary group $\mathrm{SU}(2m-1,q') = \mathrm{U}(2m-1,q') \cap \mathrm{SL}(2m-1,q)$. Finally, the unitary similitude group $\mathrm{GU}(2m-1,q')$ is the group of all $\nu \in \mathrm{GL}(2m-1,q)$ such that $\psi^\dagger(z\nu) = \gamma_\nu \psi^\dagger(z)$ for some $0 \neq \gamma_\nu \in \mathrm{GF}(q)$. Note that $\mathrm{SU}(2m-1,q') \lhd \mathrm{U}(2m-1,q') \lhd \mathrm{GU}(2m-1,q')$ with $[\mathrm{U}(2m-1,q') : \mathrm{SU}(2m-1,q')] = q'+1$ and $[\mathrm{GU}(2m-1,q') : \mathrm{U}(2m-1,q')] = q'-1$. By Chap. 2 of [KLi] we get the following:

Claim 2.3. $\widehat{\Theta}_{2m-1}(\mathrm{SU}(2m-1,q'))$ is a group between $\mathrm{SU}(2m-1,q')$ and $\mathrm{GU}(2m-1,q')$ such that $[\mathrm{GU}(2m-1,q') : \widehat{\Theta}_{2m-1}(\mathrm{SU}(2m-1,q'))] = \mathrm{GCD}(2m-1, q'+1)$.

For the above discussion and other properties of finite classical groups, see the books of Dickson [Dic], Kleidman–Liebeck [KLi] and Taylor [Tay]. For unitary groups we are

now using the notation of [Tay], whereas in [Ab5] we used the notation of [KLi]; namely, [KLi] use GU and Δ for the above U and GU respectively.

3. Compositions

Let

$$K \subset K^\sharp = K(S) \subset \Omega$$

be a chain of fields, where the element S is transcendental over K, and the field Ω is algebraically closed. For any nonzero polynomial $D = D(Y)$ of degree $P > 0$ in Y with coefficients in a subfield L of Ω, let

$$W[D] = \text{the set of all roots of } D(Y) \text{ in } \Omega$$

and note that then $L(W[D])$ is the splitting field of D over L in Ω; also note that D is separable iff $|W[D]| = P$ where $|W[D]|$ denotes the cardinality of $W[D]$. In (2.4) on p. 13 of [Ab2] we noted the following:

Composition Lemma 3.1. Let $E = E(Y)$ and $F = F(Y)$ be monic polynomials of degrees $M > 0$ and $N > 0$ in Y with coefficients in K respectively, such that $\Phi = \Phi\ (Y) = F(E(Y))$ is separable. Then F is separable and its splitting field $K(W[F])$ is contained in the splitting field $K(W[\Phi])$ of Φ. This gives a Galois theoretic epimorphism $\theta : \mathrm{Gal}(\Phi, K) \to \mathrm{Gal}(F, K)$ with $\ker \theta = \mathrm{Gal}(\Phi, K(W[F]))$. For every $u \in W[F]$, upon letting $E_u = E_u(Y) = E(Y) - u$ we have that E_u is separable. Moreover $\Phi(Y) = \prod_{u \in W[F]} E_u(Y)$ and as a disjoint union we have $W[\Phi] = \coprod_{u \in W[F]} W[E_u]$.

It follows that, for every $u \in W[F]$, the splitting field of E_u over $K(W[F])$ is contained in the splitting field of Φ over $K(W[F])$ which gives rise to a Galois theoretic epimorphism $\mathrm{Gal}(\Phi, K(W[F])) \to \mathrm{Gal}(E_u, K(W[F]))$ and hence $|\mathrm{Gal}(\Phi, K)|$ is divisible by $|\mathrm{Gal}(E_u, K(W[F]))|\,|\mathrm{Gal}(F, K)|$. Moreover, if for some u, the polynomial $E_u(Y)$ is irreducible in $K(W[F])[Y]$, then clearly $|\mathrm{Gal}(E_u, K(W[F]))|$ is divisible by the degree of $E_u(Y)$ which is M. Thus we have proved the following:

First refinement 3.2. In the situation of (3.1), if $E_u(Y)$ is irreducible in $K(W[F])[Y]$ for some $u \in W[F]$, then $|\mathrm{Gal}(\Phi, K)|$ is divisible by $M|\mathrm{Gal}(F, K)|$.

Let $F^\sharp = F^\sharp(Y) = S^N F(Y/S)$ and $\Phi^\sharp = \Phi^\sharp(Y) = F^\sharp(E(Y))$. Then $F^\sharp(Y)$ is a monic polynomial of degree N in Y with coefficients in $K[S] \subset K^\sharp$, and $u \mapsto uS$ gives a bijection $l : W[F] \to W[F^\sharp]$; consequently $F^\sharp$ is separable, its splitting field $K^\sharp(W[F^\sharp])$ over $K^\sharp$ coincides with the splitting field $K^\sharp(W[F])$ of F over $K^\sharp$, and the said bijection induces an isomorphism $l^* : \mathrm{Gal}(F, K) \to \mathrm{Gal}(F^\sharp, K^\sharp)$ of permutation groups. Likewise, $\Phi^\sharp(Y)$ is a monic polynomial of degree MN in Y with coefficients in $K[S] \subset K^\sharp$ and its Y-discriminant $\mathrm{Disc}_Y(\Phi^\sharp)$ equals some $\Delta(S) \in K[S]$; since Φ can be obtained by substituting $S = 1$ in $\Phi^\sharp$, its Y-discriminant $\mathrm{Disc}_Y(\Phi)$ equals $\Delta(1)$; consequently Φ separable $\Rightarrow$ $\Delta(1) \neq 0 \Rightarrow \Delta(S) \neq 0 \Rightarrow \Phi^\sharp$ separable. As noted in (3.1), the splitting field $K^\sharp(W[\Phi^\sharp])$ of $\Phi^\sharp$ over $K^\sharp$ contains the splitting field $K^\sharp(W[F^\sharp])$ of $F^\sharp$ over $K^\sharp$, and this gives a Galois theoretic epimorphism $\theta^\sharp : \mathrm{Gal}(\Phi^\sharp, K^\sharp) \to \mathrm{Gal}(F^\sharp, K^\sharp)$ with $\ker \theta^\sharp = \mathrm{Gal}(\Phi^\sharp, K^\sharp(W[F^\sharp]))$. If $F(Y) \neq Y^N$ then we can take $0 \neq u \in W[F]$; by the above observation $uS \in W[F^\sharp]$, and by Gauss Lemma $E(Y) - uS$ is irreducible in $K^\sharp(W[F^\sharp])[Y]$; consequently by (3.2), $|\mathrm{Gal}(\Phi^\sharp, K^\sharp)|$ is divisible by $M|\mathrm{Gal}(F^\sharp, K^\sharp)|$. Thus we have proved the following:

Second refinement 3.3. In the situation of (3.1), let $F^\sharp = F^\sharp(Y) = S^N F(Y/S)$ and $\Phi^\sharp = \Phi^\sharp(Y) = F^\sharp(E(Y))$. Then $F^\sharp(Y) \in K^\sharp[Y]$ is monic separable of degree N, and its splitting field $K^\sharp(W[F^\sharp])$ over $K^\sharp$ coincides with the splitting field $K^\sharp(W[F])$ of F over $K^\sharp$; moreover, $u \mapsto uS$ gives a bijection $l : W[F] \to W[F^\sharp]$ which induces an isomorphism $l^* : \mathrm{Gal}(F,K) \to \mathrm{Gal}(F^\sharp, K^\sharp)$ of permutation groups. Likewise, $\Phi^\sharp(Y) \in K^\sharp[Y]$ is monic separable of degree MN, its splitting field $K^\sharp(W[\Phi^\sharp])$ over $K^\sharp$ contains the splitting field $K^\sharp(W[F^\sharp])$ of $F^\sharp$ over $K^\sharp$, and this gives a Galois theoretic epimorphism $\theta^\sharp : \mathrm{Gal}(\Phi^\sharp, K^\sharp) \to \mathrm{Gal}(F^\sharp, K^\sharp)$ with $\ker \theta^\sharp = \mathrm{Gal}(\Phi^\sharp, K^\sharp(W[F^\sharp]))$; moreover, if $F(Y) \neq Y^N$ then $|\mathrm{Gal}(\Phi^\sharp, K^\sharp)|$ is divisible by $M|\mathrm{Gal}(F^\sharp, K^\sharp)|$.

Now, instead of assuming $F(Y) \neq Y^N$, assume that $E(Y) = Y^M$, take $R \in \Omega$ with $R^M = S$, and let $K^\flat = K(R)$. Then the splitting field $K^\flat(W[\Phi^\sharp])$ of $\Phi^\sharp$ over $K^\flat$ contains the splitting field $K^\flat(W[F^\sharp])$ of $F^\sharp$ over $K^\flat$ and this gives a Galois theoretic epimorphism $\theta^\flat : \mathrm{Gal}(\Phi^\sharp, K^\flat) \to \mathrm{Gal}(F^\sharp, K^\flat)$. Moreover, $v \mapsto vR$ gives a bijection $\lambda : W[\Phi] \to W[\Phi^\sharp]$ which induces an isomorphism $\lambda^\flat : \mathrm{Gal}(\Phi, K) \to \mathrm{Gal}(\Phi^\sharp, K^\flat)$ of permutation groups. Likewise, the bijection $l : W[F] \to W[F^\sharp]$ induces an isomorphism $l^\flat : \mathrm{Gal}(F,K) \to \mathrm{Gal}(F^\sharp, K^\flat)$ of permutation groups. This gives the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Gal}(\Phi, K) & \xrightarrow{\lambda^\flat} & \mathrm{Gal}(\Phi^\sharp, K^\flat) & \xrightarrow{\lambda^\sharp} & \mathrm{Gal}(\Phi^\sharp, K^\sharp) \\
\theta \downarrow & & \downarrow \theta^\flat & & \downarrow \theta^\sharp \\
\mathrm{Gal}(F, K) & \xrightarrow[l^\flat]{} & \mathrm{Gal}(F^\sharp, K^\flat) & \xrightarrow[l^\sharp]{} & \mathrm{Gal}(F^\sharp, K^\sharp)
\end{array}$$

where $\lambda^\sharp$ and $l^\sharp$ are the usual monomorphisms of permutations groups obtained by restricting field automorphisms to suitable subfields (cf. the basic extension principle on p. 93 of [Ab1]). Let $\lambda^* = \lambda^\sharp \circ \lambda^\flat$. Then $\lambda^* : \mathrm{Gal}(\Phi, K) \to \mathrm{Gal}(\Phi^\sharp, K^\sharp)$ is a monomorphism of permutation groups, and we get the condensed commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(\Phi, K) & \xrightarrow{\lambda^*} & \mathrm{Gal}(\Phi^\sharp, K^\sharp) \\
\theta \downarrow & & \downarrow \theta^\sharp \\
\mathrm{Gal}(F, K) & \xrightarrow[l^*]{} & \mathrm{Gal}(F^\sharp, K^\sharp)
\end{array}$$

where for the isomorphism l^* we have $l^* = l^\sharp \circ l^\flat$. Thus we have proved the following:

Third refinement 3.4. In the situation of (3.1) and (3.3), instead of assuming $F(Y) \neq Y^N$, assume that $E(Y) = Y^M$, take $R \in \Omega$ with $R^M = S$, and let $K^\flat = K(R)$. Then the splitting field $K^\flat(W[\Phi^\sharp])$ of $\Phi^\sharp$ over $K^\flat$ contains the splitting field $K^\flat(W[F^\sharp])$ of $F^\sharp$ over $K^\flat$ and this gives a Galois theoretic epimorphism $\theta^\flat : \mathrm{Gal}(\Phi^\sharp, K^\flat) \to \mathrm{Gal}(F^\sharp, K^\flat)$. Moreover, $v \mapsto vR$ gives a bijection $\lambda : W[\Phi] \to W[\Phi^\sharp]$ which induces an isomorphism $\lambda^\flat : \mathrm{Gal}(\Phi, K) \to \mathrm{Gal}(\Phi^\sharp, K^\flat)$ of permutation groups. Likewise, the bijection $l : W[F] \to W[F^\sharp]$ induces an isomorphism $l^\flat : \mathrm{Gal}(F,K) \to \mathrm{Gal}(F^\sharp, K^\flat)$ of permutation groups. Let $\lambda^* = \lambda^\sharp \circ \lambda^\flat$ where $\lambda^\sharp : \mathrm{Gal}(\Phi^\sharp, K^\flat) \to \mathrm{Gal}(\Phi^\sharp, K^\sharp)$ and $l^\sharp : \mathrm{Gal}(F^\sharp, K^\flat) \to \mathrm{Gal}(F^\sharp, K^\sharp)$ are the usual monomorphisms of permutation groups. Then $\lambda^* : \mathrm{Gal}(\Phi, K) \to \mathrm{Gal}(\Phi^\sharp, K^\sharp)$ is a monomorphism of permutation groups, and for the isomorphism l^* we have $l^* = l^\sharp \circ l^\flat$. Moreover, the above two diagrams commute.

4. Enlargement by homogenization

Let

$$\mathrm{GF}(q) \subset K \subset K^\sharp = K(S) \subset \Omega$$

be a chain of fields, where the element S is transcendental over K, and the field Ω is algebraically closed. As an abbreviation, for every integer $i \geq -1$ we put

$$\langle i \rangle = 1 + q + \cdots + q^i.$$

Following the definitions introduced in [Ab6], let $f = f(Y) \in K[Y]$ be a separable monic projective q-polynomial of q-prodegree n in Y over K and let $\phi = \phi(Y)$ be the subvectorial associate of f, i.e., let

$$f = f(Y) = Y^{\langle n-1 \rangle} + \sum_{i=1}^{n} a_i Y^{\langle n-1-i \rangle} \quad \text{where} \quad a_i \in K \quad \text{with} \quad a_n \neq 0$$

and

$$\phi = \phi(Y) = f(Y^{q-1}) = Y^{q^n-1} + \sum_{i=1}^{n} a_i Y^{q^{n-i}-1}.$$

Let

$$V[\phi] = \text{the set of all roots of } Y\phi(Y) \text{ in } \Omega$$

and note that then

$$V[\phi] = \{0\} \cup W[\phi].$$

Now, as noted in [Ab2], $V[\phi]$ is an n-dimensional GF(q)-vector-subspace of Ω, and in a natural manner we have $\mathrm{Gal}(\phi, K) < \mathrm{GL}(V[\phi])$. Again, as noted in [Ab2], $y \mapsto y^{q-1}$ gives a surjective map $W[\phi] \to W[f]$ whose fibres are punctured 1-spaces, and hence we may identify $W[f]$ with the projective space associated with $V[\phi]$, and after this identification we get $\mathrm{Gal}(f, K) < \mathrm{PGL}(V[\phi])$ in such a manner that the Galois theoretic epimorphism $\mathrm{Gal}(\phi, K) \to \mathrm{Gal}(f, K)$ is compatible with the canonical epimorphism $\Theta_{V[\phi]} : \mathrm{GL}(V[\phi]) \to \mathrm{PGL}(V[\phi])$.

Let $f^\sharp = f^\sharp(Y)$ be the homogenization of f, i.e., let

$$f^\sharp = f^\sharp(Y) = S^{\langle n-1 \rangle} f(Y/S) = Y^{\langle n-1 \rangle} + \sum_{i=1}^{n} a_i S^{\langle n-1 \rangle - \langle n-1-i \rangle} Y^{\langle n-1-i \rangle}$$

and note that then $f^\sharp = f^\sharp(Y) \in K^\sharp[Y]$ is a separable monic projective q-polynomial of q-prodegree n in Y over $K^\sharp$. Let

$$\phi^\sharp = \phi^\sharp(Y) = f^\sharp(Y^{q-1}) = Y^{q^n-1} + \sum_{i=1}^{n} a_i S^{\langle n-1 \rangle - \langle n-1-i \rangle} Y^{q^{n-i}-1}$$

be the subvectorial associate of $f^\sharp$.

Now $E(Y) = Y^{q-1} \in K[Y]$ is monic of degree $M = q-1$ in Y and $\phi(Y) = f(E(Y))$. Taking $R \in \Omega$ with $R^{q-1} = S$, we see that $v \mapsto vR$ gives a GF(q)-linear bijection $\Lambda : V[\phi] \to V[\phi^\sharp]$. Since $|\mathrm{GL}(V[\phi^\sharp])| = (q-1)|\mathrm{PGL}(V[\phi^\sharp])|$, we also see that: if $H < H^\sharp < \mathrm{GL}(V[\phi^\sharp])$ with $\Theta_{V[\phi^\sharp]}(H) = \Theta_{V[\phi^\sharp]}(H^\sharp)$ and $|H^\sharp| \geq (q-1)|\Theta_{V[\phi^\sharp]}(H^\sharp)|$, then $H^\sharp = \widehat{\Theta}_{V[\phi^\sharp]}(H)$. Therefore by (3.1), (3.3) and (3.4) we get the following:

Enlargement principle 4.1. Take $R \in \Omega$ with $R^{q-1} = S$. Then $v \mapsto vR$ gives a GF(q)-linear bijection $\Lambda : V[\phi] \to V[\phi^\sharp]$. Moreover, for the induced isomorphism $\Lambda^* : \mathrm{GL}(V[\phi]) \to \mathrm{GL}(V[\phi^\sharp])$ we have $\mathrm{Gal}(\phi^\sharp, K^\sharp) = \widehat{\Theta}_{V[\phi^\sharp]}(\Lambda^*(\mathrm{Gal}(\phi, K)))$.

As an immediate consequence of (4.1) we have the following:

COROLLARY 4.2

If for some basis $\beta = (\beta_1, \ldots, \beta_n)$ of $V[\phi]$ we have $\Lambda^*_\beta(\mathrm{Gal}(\phi, K)) = H < \mathrm{GL}(n, q)$, then for the basis $\beta^\sharp = (\beta_1 R, \ldots, \beta_n R)$ of $V[\phi^\sharp]$ we have $\Lambda^*_{\beta^\sharp}(\mathrm{Gal}(\phi^\sharp, K^\sharp)) = \widehat{\Theta}_n(H)$.

5. Symplectic equations

Let e be an integer with $1 \le e \le m-1$, and let

$$\mathrm{GF}(q) \subset k_p \subset K_e = k_p(X, T_1, \ldots, T_e) \subset K_e^\sharp = K_e(S) \subset \Omega$$

be a chain of fields, where the elements $X, S, T_1, \ldots, T_e$ are algebraically independent over k_p, and the field Ω is algebraically closed. Consider the separable monic projective q-polynomial

$$f_e = f_e(Y) = Y^{\langle 2m-1\rangle} + 1 + XY^{\langle m-1\rangle} + \sum_{i=1}^{e} (T_i^{q^i} Y^{\langle m-1+i\rangle} + T_i Y^{\langle m-1-i\rangle})$$

of q-prodegree $2m$ in Y over K_e, and let

$$\phi_e = \phi_e(Y) = f_e(Y^{q-1}) = Y^{q^{2m}-1} + 1 + XY^{q^m-1} + \sum_{i=1}^{e} (T_i^{q^i} Y^{q^{m+i}-1} + T_i Y^{q^{m-i}-1})$$

be the subvectorial associate of f_e. Let

$$\begin{aligned} f_e^\sharp = f_e^\sharp(Y) &= S^{\langle 2m-1\rangle} f_e(Y/S) \\ &= Y^{\langle 2m-1\rangle} + S^{\langle 2m-1\rangle} + S^{\langle 2m-1\rangle - \langle m-1\rangle} XY^{\langle m-1\rangle} \\ &\quad + \sum_{i=1}^{e} (S^{\langle 2m-1\rangle - \langle m-1+i\rangle} T_i^{q^i} Y^{\langle m-1+i\rangle} + S^{\langle 2m-1\rangle - \langle m-1-i\rangle} T_i Y^{\langle m-1-i\rangle}) \end{aligned}$$

be the homogenization of f_e, and let

$$\begin{aligned} \phi_e^\sharp = \phi_e^\sharp(Y) &= f_e^\sharp(Y^{q-1}) \\ &= Y^{q^{2m}-1} + S^{\langle 2m-1\rangle - \langle m-1\rangle} XY^{q^m-1} \\ &\quad + \sum_{i=1}^{e} (S^{\langle 2m-1\rangle - \langle m-1+i\rangle} T_i^{q^i} Y^{q^{m+i}-1} + S^{\langle 2m-1\rangle - \langle m-1-i\rangle} T_i Y^{q^{m-1}-1}) \end{aligned}$$

be the subvectorial associate of $f_e^\sharp$.

The following assertion was proved in Theorem 6.2 of [Ab3] when k_p is algebraically closed and $m > 2$; the assumption of k_p being algebraically closed was removed in Remark 4.3 of [AL1], and the assumption of m being greater than 2 was removed in Remark (5.3) of [AL2].

Assertion 5.1. For some basis β of $V[\phi_e]$ we have $\Lambda^*_\beta(\mathrm{Gal}(\phi_e, K_e)) = \mathrm{Sp}(2m, q)$.

By applying (4.2) with $n = 2m$ and $\phi = \phi_e$, as an immediate consequence of (5.1) we get the following:

COROLLARY 5.2

For some basis $\beta^\sharp$ of $V[\phi_e^\sharp]$ we have $\Lambda^*_{\beta^\sharp}(\mathrm{Gal}(\phi_e^\sharp, K_e^\sharp)) = \widehat{\Theta}_{2m}(\mathrm{Sp}(2m,q))$; see (2.1).

This explains Case (′) of [AL1] and [AL2].

6. Orthogonal equations

Let e be an integer with $1 \le e \le m-1$, and let

$$\overline{k}_p \subset \overline{K}_e = \overline{k}_p(T_1,\dots,T_e) \subset \overline{K}_e^\sharp = \overline{K}_e(S) \subset \Omega$$

be a chain of fields, where the elements $S, T_1, \dots, T_e$ are algebraically independent over the algebraically closed field $\overline{k}_p$ of characteristic p, and the field Ω is algebraically closed. Consider the separable monic projective q-polynomial

$$f_e^- = f_e^-(Y) = Y^{\langle 2m-1\rangle} - 1 + \sum_{i=1}^{e}(T_i^{q^i} Y^{\langle m-1+i\rangle} - T_i Y^{\langle m-1-i\rangle})$$

of q-prodegree $2m$ in Y over $\overline{K}_e$, and let

$$\phi_e^- = \phi_e^-(Y) = f_e^-(Y^{q-1}) = Y^{q^{2m}-1} - 1 + \sum_{i=1}^{e}(T_i^{q^i} Y^{q^{m+i}-1} - T_i Y^{q^{m-i}-1})$$

be the subvectorial associate of f_e^-. Let

$$\begin{aligned} f_e^{-\sharp} = f_e^{-\sharp}(Y) &= S^{\langle 2m-1\rangle} f_e^-(Y/S) \\ &= Y^{\langle 2m-1\rangle} - S^{\langle 2m-1\rangle} + \sum_{i=1}^{e} \\ &\quad \times (S^{\langle 2m-1\rangle - \langle m-1+i\rangle} T_i^{q^i} Y^{\langle m-1+i\rangle} - S^{\langle 2m-1\rangle - \langle m-1-i\rangle} T_i Y^{\langle m-1-i\rangle}) \end{aligned}$$

be the homogenization of f_e^-, and let

$$\begin{aligned} \phi_e^{-\sharp} = \phi_e^{-\sharp}(Y) &= f_e^{-\sharp}(Y^{q-1}) \\ &= Y^{q^{2m}-1} - S^{\langle 2m-1\rangle} \\ &\quad + \sum_{i=1}^{e}(S^{\langle 2m-1\rangle - \langle m-1+i\rangle} T_i^{q^i} Y^{q^{m+i}-1} - S^{\langle 2m-1\rangle - \langle m-1-i\rangle} T_i Y^{q^{m-i}-1}) \end{aligned}$$

be the subvectorial associate of $f_e^{-\sharp}$.

The following assertion was proved in Theorem 6.2 of [Ab4].

Assertion 6.1. If $m > 3$ and $p > 2 \le e$, then for some basis β of $V[\phi_e^-]$ we have $\Lambda^*_\beta(\mathrm{Gal}(\phi_e^-, \overline{K}_e)) = \Omega^-(2m,q)$.

By applying (4.2) with $n = 2m$ and $\phi = \phi_e^-$, as an immediate consequence of (6.1) we get the following:

COROLLARY 6.2

If $m > 3$ and $p > 2 \le e$, then for some basis $\beta^\sharp$ of $V[\phi_e^{-\sharp}]$ we have $\Lambda^*_{\beta^\sharp}(\mathrm{Gal}(\phi_e^{-\sharp}, \overline{K}_e^\sharp)) = \Theta_{2m}(\Omega^-(2m,q))$; see (2.2).

7. Unitary equations

Let e be an integer with $1 \leq e \leq m-1$, and let

$$\overline{k}_p \subset \overline{K}_e = \overline{k}_p(T_1, \ldots, T_e) \subset \overline{K}_e^\sharp = \overline{K}_e(S) \subset \Omega$$

be a chain of fields, where the elements $S, T_1, \ldots, T_e$ are algebraically independent over the algebraically closed field $\overline{k}_p$ of characteristic p, and the field Ω is algebraically closed. Consider the separable monic projective q-polynomial

$$f_e^\dagger = f_e^\dagger(Y) = Y^{\langle 2m-2\rangle} + 1 + \sum_{i=1}^{e}(T_i^{q'q^{i-1}} Y^{\langle m-2+i\rangle} + T_i Y^{\langle m-1-i\rangle})$$

of q-prodegree $2m-1$ in Y over $\overline{K}_e$, and let

$$\phi_e^\dagger = \phi_e^\dagger(Y) = f_e^\dagger(Y^{q-1}) = Y^{q^{2m-1}-1} + 1 + \sum_{i=1}^{e}(T_i^{q'q^{i-1}} Y^{q^{m-1+i}-1} + T_i Y^{q^{m-i}-1})$$

be the subvectorial associate of $f_e^\dagger$. Let

$$\begin{aligned} f_e^{\dagger\sharp} = f_e^{\dagger\sharp}(Y) &= S^{\langle 2m-2\rangle} f_e^\dagger(Y/S) \\ &= Y^{\langle 2m-2\rangle} + S^{\langle 2m-2\rangle} + \sum_{i=1}^{e} \\ &\quad \times (S^{\langle 2m-2\rangle - \langle m-2+i\rangle} T_i^{q'q^{i-1}} Y^{\langle m-2+i\rangle} + S^{\langle 2m-2\rangle - \langle m-1-i\rangle} T_i Y^{\langle m-1-i\rangle}) \end{aligned}$$

be the homogenization of $f_e^\dagger$, and let

$$\begin{aligned} \phi_e^{\dagger\sharp} = \phi_e^{\dagger\sharp}(Y) &= f_e^{\dagger\sharp}(Y^{q-1}) \\ &= Y^{q^{2m-1}-1} + S^{\langle 2m-2\rangle} + \sum_{i=1}^{e} \\ &\quad \times (S^{\langle 2m-2\rangle - \langle m-2+i\rangle} T_i^{q'q^{i-1}} Y^{q^{m-1+i}-1} + S^{\langle 2m-2\rangle - \langle m-1-i\rangle} T_i Y^{q^{m-i}-1}) \end{aligned}$$

be the subvectorial associate of $f_e^{\dagger\sharp}$.

The following assertion was proved in Theorem 6.2 of [Ab5].

Assertion 7.1. For some basis β of $V[\phi_e^\dagger]$ we have $\Lambda_\beta^*(\mathrm{Gal}(\phi_e^\dagger, \overline{K}_e)) = \mathrm{SU}(2m-1, q')$.

By applying (4.2) with $n = 2m-1$ and $\phi = \phi_e^\dagger$, as an immediate consequence of (7.1) we get the following:

COROLLARY 7.2

For some basis $\beta^\sharp$ of $V[\phi_e^{\dagger\sharp}]$ we have $\Lambda_{\beta^\sharp}^*(\mathrm{Gal}(\phi_e^{\dagger\sharp}, \overline{K}_e^\sharp)) = \widehat{\Theta}_{2m-1}(\mathrm{SU}(2m-1, q'))$; see (2.3).

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Remarks on the wonderful compactification of semisimple algebraic groups

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Abstract. We prove that if G is a semisimple algebraic group of adjoint type over the field of complex numbers, H is the subgroup of all fixed points of an involution σ of G that is induced by an involution $\hat{\sigma}$ of the simply connected covering $\hat{G}$ of G, then the wonderful compactification $\overline{G/H}$ of the homogeneous space G/H is isomorphic to the G.I.T quotient $\overline{G}^{ss}(L)//H$ of the wonderful compactification $\overline{G}$ of G for a suitable choice of a line bundle L on $\overline{G}$. We also prove a functorial property of the wonderful compactifications of semisimple algebraic groups of adjoint type.

Keywords. Involution; wonderful compactification; symmetric varieties; special dominant weight; line bundle.

1. Introduction

In [D-P], what are known as "wonderful compactifications" of symmetric varieties were constructed, and studied by De Concini and Procesi. More precisely, if G is a semisimple algebraic group of adjoint type over the field of complex numbers, H is the subgroup of all fixed points of an involution σ of G that is induced by an involution $\hat{\sigma}$ of the simply connected covering $\hat{G}$ of G, then, they have constructed a complete embedding $\overline{G/H}$ of the homogeneous space G/H, with boundary being a union of normal crossing divisors. In particular, one gets such a compactification $\overline{G}$ for the group G (G being considered as $(G \times G)/\Delta(G)$).

Now, we consider the subgroup H of G as above and view $\overline{G}$ as a H variety (H acting on the right). We also have H linearised ample line bundles L, and one can therefore take the G.I.T quotients [cf [GIT]] $\overline{G}^{ss}(L)//H$ of $\overline{G}$ (it is known that the connected component of the identity element in $H = G^{\sigma}$ is reductive and hence the G.I.T qoutient $\overline{G}^{ss}(L)//H$ makes sense). This way also one obtains a compactification of G/H. A natural question is to get an explicit relationship between these compactifications and the "wonderful compactifications".

The aim of this paper is to prove the following result:

(a) There is a G-linearised ample line bundle L on $\overline{G}$ such that $\overline{G}^{ss}(L)//H$ is isomorphic to $\overline{G/H}$.
Continuing this line of investigation, we also obtain
(b) A natural functorial property of the wonderful compactifications $\overline{G}$ of G. [cf Theorem 4.7 for precise statement.]

The layout of this paper is as follows:

Section 2 consists of notations and basic theorems. In § 3, we prove result (a). In § 4, we prove result (b).

2. Notations and basic theorems

Through out § 2 *and* 3, *we fix the following notations*. Let G be a semisimple algebraic group of adjoint type over the field of complex numbers. Let $\pi : \hat{G} \longrightarrow G$ be a simply connected covering of G. Let $\sigma : G \longrightarrow G$ be an automorphism of order two that is induced by an involution $\hat{\sigma}$ of $\hat{G}$, let T be a σ stable maximal torus of G such that the dimension of the subtorus $T_1 = \{t \in T : \sigma(t) = t^{-1}\}$ is maximal and let $H = G^\sigma$, denote the invariants of σ in G. Also, let H' denote the invariants of $\hat{\sigma}$ in $\hat{G}$ and let $\hat{H}$ denote the normaliser of H' in $\hat{G}$. We note that $\hat{H}$ is actually the pullback $\pi^{-1}(H)$. The group of characters of H' (resp. of $\hat{H}$) is denoted by $X(H')$ (resp. $X(\hat{H})$).

We have

Lemma 2.1. [*cf* [D-P] (*p*-4)]. *One can choose the set* Φ^+ *of positive roots* (*with respect to* T *as above*) *in such a way that:*

$$\text{if } \alpha \in \Phi^+, \text{ then, either } \sigma(\alpha) = \alpha \text{ or } \sigma(\alpha) \in \Phi^-.$$

Let Φ^+ be the set of positive roots as in Lemma 2.1, B be the corresponding Borel subgroup of G containing T. Let $\Phi_0 = \{\alpha \in \Phi : \sigma(\alpha) = \alpha\}$, $\Phi_1 = \Phi - \Phi_0$.

Let $\Gamma \subset \Phi^+$ denote the set of simple roots, let $\Gamma_0 = \Gamma \cap \Phi_0$ and let $\Gamma_1 = \Gamma \cap \Phi_1$. We label these sets by:

$$\Gamma_0 = \{\beta_1, \beta_2, \ldots, \beta_k\} \text{ and } \Gamma_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_j\}.$$

Let $(.,.)$ denote the positive definite bilinear form on $E =$ (Root lattice) $\otimes \mathbb{R}$ induced by the Killing form of the Lie algebra of G and define $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$. Now, consider the fundamental weights. Since, they form a dual basis of the simple co-roots, we also divide them:

$$\{\zeta_1, \ldots, \zeta_k\} \text{ and } \{\varpi_1, \ldots, \varpi_j\},$$

where ζ_i is dual to β_i and ϖ_i is dual to α_i.

Using the Killing form, one can see that σ induces a permutation $\tilde{\sigma}$ of order two in the indices $\{1, 2, \ldots, j\}$ such that $\sigma(\varpi_i) = -\varpi_{\tilde{\sigma}(i)}$. For a proof, one can see [[D-P] (pp 5–6)]. We recall from [D-P], the number l denotes the rank (G/H) and we have

$$\begin{aligned} l &= \text{the number of } \tilde{\sigma} \text{ orbits in } \{1, 2, \ldots, j\} \\ &= \dim(T_1). \end{aligned}$$

Lemma 2.2. [*cf* (*p*-5) [*D-P*]]. *For every* $\alpha_i \in \Gamma_1$ *we have that* $\sigma(\alpha_i)$ *is of the form* $-(\alpha_{\tilde{\sigma}(i)} + \sum_s n_{i,s}\beta_s)$ *where* $n_{i,s}$*'s are nonnegative integers.*

DEFINITION 2.3

A dominant weight λ is special if it is of the form $\sum_{r=1}^{j} n_r \varpi_r$ with $n_r = n_{\tilde{\sigma}(r)}$. A special weight is regular if $n_r \neq 0$ for all $r = 1, 2, \ldots, j$.

One can check that λ is special if and only if $\sigma(\lambda) = -\lambda$.

We have

Lemma 2.4. [*cf* [*D-P*] (*p*-6)]. *Let* λ *be a dominant weight and* V_λ *the corresponding irreducible representation of* $\hat{G}$ *with highest weight* λ. *Then if* $V_\lambda^{\hat{H}}$ *denotes the subspace of* $\hat{H}$ *invariant vectors in* V_λ, *one has* $\dim V_\lambda^{\hat{H}} \leq 1$, *further, if* $V_\lambda^{\hat{H}} \neq 0$, *then,* λ *is special.*

In [D-P], the Lemma (2.4) is stated in terms of the group H'. Since any $\hat{H}$ invariant vector in V_λ is also a H' invariant vector in V_λ and since we will work with the group $\hat{H}$, we have stated it in terms of $\hat{H}$.

We denote the weight lattice of G with respect to T by P. We denote the set of all dominant weights by $P_{\geq 0}$. We denote the lattice spanned by $\{\alpha_i - \sigma(\alpha_i) : i = 1, 2, \ldots, l\}$ by P_1. Let P_{sp} denote the sublattice of special weights and P_{spi} denote the lattice spanned by the dominant weights λ such that $V_\lambda^{H'} \neq 0$. We denote the usual dominant ordering on the weight lattice P by $\leq$ and we order P_{sp} by $\lambda \leq_H \mu$ if $\mu - \lambda$ is a nonnegative integral linear combination of the elements $\alpha_i - \sigma(\alpha_i)$, $i = 1, 2, \ldots, l$.

Let $X = \overline{\hat{G}/\hat{H}} = \overline{G/H}$ denote the wonderful compactification of the symmetric variety $\hat{G}/\hat{H} = G/H$. Let $\lambda \in P_{\mathrm{spi}}$. Let L_λ denote the line bundle on X associated to λ and let $\{D_i : i \in \{1, 2, \ldots, l\}\}$ be the G stable divisors of X, $l = \mathrm{rank}(G/H)$.

Then, we have

Theorem 2.5. [*cf* [*D-P*] (*pp* 30–31)] 1. $H^0(X, L_\lambda) \neq 0$ *if and only if* $\lambda = \nu + \sum_{i=1}^{l} t_i(\alpha_i - \sigma(\alpha_i))$ *for some dominant weight* ν *and* $t_i \in \mathbb{Z}_{\geq 0}$.

2. *Further, if* $H^0(X, L_\lambda) \neq 0$, *and if* V_ν *is the irreducible* $\hat{G}$ *module of highest weight* ν, *then* $H^0(X, L_\lambda) = \oplus_\nu V_\nu^*$, *where the sum is taken over all dominant weights* ν *of the form* $\nu = \lambda - \sum_{i=1}^{l} t_i(\alpha_i - \sigma(\alpha_i))$, $t_i \in \mathbb{Z}_{\geq 0}$.

Set $\lambda_i = \varpi_i + \varpi_{\tilde{\sigma}(i)}$ for $i \in \{1, 2, \ldots, j\}$. Since the number l is the number of $\tilde{\sigma}$ orbits in the set $\{1, 2, \ldots, j\}$, we can index the λ_i's by the set $\{1, 2, \ldots, l\}$.

Now, consider the group $G \times G$ with the involution

$$\tau : G \times G \longrightarrow G \times G \text{ defined by } \tau(g, h) = (h, g). \tag{2.5.1}$$

It is clear that the invariants of τ in $G \times G$ is the diagonal $\Delta(G) = \{(g, g) : g \in G\}$.

The map

$$F : G \longrightarrow (G \times G)/\Delta(G) \text{ defined by } F(g) = \text{ the coset } (g, e)\Delta(G)$$

gives an isomorphism of G onto $(G \times G)/\Delta(G)$.

(2.5.2) In this situation, we take $T \times T$ as a τ stable maximal torus of $G \times G$, where T is a σ stable maximal torus of G as in the first paragraph of § 2, $B \times B^-$ to be the Borel subgroup of $G \times G$, where B is a Borel subgroup of G containing T with Φ^+ as in Lemma 2.1. Observe that if ν is any dominant weight (resp. regular dominant) with respect to B, then the pair $(\nu, -\nu)$ is a special dominant weight (resp. regular special dominant) with respect to the involution τ of $G \times G$ and the Borel subgroup $B \times B^-$ of $G \times G$.

3. Wonderful compactification as a quotient

In this section, we prove that there is an ample line bundle L on $\overline{G}$ such that the G.I.T. quotient $(\overline{G})^{ss}(L)//H$ is isomorphic to the complete symmetric variety $\overline{G/H}$ by the following three steps.

Step (1) (PROPOSITION 3.1)

If λ, $\mu \in P_{\mathrm{spi}}$ *are two dominant weights, then,* $\hat{H}$ *acts on* $V_\lambda^{H'}$ *and* $V_\mu^{H'}$ *via the same character if and only if* $\lambda - \mu \in P_1$.

Step (2) (*Lemma* 3.2). (i) *There exists a regular dominant weight* $\mu \in P$ *which can be written as*

$$\mu = \sum_{i=1}^{l} a_i(\varpi_i + \varpi_{\bar{\sigma}(i)}) + \sum_{s=1}^{k} b_s\beta_s$$

with each a_i, b_s *necessarily positive integers having the following properties:*

(ii) *Set* $\mu_1 := \sum_{i=1}^{l} a_i(\varpi_i + \varpi_{\bar{\sigma}(i)})$. *Then for every positive integer n and for every special weight* ν, $\nu \leq n\mu$ *if and only if* $\nu \leq n\mu_1$.

(iii) *We also have* $\mu_1 \in P_{\text{spi}} \cap X(T)$ *and the action of* $\hat{H}$ *on* $V_{\mu_1}^{H'}$ *is trivial.*

(iv) *If* $\nu \in P_{\text{spi}}$ *and the action of* $\hat{H}$ *on* $V_{\nu}^{H'}$ *is trivial, then* $\nu \leq n\mu$ *if and only if* $\nu \leq_H n\mu_1$.

Step (3) (Theorem 3.3). *Let* μ *be a regular dominant weight as in step* (2) *and let* $L := L_{(\mu,-\mu)}$. *Then* $\overline{G}^{ss}(L)//\hat{H}$ *is isomorphic to* $\overline{G/H}$.

We now prove step (1) (Proposition 3.1).

We have

$$P_1 \subset P_{\text{spi}} \subset P_{\text{sp}}. \tag{3.0.1}$$

Here, the first inclusion is because the divisor D_i of $X = \overline{G/H}$ is locally defined by $\alpha_i - \sigma(\alpha_i)$ and the second inclusion is by Lemma 2.4.

For any character χ of $\hat{H}$, $E_\chi := (\hat{G} \times \mathbb{C})/\hat{H}$ denotes $\hat{H}$ orbit equivalence classes of $\hat{G} \times \mathbb{C}$ for the action of $h \in \hat{H}$, $(g,z) \in G \times \mathbb{C}$, defined by $h(g,z) = (gh^{-1}, \chi(h)z)$.

PROPOSITION 3.1

Let $\lambda, \mu \in P_{\text{spi}}$ *be two dominant weights. Then* $\hat{H}$ *acts on* $V_\lambda^{H'}$ *and* $V_\mu^{H'}$ *via the same character if and only if* $\lambda - \mu \in P_1$.

Proof. 3.1.1. We know that we have a natural inclusion $P_{\text{spi}} \subset \text{Pic}(X)$. Now, we prove that there is a natural exact sequence of abelian groups

$$(0) \longrightarrow P_1 \longrightarrow P_{\text{spi}} \longrightarrow X(\hat{H}).$$

To do this, we consider the homomorphism $\epsilon_{\hat{H}} : X(\hat{H}) \longrightarrow \text{Pic}(G/\hat{H})$ defined by $\epsilon_{\hat{H}}(\chi) := E_\chi$. By Proposition [[3.2] (i) (pp 81–82) [KSS]], the sequence

$$X(\hat{G}) \xrightarrow{\text{res}} X(\hat{H}) \xrightarrow{\epsilon_{\hat{H}}} \text{Pic}(\hat{G}/\hat{H}) \xrightarrow{\pi_{\hat{H}}^*} \text{Pic}(\hat{G})$$

is exact. Since $\hat{G}$ is simply connected semisimple algebraic group, we have $X(\hat{G}) = (0)$ and $\text{Pic}(\hat{G}) = (0)$ so the above exact sequence becomes

$$X(\hat{H}) \xrightarrow{\sim} \text{Pic}(\hat{G}/\hat{H}). \tag{3.1.2}$$

Since $\hat{G}/\hat{H} = G/H \subset X$, the isomorphism in (3.1.2) gives a homomorphism

$$\text{Pic}(X) \longrightarrow X(\hat{H}) \tag{3.1.3}$$

whose kernel is the lattice of the divisors supported on $X - G/H$. This lattice is P_1 by Corollary 8.2.3 of [D-P] [cf p-29 [D-P]].

Thus, from (3.0.1), (3.1.1) and (3.1.3), we have the required exact sequence

$$(0) \longrightarrow P_1 \longrightarrow P_{\mathrm{spi}} \xrightarrow{\psi} X(\hat{H}). \tag{3.1.4}$$

We now describe the map $\psi : P_{\mathrm{spi}} \longrightarrow X(\hat{H})$. Let $\lambda \in P_{\geq 0} \cap P_{\mathrm{spi}}$. Then $V_\lambda^{H'}$ is one dimensional. We note that by Lemma 1.6 of [D-P], the lattice P_{spi} is stable under the Weyl involution i. We claim that $\hat{H}$ acts on $V_\lambda^{H'}$ via the character $\psi(i(\lambda))$, where i is the Weyl involution. For this claim, without loss of generality, we may assume that λ is a regular special dominant weight. By Theorem 2.5, we have $V_\lambda \subset H^0(X, L_{i(\lambda)})$. By construction of X in [D-P], the evaluation map

$$\mathrm{ev} : X \times V_\lambda \longrightarrow L_{i(\lambda)}$$

is surjective and $\hat{G}$ equivariant. Therefore, the map $\mathrm{ev} : [\hat{H}] \times V_\lambda \longrightarrow L_{i(\lambda)} | [\hat{H}]$ is a nonzero H' invariant linear form on V_λ. By reductivity of H', the restriction of this linear form to $V_\lambda^{H'}$ is nonzero and therefore we get a $\hat{H}$ equivariant isomorphism between $V_\lambda^{H'}$ and $L_{i(\lambda)} | [\hat{H}]$. Since for any two elements $\lambda, \mu \in P_{\mathrm{spi}} \cap P_{\geq 0}$, $V_\lambda^{H'}$ is isomorphic to $V_\mu^{H'}$ as $\hat{H}$ modules if and only if $V_{i(\lambda)}^{H'}$ is isomorphic to $V_{i(\mu)}^{H'}$ as $\hat{H}$ modules, by (3.1.4), the proposition follows. □

Thus, we have proved step (1).

We now prove step (2) (*Lemma* 3.2).

We now recall from § 2, the usual dominant ordering on the weight lattice P and denote it by $\leq$. On the other hand, we order P_{sp} by setting $\lambda \leq_H \mu$ if $\mu - \lambda$ is a nonnegative integral linear combination of the elements $\alpha_i - \sigma(\alpha_i)$, $i = 1, 2, \ldots, l$. Notice that the set of fundamental weights $\{\varpi_1, \ldots, \varpi_j\}$ and that of simple roots $\{\beta_1, \ldots, \beta_k\}$ are mutually orthogonal and each linearly independent, together they form a basis of $P \otimes \mathbb{Q}$.

We have

Lemma 3.2. (i) *There exists a regular dominant weight $\mu \in P$ which can be written as*

$$\mu = \sum_{i=1}^{l} a_i(\varpi_i + \varpi_{\bar{\sigma}(i)}) + \sum_{s=1}^{k} b_s \beta_s$$

with each a_i, b_s necessarily positive integers having the following properties:
(ii) *Set $\mu_1 = \sum_{i=1}^{l} a_i(\varpi_i + \varpi_{\bar{\sigma}(i)})$. Then for every positive integer n and for every special weight ν, $\nu \leq n\mu$ if and only if $\nu \leq n\mu_1$.*
(iii) *We also have $\mu_1 \in P_{\mathrm{spi}} \cap X(T)$ and the action of $\hat{H}$ on $V_{\mu_1}^{H'}$ is trivial.*
(iv) *If $\nu \in P_{\mathrm{spi}}$ and the action of $\hat{H}$ on $V_\nu^{H'}$ is trivial, then $\nu \leq n\mu$ if and only if $\nu \leq_H n\mu_1$.*

Proof. (i) Set ρ_0 equal to the sum of all positive roots which are in the linear span of $\{\beta_1, \ldots, \beta_k\}$. Clearly $\langle \rho_0, \beta_s \rangle = 2$ while $\langle \rho_0, \alpha_i \rangle \leq 0$. Then, we choose the a_i's large enough so that $a_i > -\min\{\langle \rho_0, \alpha_i \rangle, \langle \rho_0, \alpha_{\bar{\sigma}(i)} \rangle\}$. The rest of (1) is immediate.

(ii) This is a computation. $\nu \leq n\mu_1$ implies $\nu \leq n\mu$ is trivial. We now prove the other side for the case $n = 1$, since the arguement is similar to all positive integers n. Set $\mu_0 = \mu - \mu_1$. Assume ν is special and $\nu \leq \mu$. Write

$$\nu = \mu - \sum_{i=1}^{j} x_i \alpha_i - \sum_{s=1}^{k} y_s \beta_s, x_i, y_s \in \mathbb{Z}_{\geq 0}.$$

Applying σ, we have

$$-\nu = -\mu_1 + \mu_0 - \sum_{i=1}^{j} x_i\sigma(\alpha_i) - \sum_{s=1}^{k} y_s\beta_s.$$

Thus, we have

$$2\mu_0 = \sum_{i=1}^{j} x_i(\alpha_i + \sigma(\alpha_i)) + 2\sum_{s=1}^{k} y_s\beta_s.$$

Hence, we have

$$2\mu_0 = \sum_{i=1}^{j} x_i(\alpha_i - \alpha_{\bar{\sigma}(i)}) - \sum_{s=1}^{k}\left(\sum_{i=1}^{j} x_i n_{i,s}\right)\beta_s + 2\sum_{s=1}^{k} y_s\beta_s.$$

From this it is clear that $b_s \leq y_s$ for every s. Thus, we have $\nu \leq \mu_1$.

(iii) Since $P_{\mathrm{spi}} \cap X(T)$ is a sublattice of P_{sp} of finite index, we can choose a_i's so that $\mu_1 \in P_{\mathrm{spi}} \cap X(T)$. Since H' is a subgroup of $\hat{H}$ of finite index, we can choose a_i's so that $\hat{H}/H'$ act trivially on $V_{\mu_1}^{H'}$.

(iv) follows from (3) and Proposition 3.1. □

We now prove step (3) (*Theorem* 3.3).

Let G, T, B, H, H' and $\hat{H}$ be as in § 2. Let $\mu = \sum_{r=1}^{l} a_r(\varpi_r + \varpi_{\bar{\sigma}(r)}) + \sum_{s=1}^{k} b_s\beta_s$ be the regular dominant weight (with respect to T and B) as in Lemma 3.2(1). With this μ, let $\mu_1 := \sum_{r=1}^{l} a_r(\varpi_r + \varpi_{\bar{\sigma}(r)})$ be the regular special dominant weight (with respect to our fixed involution σ). Let $X = \hat{G}/\hat{H} = G/H$, $Y = \overline{G}$, and let $M = L_{\mu_1}$ be the very ample line bundle on X defined by μ_1. Let $L = L_{(\mu,-\mu)}$ be the very ample line bundle on Y defined by regular special dominant weight $(\mu, -\mu)$ with respect to the twisted involution τ of $G \times G$ [cf (2.5.2)] and the Borel subgroup $B \times B^-$ of $G \times G$. Let $H^0(X, M^n)$ denote the space of global sections of the line bundle M^n. We note that Y is naturally a $\hat{G} \times \hat{G}$ space and so the vector space $H^0(Y, L^n)$ is a $\hat{G} \times \hat{G}$ module and in particular a $e \times \hat{H}$ module. Let $H^0(Y, L^n)^{\hat{H}}$ denote the $\hat{H}$ invariant global sections of L^n on Y, where the action of $\hat{H}$ on $H^0(Y, L^n)$ is through $e \times \hat{H}$.

We note that the pull backs of the line bundles L^n (resp. M^n) are trivial on G (resp. G/H).

Then, we have

Theorem 3.3. *The G.I.T. quotient $Y_{\hat{H}}^{ss}(L)//\hat{H}$ is isomorphic to the polarised variety (X, M). More precisely, the graded k algebra $\oplus_{n\in\mathbb{Z}_{\geq 0}} H^0(X, M^n)$ is isomorphic to the graded k algebra $\oplus_{n\in\mathbb{Z}_{\geq 0}} H^0(Y, L^n)^{\hat{H}}$.*

Proof. We will obtain, for every $n \in \mathbb{Z}_{\geq 0}$, an isomorphism

$$\phi_n : H^0(X, M^n) \longrightarrow H^0(Y, L^n)^{\hat{H}},$$

of G modules. We will show that the following diagram commutes

$$\begin{array}{ccc} H^0(X, M^m) \otimes H^0(X, M^n) & \longrightarrow & H^0(X, M^{m+n}) \\ \downarrow & & \downarrow \\ H^0(Y, L^m)^{\hat{H}} \otimes H^0(Y, L^n)^{\hat{H}} & \longrightarrow & H^0(Y, L^{m+n})^{\hat{H}}. \end{array}$$

Here the horizontal arrows are the natural surjective maps, the vertical map on the left is $\phi_m \otimes \phi_n$ and the vertical map on the right is ϕ_{m+n}. This will clearly prove the theorem.

To define ϕ_n, we will first get for every n, injective maps ψ_n and θ_n from $H^0(Y, L^n)^{\hat{H}}$ and $H^0(X, M^n)$ respectively into the $\mathbb{C}$ algebra $R := \mathbb{C}[G]^{\hat{H}}$ of $\hat{H}$ invariant regular functions on G, where the action of $\hat{H}$ on $\mathbb{C}[G]$ is through the right regular action of H on G and the homomorphism $\pi|\hat{H} : \hat{H} \longrightarrow H$. We will then show that the images of ψ_n and θ_n are equal. We can then define ϕ_n as $\psi_n^{-1}\theta_n$. That the diagram above is commutative will follow once we show that the following diagram and its analogue for θ_n are commutative:

$$\begin{array}{ccc} H^0(Y, L^m)^{\hat{H}} \otimes H^0(Y, L^n)^{\hat{H}} & \longrightarrow & H^0(Y, L^{m+n})^{\hat{H}} \\ \downarrow & & \downarrow \\ R \otimes R & \longrightarrow & R \end{array} . \tag{3.3.1}$$

Here, the top horizontal arrow is the natural surjective map, the bottom horizontal arrow is the multiplication in R, the left vertical map is $\psi_m \otimes \psi_n$ and the right vertical map is ψ_{m+n}.

We will now obtain ψ_n. Let D be a divisor in Y defining L. Then $H^0(Y, L^n)$ is the space of rational functions on Y that have poles of order atmost n on D. Since μ lies in the root lattice, the pull back of the line bundle $L := L(\mu, -\mu)$ is trivial on G and so D can be chosen so that support of D lies in the complement of G in Y. (For a proof: let D_1, $D_2, \ldots, D_l$ be the irreducible divisors of Y which do not meet G. Write $D = \sum_i m_i D_i + \sum_i n_i D_i'$, with prime divisors D_i''s meeting G. As L is trivial on G, $D \cap G$ is a principal divisor and hence there exists a rational function f on G such that $\sum_i n_i(D_i' \cap G) = D \cap G = \mathrm{div}_G(f)$. Therefore, we have $\mathrm{div}_Y(f) = \sum_i e_i D_i + \sum_i n_i D_i'$ for some integers $e_1, e_2, \ldots, e_l$. Thus, $D - \mathrm{div}_Y(f) = \sum_i (m_i - e_i) D_i$ (where $\mathrm{div}_Y(f)$ denotes the principal divisor defined by the rational function f in Y) is linearly equivalent to D, and the support $D - \mathrm{div}_Y(f)$ lies in the complement of G in Y).

Since G is dense in Y, we have therefore identified $H^0(Y, L^n)$ as a subspace of the ring $k[G]$ of regular functions on G. By taking $\hat{H}$ invariants, we obtain ψ_n. It should be clear from our definition of ψ_n that the diagram 3.3.1 commutes.

We will now obtain θ_n. This construction is parallel to that of ψ_n. Let D be a divisor in $X := \overline{G/H}$ defining M. Then $H^0(X, M^n)$ is the space of rational functions on X that have poles of order atmost n on D. By the hypothesis of μ_1 as in Lemma 3.2, we see that the pull back of M is trivial on $\hat{G}/\hat{H} = G/H$, and so D can be chosen so that the support of D lies in the complement of G/H in X. Since G/H is dense in X, we have therefore identified $H^0(X, M^n)$ as a subspace of the $\mathbb{C}$-algebra $\mathbb{C}[G/H]$ of regular functions on the affine variety G/H. Now $\mathbb{C}[G/H]$ is just the $\mathbb{C}$-algebra $\mathbb{C}[G]^H = R$ of H invariant regular functions on G. Since $\hat{H}$ on through the homomorphism $\pi|\hat{H} : \hat{H} \longrightarrow H$, we have $R = \mathbb{C}[G]^{\hat{H}}$. We have therefore obtained θ_n. It should be clear from the definition of θ_n that the analogue of (3.3.1) for θ_n is a commutative diagram.

It only remains to show that the images of ψ_n and θ_n are equal. For this, we first write R as a multiplicity free direct sum of irreducible G modules. We will then show that the images of ψ_n and θ_n are equal.

It is known that $\mathbb{C}[G] = \oplus_\nu (V_\nu^* \otimes V_\nu)$ as $G \times G$ modules, where the sum runs over all dominant weights of G (with respect to the maximal torus and Borel subgroups that we have fixed as in Lemma 2.1) that are divisible by the order of the finite group of weight lattice modulo the root lattice of G.

We now take $\hat{H}$ invariants on both sides of the above equation. The left hand side becomes R. By [D-P], if $V_\nu^* \otimes (V_\nu)^{\hat{H}} \neq 0$, then it is isomorphic to V_ν^* and ν is special. This is an immediate consequence of the fact that the action of H on $V_\nu^* \otimes V_\nu$ induced by the right action of $\hat{H}$ on Y is the action of $\hat{H}$ on the right factor V_ν of $V_\nu^* \otimes V_\nu$.

Since $\dim V_\nu^{\hat{H}} \leq 1$ for any ν, we have

$$(V_\nu^* \otimes V_\nu)^{\hat{H}} \neq 0 \text{ implies } (V_\nu^* \otimes V_\nu)^{\hat{H}} \xrightarrow{\sim} V_\nu^* \text{ and } \nu \in P_{\mathrm{spi}}. \tag{3.3.2}$$

Thus $R = \oplus_\nu V_\nu^* = \oplus_\nu V_\nu^* \otimes (V_\nu)^{\hat{H}}$, where the sum runs over certain special dominant weights, and so R is a multiplicity free direct sum of certain irreducible G modules.

Now, we show that the images of θ_n and ψ_n are equal and this will complete the proof of the theorem. By Theorem 2.5 applying to the involution σ and the symmetric variety X, we have $H^0(X, M^n) = \oplus_{\nu \leq_H n\mu_1} V_\nu^*$, where the sum runs over all special dominant weights $\nu \leq_H n\mu_1$. Therefore, by the above decomposition of $R = \oplus_\nu V_\nu^* \otimes (V_\nu)^{\hat{H}}$,

$$\text{the image of } \theta_n \text{ is just the } \hat{H} \text{ invariants} \quad \bigoplus_{\text{special } \nu \leq_H n\mu_1} V_\nu^* \otimes (V_\nu)^{\hat{H}}. \tag{3.3.3}$$

On the other hand, by Theorem 2.5, applying to the involution τ of $G \times G$, and to the symmetric variety $Y = \overline{G}$, we also have

$$H^0(Y, M^n) = \bigoplus_\nu V_\nu^* \otimes V_\nu, \tag{3.3.4}$$

where the sum runs over all dominant weights $\nu \leq n\mu$.

By Lemma 3.2 (4), we have

$$\nu \leq n\mu \text{ if and only if } \nu \leq_H n\mu_1 \text{ for any dominant weight } \nu \in P_{\mathrm{spi}}. \tag{3.3.5}$$

Thus, from (3.3.3), (3.3.4) and (3.3.5) the image of ψ_n and the image of θ_n are equal. Hence the theorem. □

4. Functoriality of the wonderful compactifications

In this section, we characterise when a morphism of semisimple algebraic groups of adjoint type $\phi : H \longrightarrow G$ can be extended to a $H \times H$ equivariant morphism of the wonderful compactifications $\phi^\wedge : \overline{H} \longrightarrow \overline{G}$. [cf. Theorem 4.7]

We first recall some facts from [D-P]. We fix some notations which will be used in the paragraph below. Let G be a semisimple algebraic group of adjoint type, T a maximal torus of G, B a Borel subgroup of G containing T, B^- the opposite Borel subgroup of G determined by T and B, U and U^- be the unipotent radicals of B and B^- respectively. Also, let $\Gamma_B = \{\beta_1, \beta_2, \ldots, \beta_l\}$ denote the set of simple roots of G with respect to B and T.

By construction in [D-P], the wonderful compactification of G is the scheme theoretic closure of $G \times G$ orbit of the line through the identity element $1^\sim$ of $\mathrm{End}(V_\lambda)$ in the projective space $\mathbb{P}(\mathrm{End}(V_\lambda))$ where λ is an arbitrary regular dominant weight with respect to T and B. In the case of the involution $\tau((g,h)) = (h,g)$ (as in 2.6.1), we have: $(T \times T)_1 = \{(t,s) \in T \times T : \tau((t,s)) = (t^{-1}, s^{-1})\} = \{(t, t^{-1}) : t \in T\}$ and

$$\begin{aligned}\overline{(\beta_i, 0)} &= \tfrac{1}{2}[(\beta_i, 0) - \tau((\beta_i, 0))] = \tfrac{1}{2}[(\beta_i, 0) - (0, \beta_i)] \\ &= \tfrac{1}{2}[(0, -\beta_i) - \tau((0, -\beta_i))] = \overline{(0, -\beta_i)}\end{aligned}$$

for every $i \in \{1,2,\dots,l\}$. Therefore

$$(t,t^{-1})^{-2[\overline{(\beta_i,0)}]} = t^{-\beta_i}(t^{-1})^{\beta_i} = t^{-2\beta_i} \tag{4.0.1}$$

for every $t \in T$ and $i \in \{1,2,\dots,l\}$. Let Φ_1 be the set of roots that are not fixed by τ. We then have

$$\Phi_1^- = \{\nu \in \Phi^-(B \times B^-) : \tau(\nu) \neq \nu\} = \{(-\beta,0) : \beta \in \Phi^+(B)\} \\ \cup \{(0,\beta) : \beta \in \Phi^+(B)\}$$

and therefore the unipotent subgroup of $G \times G$ generated by the root groups $\{G_{a,\nu} : \nu \in \Phi_1^-\}$ is $U^- \times U$.

Here, we fix λ a regular dominant weight of G with respect to B. Let $p \in (\mathrm{End}(V_\lambda))^*$ be the linear function on $\mathrm{End}(V_\lambda)$ defined by $p(v) =$ the coefficient of the highest weight vector $v_{(\lambda,-\lambda)}$ in the expression of v (in terms of $T \times T$ weight vectors) for all $v \in \mathrm{End}(V_\lambda)$ and let $V = \{x \in \overline{G} : p(x) \neq 0\}$ be the open set of $\overline{G}$ defined by the nonvanishing of p. Since p is a lowest weight vector of the $G \times G$ module $(\mathrm{End}(V_\lambda))^*$, the open set V of $\overline{G}$ is $B^- \times B$ stable. Consider the action of $T \times T$ on the affine space $\mathbb{A}^l$ (l denoting the rank of G) defined by

$$(t,s) \cdot (x_1,x_2,\dots,x_l) = ((ts^{-1})^{-\beta_1}x_1, (ts^{-1})^{-\beta_2}x_2, \dots, (ts^{-1})^{-\beta_l}x_l).$$

This action will induce an action of $(T \times T)_1$ on the affine space $\mathbb{A}^l$, namely

$$(t,t^{-1}) \cdot (x_1,x_2,\dots,x_l) = (t^{-2\beta_1}x_1, t^{-2\beta_2}x_2, \dots, t^{-2\beta_l}x_l), \tag{4.0.2}$$

for $(t,t^{-1}) \in (T \times T)_1$ and $(x_1,x_2,\dots,x_l) \in \mathbb{A}^l$. By Lemma 2.2 of [[p. 11] [D-P]] and (4.0.1), we have a $(T \times T)_1$ equivariant isomorphism

$$\begin{aligned} f :\ &\text{closure of } (T \times T)_1 \text{ orbit of the line through the identity} \\ &\text{element } 1^\sim \text{ of } \mathrm{End}(V_\lambda) \text{ in } V \xrightarrow{\sim} \mathbb{A}^l. \end{aligned} \tag{4.0.3}$$

Now, consider the action of $B^- \times B$ on $U^- \times U \times \mathbb{A}^l$ as follows:

For $(u_1^- t, su_1) \in B^- \times B$ with $u_1^- \in U^-, u_1 \in U, t,s \in T$
and $(u^-,u,x) \in U^- \times U \times \mathbb{A}^l$
we define $(u_1^- t, su_1)(u^-,u,x) = (u_1^- u^-, uu_1^{-1}, (t,s)x)$.

With this action of $B^- \times B$ on $U^- \times U \times \mathbb{A}^l$ and for the canonical action of $B^- \times B$ on V, [by Proposition [2.3] [p. 12] [D-P]] the morphism $\psi : U^- \times U \times \mathbb{A}^l \longrightarrow V$ defined by

$$\begin{aligned} &\psi((u^-,u,x)) = u^- f^{-1}(x) u^{-1} [f \text{ as in (4.0.3)}] \text{ is a} \\ &B^- \times B \text{ equivariant isomorphism.} \end{aligned} \tag{4.0.4}$$

Let $\{D_i' : i \in \{1,2,\dots,l\}\}$ be the set of all $G \times G$ divisors of $\overline{G}$ [cf Theorem 3.1 [p. 14] [D-P]]. By Theorem 3.1 [p. 14] of [D-P], we also have

$$\psi^{-1}(D_i' \cap V) = U^- \times U \times \mathbb{A}_i^{l-1} \text{ where } \mathbb{A}_i^{l-1} = \{(x_1,x_2,\dots,x_l) \in \mathbb{A}^l : x_i = 0\}. \tag{4.0.5}$$

We also have properties (4.0.3), (4.0.4) and (4.0.5) for the wonderful compactification $\overline{H}$ of H. In the case of $\overline{H}$, we denote the open subset of $\overline{H}$ similar to the open set V of $\overline{G}$ by

V_H, $(T\times T)_1$ by $(T_H\times T_H)_1$, $\mathbb{A}^l$ by $\mathbb{A}^{l_H}$, the $H\times H$ stable divisors of $\overline{H}$ by $D_1, D_2, \ldots, D_{l_H}$ and so on.

We use the following basic result from the paper [C] in our proof.

Theorem 4.1. [*cf* [*p.* 17] [C]]. *A connected semisimple algebraic group G has a finite set of connected closed normal subgroups $G_1, G_2, \ldots, G_k$ such that:*

(a) *each G_i is simple.*
(b) $[G_i, G_j] = 1$ *for* $i \neq j$.
(c) *the multiplication map $\mu : \Pi_{i=1}^k G_i \longrightarrow G$ is an isogeny.*

Now onwards, J will denote the set $\{1,2,\ldots,k\}$ where k is as in Theorem 4.1.

Note 4.2. If G is a semisimple algebraic group of *adjoint type*, then the multiplication map $\mu : \Pi_{i\in J} G_i \longrightarrow G$ is an isomorphism and any closed normal subgroup of G is isomorphic to $\Pi_{i\in I} G_i$ (via μ^{-1}) for some appropriate subset I of J.

Proof of Note 4.2. By Theorem 4.1, $\mu : \Pi_{i\in J} G_i \longrightarrow G$ is an isogeny. Since G is of adjoint type, each G_i is of adjoint type, and so is the product $\Pi_{i\in J} G_i$. The kernel $K = \ker\mu$ being a finite normal subgroup of $\Pi_{i\in J} G_i$, we must have $K = 1$. Hence μ is an isomorphism.

Let N be any closed normal subgroup of G. Then, the connected component of the identity N^0 is isomorphic to $\Pi_{i\in I} G_i$ for some $I \subseteq J$. Therefore, G/N^0 is isomorphic to $((\Pi_{i\in J} G_i)/(\Pi_{i\in I} G_i) \xrightarrow{\sim} \Pi_{i\notin I} G_i)$. So, N/N^0 is isomorphic to a finite normal subgroup of $\Pi_{i\notin I} G_i$. Now, $\Pi_{i\notin I} G_i$ being semisimple of adjoint type, $N/N^0 = 1$.

In the paragraph below, we will make some elementary observations which will be used later.

Remark 4.3. It is known that there exist a maximal torus T_H of H, a Borel subgroup B_H of H containing T_H and a maximal torus T_G of G containing T_H. Let Φ_H (resp. Φ_G) denote the set of all roots of H (resp. G) with respect to T_H (resp. T_G). The set Φ_G is the disjoint union of the following two sets: $\Phi_G^0 := \{\beta\in\Phi_G : i^*(\beta) = 0\}$ and $\Phi_G^1 := \{\beta\in\Phi_G : i^*(\beta) \neq 0\}$. It is easy to see that the one dimensional root groups $\{G_{a,\beta} : \beta\in\Phi_G^0\}$ generate a reductive subgroup of G, say G_0, and let $\Phi_{G_0}^+$ denote the set of all positive roots with respect to a Borel subgroup of G_0 containing the maximal torus $(T_G\cap G_0)^0$ (= the connected component of identity element in $T_G\cap G_0$) of G_0. By identifying $X(T_H)$ with $\mathbb{Z}^{l_H}$ via the basis Γ_{B_H}, there is a canonical total order "$>$" on $X(T_H)$ induced by the lexicographic order on $\mathbb{Z}^{l_H}$.

We fix T_H, B_H, T_G as in remark above and hence we fix G_0 and $\Phi_{G_0}^+$ also as above. For $\beta\in X(T_G)$, we denote $i^*(\beta) > 0$ to represent $i^*(\beta)$ is positive in the canonical total order "$>$" on $X(T_H)$.

With these notations, we have

Lemma 4.4. *The set of roots $\Phi_G^+ := \{\beta\in\Phi_G^1 : i^*(\beta) > 0\} \cup \Phi_{G_0}^+$ is the set of positive roots of G with respect to a Borel subgroup B_G of G containing T_G and B_H.*

Proof of Lemma. For any root $\alpha\in\Phi_G^+$, we denote the one dimensional root space corresponding to α by $\mathcal{G}_\alpha$ and let X_α denote a nonzero element of $\mathcal{G}_\alpha$. Let $L := \oplus_{\alpha\in\Phi_G^+}\mathcal{G}_\alpha$. Since L is a Lie(T_G) submodule of Lie(G), Lie(T_G) is abelian, dim(Lie(T_G))+ dimension maximal nilpotent subalgebra (namely nilpotent radical of a Borel subalgebra) is

equal to the dimension of a Borel subgroup of G and the sum $L \oplus \mathrm{Lie}(T_G)$ is direct, to prove the claim, it is sufficient to prove that L is a maximal nilpotent subalgebra of $\mathrm{Lie}(G)$.

We will now prove some observations which will be used to prove that L is a maximal nilpotent subalgebra of $\mathrm{Lie}(G)$.

(4.4.1) If $\nu = \sum_{\alpha\in\Phi_G^+} m_\alpha\alpha$, with every m_α is a nonnegative integer and at least one m_α is positive, then we must have $\nu \neq 0$.

Proof of observation 4.4.1. Let $\nu = \sum_{\alpha\in\Phi_G^+} m_\alpha\alpha$ be such that each m_α is a nonnegative integer and atleast one m_α is positive. If every α such that m_α is positive belongs to $\Phi_{G_0}^+$, then the sum $\sum_{\alpha\in\Phi_G^+} m_\alpha\alpha$ is nonzero, since $\Phi_{G_0}^+$ is precisely the set of positive roots of G_0 with respect to a Borel subgroup of G_0 containing $T_G \cap G_0$. Otherwise, the sum $\sum_{\alpha\in\Phi_G^+} m_\alpha\alpha$ is strictly positive in the canonical total ordering ">" and hence the sum $\sum_{\alpha\in\Phi_G^+} m_\alpha\alpha$ is nonzero. □

(4.4.2) If α and β are two elements of Φ_G^+ such that $\alpha+\beta \in \Phi_G$, then we must have $\alpha+\beta\in\Phi_G^+$. The proof of this follows immediately from (4.4.1).

(4.4.3) We have $\Phi_G^+ \cup -\Phi_G^+ = \Phi_G$, where $-\Phi_G^+ := \{\alpha\in\Phi_G : -\alpha\in\Phi_G^+\}$.

Proof of the observation 4.4.3. If $\alpha\in\Phi_G$, then either we have $\alpha\in\Phi_G^0$ or we have $\alpha\in\Phi_G^1$. In the first case, either we have $\alpha\in\Phi_{G_0}^+\subset\Phi_G^+$ or we have $-\alpha\in\Phi_{G_0}^+\subset\Phi_G^+$ (since $\Phi_{G_0}^+$ is precisely the set of all positive roots of G_0 with respect to a Borel subgroup of G_0 containing $T\cap G_0$). In the second case, we have $i^*(\alpha)\neq 0$. Since ">" is a total order on $X(T_H)$, we have either $i^*(\alpha)>0$ or $i^*(-\alpha) = -i^*(\alpha)>0$. Therefore, either we have $\alpha\in\Phi_G^+$ or we have $-\alpha\in\Phi_G^+$. □

From the observations (4.4.1), (4.4.2) and (4.4.3), it is easy to see that L is a nilpotent Lie algebra of $\mathrm{Lie}(G)$ and Φ_G^+ is the set of positive roots with respect to the Borel subalgebra $L\oplus\mathrm{Lie}(T_G)$ of $\mathrm{Lie}(G)$.

Hence the lemma. □

We have

PROPOSITION 4.5

Let H be a semisimple subgroup of adjoint type of a semisimple algebraic group of adjoint type G. Then, there is a pair of triples (T_H, B_H, B_H^-) and (T_G, B_G, B_G^-) in H and G respectively such that the following hold:

(a) $T_H\subseteq T_G$, $B_H\subseteq B_G$ *and* $B_H^-\subseteq B_G^-$,

(b) *the intersection of the two monoids,*

$\mathbb{Z}_{\geq 0}$ *span of* $i^*(\Gamma_{B_G})\cap\mathbb{Z}_{\geq 0}$ *span of* $i^*(\Gamma_{B_G^-} = -\Gamma_{B_G})$ *is zero.*

(Here, by a triple (T,B,B^-), we mean a maximal torus T, a Borel subgroup B, opposite Borel subgroup B^- determined by T and B and i^* denotes the restriction map $X(T_G)\longrightarrow X(T_H)$, where $X(T_G)$ and $X(T_H)$ denoting the character groups of T_G and T_H respectively

and Γ_{B_G} and $\Gamma_{B_G^-}$ denoting the sets of simple roots of G with respect to B_G and B_G^- respectively.)

Proof of Proposition. Let T_H, B_H be as in Remark 4.3. Let B_G be the Borel subgroup of G corresponding to our Φ_G^+ as in Lemma 4.4.

We now prove that $\mathrm{Lie}(B_H)$ is contained in $\mathrm{Lie}(B_G)$. For, let α be a positive root of G with respect to B_H and T_H. Write $X_\alpha \in \mathrm{Lie}(B_H) \subseteq \mathrm{Lie}(G)$ as a linear combination $\sum_{\beta\in\Phi_G} c_\beta X_\beta$, with $c_\beta \in \mathbb{C}$ for each $\beta \in \Phi_G$. For any $\beta \in \Phi_G$ such that c_β is nonzero, we must have $i^*(\beta) = \alpha$ and hence $\beta \in \Phi_G^+$.

Therefore, B_H is contained in B_G. The proof of B_G^- containing B_H^- is similar to the proof of $B_H \subseteq B_G$. Thus, we have proved (a).

The proof of (b) follows from the fact that a nonzero element of $X(T_H)$ cannot be both positive and negative in the lexicographic ordering.

Hence the proposition. □

We will now make a remark which will be used in Theorem 4.7.

Remark 4.6. If $H = H' \times K$, where H', K, H are semisimple algebraic groups of adjoint type, then, there is a canonical isomorphism of the wonderful compactification $\overline{H}$ onto the product of the wonderful compactifications $\overline{H'} \times \overline{K}$ of H' and K.

Proof. There is a natural Segre map $\overline{H'} \times \overline{K} \longrightarrow \overline{H' \times K} = \overline{H}$ induced by $\mathrm{End}(V_\lambda) \times \mathrm{End}(V_\mu) \longrightarrow \mathrm{End}(V_{\lambda+\mu}) = \mathrm{End}(V_\lambda) \otimes \mathrm{End}(V_\mu)$, where λ, μ are regular dominant weights for H' and K respectively. It is easy to see that the Segre map is a $H \times H$ equivariant isomorphism, and we identify $\overline{H}$ with $\overline{H'} \times \overline{K}$ through this isomorphism. We denote by $\hat{i}$ the map $\overline{H'} \longrightarrow \overline{H}$ given by $x \mapsto (x, e)$, where e is the identity element of K. □

Now, for a homomorphism $\phi : H \longrightarrow G$ of semisimple algebraic groups of adjoint type, we define a natural morphism $\hat{i} : \overline{\phi(H)} \longrightarrow \overline{H}$ which will be used in the statement of Theorem 4.7.

To do this, let $\phi : H \longrightarrow G$ be a homomorphism of semisimple algebraic groups of adjoint type, let K be the kernel of the homomorphism and let $\phi(H)$ denote the image of H in G. Since the homomorphism $\phi : H \longrightarrow \phi(H)$ is surjective, by Note 4.2, H must be isomorphic to $\phi(H) \times K$. Thus by Remark 4.4 applying to the situation $H' = \phi(H)$, we have a natural map $\overline{\phi(H)} \longrightarrow \overline{H} = \overline{\phi(H)} \times \overline{K}$ given by $x \mapsto (x, e)$, where e is the identity element of K. We denote this map by $\hat{i}$.

We now prove the main theorem.

Now, let $\phi : H \longrightarrow G$ be a homomorphism of semisimple algebraic groups of adjoint type. Consider the natural map $\overline{\phi(H)} \longrightarrow \overline{H} = \overline{\phi(H)} \times \overline{K}$ sending x to (x, e), where e is the identity element of K, the kernel of the homomorphism ϕ. We denote this natural map $\overline{\phi(H)} \longrightarrow \overline{H} = \overline{\phi(H)} \times \overline{K}$ sending x to (x, e) by $\hat{i}$ and we use in the statement of the following theorem.

Theorem 4.7. *For a morphism $\phi : H \longrightarrow G$ of semisimple algebraic groups of adjoint type, the following statements are equivalent:*

(1) *ϕ can be extended to a $H \times H$ equivariant morphism $\hat{\phi} : \overline{H} \longrightarrow \overline{G}$ with the property that the composition $\hat{\phi} \circ \hat{i} : \overline{\phi(H)} \longrightarrow \overline{G}$ (where $\hat{i}$ is as above) is an isomorphism onto the scheme theoretic closure of $\phi(H)$ in $\overline{G}$.*

(2) *There are two pairs (T_H, B_H), (T_G, B_G) in H and G respectively such that $\phi(T_H) \subseteq T_G$, $\phi(B_H) \subseteq B_G$ and $\phi^*(\Gamma_{B_G}) \subseteq \mathbb{Z}_{\geq 0}$ span of Γ_{B_H}. (Here, by a tuple (T, B), we mean a maximal torus T and a Borel subgroup B containing T, Γ_{B_H}* (resp. Γ_{B_G}) *denotes the set of simple roots of H* (resp. G) *with respect to T_H and B_H* (resp. T_G *and* B_G).)

Proof. We will first reduce to the case when $\phi = i$ is an inclusion. For this, we first note that if ϕ is a surjection, then both the statements (1) and (2) hold: statement (1) can be seen to be true from Remark 4.6, and statement (2) holds obviously. For a general ϕ, we can break it up as $i \circ \pi$ where π is a surjection and i is an inclusion. It is easy to see that statement (1) (respectively (2)) for i is equivalent to statement (1) (respectively (2)) for ϕ (because both statements (1) and (2) are true for π as we have seen above). We are therefore justified in assuming from now on that H is a subgroup of G and ϕ is the inclusion.

(1)$\Longrightarrow$(2):

Let $\hat{\phi} : \overline{H} \xrightarrow{\sim} Y$ be a $H \times H$ equivariant isomorphism such that the restriction of $\hat{\phi}$ to the group H is the inclusion $\phi : H \hookrightarrow G$.

We start by making some observations. Fix triples (T_H, B_H, B_H^-) and (T_G, B_G, B_G^-) satisfying $T_H \subseteq T_G$, $B_H \subseteq B_G$ and $B_H^- \subseteq B_G^-$ (such triples exist by Proposition 4.5(a)). Let U and U^- be the unipotent radicals of B_G and B_G respectively. Let U_H and U_H^- be unipotent radicals of B_H and B_H^- respectively. Let Y be the scheme theoretic closure of H in $\overline{G}$. Let λ be a regular dominant weight with respect to the Borel subgroup B_G of G. Let V be the $B_G^- \times B_G$ stable open subset of $\overline{G}$ as in the beginning of this section. Set $V_Y := V \cap Y$. Since V is a $B^- \times B$ stable open subset of $\overline{G}$, $B_H^- \times B_H$ is contained in $B_G^- \times B_G$, Y is closed in $\overline{G}$ and the isomorphism ψ of (4.0.4) is $B_H^- \times B_H$ equivariant (in fact $B^- \times B$ equivariant) $\psi^{-1}(Y \cap V)$ is $B_H^- \times B_H$ stable closed subset of $U^- \times U \times \mathbb{A}^l$. Consider the map $U_H^- \times U_H \times (T_H \times T_H)_1 \longrightarrow U^- \times U \times \mathbb{A}^l$ defined by $(u^-, u, (t, t^{-1})) \mapsto (u^-, u, (t^{-2\beta_1}, \ldots, t^{-2\beta_l}))$. Since $Y \cap V$ is an open subset of Y, we have

$$\dim(Y \cap V) = \dim(Y) = \dim(U_H^- \times U_H \times (T_H \times T_H)_1).$$

(4.3.1) Therefore, the $U_H^- \times U_H \times (T_H \times T_H)_1$ orbit of the point $(1,1), (1,1,\ldots,1))$ in $\psi^{-1}(Y \cap V)$ is open and dense in $\psi^{-1}(V_Y = Y \cap V)$. Since U_H^- (resp. U_H) is closed in U^- (resp. in U), the closure of $U_H^- \times U_H \times (T_H \times T_H)_1$ orbit of $((1,1), (1,1,\ldots,1))$ in $U^- \times U \times \mathbb{A}^l$ is $U_H^- \times U_H \times Z$, where Z is the closure of the $(T_H \times T_H)_1$ orbit of $(1,1,\ldots,1)$ in $\mathbb{A}^l$. Therefore, by (4.3.1), we have $\psi^{-1}(V_Y = Y \cap V) = U_H^- \times U_H \times Z$; note that $U_H \times U_H \times Z$ is closed in $U^- \times U \times \mathbb{A}^l$ since $U_H^- \times U_H$ is closed in $U^- \times U$ and Z is closed in $\mathbb{A}^l$.

By hypothesis, Y is isomorphic to $\overline{H}$ and therefore smooth. So, V_Y is smooth and so also Z (being a factor of V_Y). Being a smooth toric $(T_H \times T_H)_1$ variety, Z is isomorphic to the product of an affine space and a torus (cf Proposition 2.1 [p. 29] [Fu]). We will show below the following:

(a) Z is in fact an affine space.
(b) The image of $1 \times 1 \times \mathbb{A}^{l_H}$ under $\hat{\phi} \circ \psi_H$ equals the image of $1 \times 1 \times Z$ under ψ (as subsets of Y).

We then prove (1)$\Longrightarrow$(2) by using (a) and (b).

We now prove (a). Consider the weights of $(T_H \times T_H)_1$ module. Since $k[Z]$ is a k-algebra, this set of weights is a monoid. Since the restrictions of the co-ordinate functions $x_1, x_2, \ldots, x_l$ of $\mathbb{A}^l$ (as in (4.0.2)) generate the k-algebra $k[Z]$, the above

mentioned monoid is generated by the $(T_H \times T_H)_1$ weights of the restrictions of the functions $x_1, x_2, \ldots, x_l$ and hence the monoid is generated by $\{\phi^*(2\beta_k) : \beta_k \in \Gamma_{B_G}\}$. By Lemma 4.3(b), there is no nonzero weight χ such that both χ and $-\chi$ belong to this monoid. Thus, the units of the k-algebra are constants and hence the torus part of Z is empty, which proves (a).

We now prove (b).

Let $z_1, z_2, \ldots, z_{l_H}$ be l_H co-ordinate functions of the affine space Z which are $(T_H \times T_H)_1$ weight vectors (this exist since Z is a $(T_H \times T_H)$ toric variety that is an affine space). Let z_0 denote the unique point of Z where all the z_i's vanish. It is clear that z_0 is a $T_H \times T_H$ fixed point of Z. Since the divisors defined by the z_i's are l_H distinct $T_H \times T_H$ stable irreducible divisors of $1 \times 1 \times Z$, each intersection $\hat{\phi}(D_i) \cap \psi((1,1) \times Z)$ is nonempty and these divisors $\hat{D}_i \cap \psi((1,1) \times Z)$ of $\psi((1,1) \times Z)$ are just the divisors defined by the z_i's.

(4.5.1) Therefore, the $T_H \times T_H$ fixed point $\psi((1,1,z_0))$ must lie in the intersection $\cap_{i=1}^{l_H} \hat{\phi}(D_i)$.

Now, consider the isomorphism ψ_H of (4.0.4) and the point $(1,1,0)$ of $U_H^- \times U_H \times \mathbb{A}^{l_H}$ (in the case of $\overline{H}$). Since $\psi_H((1,1,0))$ is a $T_H \times T_H$ fixed point of $\cap_{i=1}^{l_H} D_i = (H \times H)/(B_H \times B_H^-)$, we have

$$\psi_H((1,1,0)) = (w, w')(B_H \times B_H^-)$$

for some $w, w' \in W_H$. Since the isotropy of $\psi_H((1,1,0))$ in $U_H^- \times U_H$ is identity, we have $wB_Hw^{-1} \cap U_H^- = (1)$ and $w'B_H^-(w')^{-1} \cap U_H = (1)$. Therefore the set of roots $\{\alpha \in \Phi^+(B_H) : w(\alpha) \in \Phi^-(B_H)\}$ is empty and hence w is the identity element of W_H and $wB_Hw^{-1} = B_H$. Similarly, w' is the identity element of W_H and $w'B_H^-(w')^{-1} = B_H^-$. Hence the isotropy of $\psi_H((1,1,0))$ in $H \times H$ is $B_H \times B_H^-$. Since $\hat{\phi} : \overline{H} \xrightarrow{\sim} Y$ is a $H \times H$ equivariant isomorphism and $\psi((1,1) \times Z)$ contains a $T_H \times T_H$ fixed point of $\cap_{i=1}^{l_H} \hat{\phi}(D_i) \xrightarrow{\hat{\phi}^{-1}} \cap_{i=1}^{l_H} D_i = (H \times H)/(B_H \times B_H^-)$ (by (4.5.1)), namely $\psi((1,1,z_0))$, by the above argument (since the isotropy of $\psi(1,1,z_0)$ in $U_H^- \times U_H$ is trivial), the isotropy of $\psi((1,1,z_0))$ in $H \times H$ is $B_H \times B_H^-$. Since $\cap_{i=1}^{l_H} D_i = (H \times H)/(B_H \times B_H^-)$ is the unique closed $H \times H$ orbit of $\overline{H}$, $\psi_H((1,1,0))$ is the unique point of $\overline{H}$ whose isotropy in $H \times H$ is $B_H \times B_H^-$. Since $\hat{\phi} : \overline{H} \xrightarrow{\sim} Y$ is a $H \times H$ equivariant isomorphism, $\psi((1,1,z_0))$ is the unique point of Y whose isotropy in $H \times H$ is $B_H \times B_H^-$ and $\hat{\phi}(\psi_H((1,1,0))) = \psi((1,1,z_0))$. Therefore, $\hat{\phi}(\psi_H((1,1) \times \mathbb{A}^{l_H}) \cap \psi((1,1) \times Z))$ is a smooth affine $(T_H \times T_H)_1$ toric variety containing the unique $(T_H \times T_H)_1$ fixed point $\psi((1,1,z_0))$ of $\psi((1,1) \times Z)$. Hence, [by Proposition 2.1 [p. 29 [Fu]], $\hat{\phi}|\psi_H((1,1) \times \mathbb{A}^{l_H}) : \psi_H((1,1) \times \mathbb{A}^{l_H}) \xrightarrow{\sim} \psi((1,1) \times Z)$ is a $(T_H \times T_H)_1$ equivariant isomorphism.

We now complete the proof of (1)$\Longrightarrow$(2).

Let $x_1, x_2, \ldots, x_l$ be the coordinate functions of the affine space $\mathbb{A}^l$ as in (4.0.2) and let $y_1, y_2, \ldots, y_{l_H}$ (in the case of $\overline{H}$) be the corresponding coordinate functions of the affine space $\mathbb{A}^{l_H}$. Then, we have

$$(t, t^{-1})x_i = t^{2\beta_i}x_i \text{ for all } i \in \{1, 2, \ldots, l\},\ t \in T \text{ and}$$
$$(t, t^{-1})y_i = t^{2\alpha_i}y_i \text{ for all } i \in \{1, 2, \ldots, l_H\},\ t \in T_H.$$

Since each x_i is a $(T \times T)_1$ weight vector, each x_i is also a $(T_H \times T_H)_1$ weight vector and hence $(\hat{\phi} \circ \psi_H)^*(x_i|\psi((1,1) \times Z))$ is also a $(T_H \times T_H)_1$ weight vector of the the co-ordinate ring of $\mathbb{A}^{l_H}$ and therefore $(\hat{\phi} \circ \psi_H)^*(x_i|\psi((1,1) \times Z))$ is a monomial $\prod_{r=1}^{l_H} y_r^{m_{i,r}}$, with each $m_{i,r} \in \mathbb{Z}_{\geq 0}$. Hence, for every $t \in T_H$, we have

$$
\begin{aligned}
t^{2\beta_i}(\hat{\phi}\circ\psi_H)^*(x_i|\psi((1,1)\times Z)) &= (t,t^{-1}).(\hat{\phi}\circ\psi_H)^*(x_i|\psi((1,1)\times Z))\\
&= (t,t^{-1})\left(\prod_{r=1}^{l_H} y_r^{m_{i,r}}\right) = \prod_{r=1}^{l_H} t^{2\alpha_r} y_r^{m_{i,r}}\\
&= t^{2\sum_{r=1}^{l_H} m_{i,r}\alpha_r}(\hat{\phi}\circ\psi_H)^*(x_i|\psi((1,1)\times Z))
\end{aligned}
$$

and therefore $2\beta_i|T_H = 2(\sum_{r=1}^{l_H} m_{i,r}\alpha_r)$. Thus, we have

$$\beta_i|T_H = \sum_{r=1}^{l_H} m_{i,r}\alpha_r \in \mathbb{Z}_{\geq 0} \text{ span of } \Gamma_{B_H}.$$

Therefore, we have $\phi^*(\Gamma_{B_G}) \subseteq \mathbb{Z}_{\geq 0}$ span of Γ_{B_H}.

(2)$\Longrightarrow$(1) :

Let (T_H, B_H) and (T_G, B_G) be a pair of tuples in H and G respectively such that $T_H \subseteq T_G$, $B_H \subseteq B_G$ and $i^*(\Gamma_{B_G}) \subseteq \mathbb{Z}_{\geq 0}$ span of Γ_{B_H}.

We first recall some facts from [D-P]. By construction in [D-P], the wonderful compactification $\overline{G}$ is the scheme theoretic closure of $G \times G$ orbit of the line through the identity element 1_λ of $\mathrm{End}(V_\lambda)$ in $\mathbb{P}(\mathrm{End}(V_\lambda))$, where λ is a regular dominant weight of G with respect to T_G and B_G. For our convenience, we choose λ so that λ is a character of T_G.

We will prove the following:

(c) $\mathrm{End}(V_\lambda)$ is a direct sum of $\mathrm{End}(V_\delta)$ and a $H \times H$ module W, where δ is the restriction of λ to T_H.

(d) The identity element 1_λ of $\mathrm{End}(V_\lambda)$ is a sum of two vectors v and w with v a nonzero vector in $\mathrm{End}(V_\delta)$ and w is a linear combination $(T_H \times T_H)_1$ weight vectors of weight $(\delta - \sum_{i=1}^{l_H} m_i\alpha_i, -(\delta - \sum_{i=1}^{l_H} m_i\alpha_i))$, where α_i's are simple roots of H with respect to T_H, B_H above and m_i's are nonnegative integers.

We will then prove (2)$\Longrightarrow$(1) by using (c), (d) and Lemma 4.1 of [D-P] [cf pp [16–17] [D-P]].

We now prove (c):

Since $\phi^*(\Gamma_{B_G}) \subseteq \mathbb{Z}_{\geq 0}$ span of Γ_{B_H}, we must have $B_H^- \subseteq B_G^-$ and hence $B_H \times B_H^- \subseteq B_G \times B_G^-$. Therefore, the smallest $H \times H$ submodule of $\mathrm{End}(V_\lambda)$ containing the highest weight vector $v_\lambda \otimes v_{-\lambda}$ is the $H \times H$ irreducible submodule of $\mathrm{End}(V_\lambda)$ of highest weight $(\delta, -\delta)$, namely $\mathrm{End}(V_\delta)$. By, complete reducibility of the $H \times H$ module $\mathrm{End}(V_\lambda)$, there is a $H \times H$ submodule W of $\mathrm{End}(V_\lambda)$ such that $\mathrm{End}(V_\lambda) = \mathrm{End}(V_\delta) \oplus W$.

Thus (c) is proved.

We now prove (d):

Since the coefficient of the highest weight vector $v_\lambda \otimes v_{-\lambda}$ in the expression of 1_λ, the identity element of $\mathrm{End}(V_\lambda)$ (in terms of $T_G \times T_G$ weight vectors) is nonzero, 1_λ must be a sum of a nonzero vector v in $\mathrm{End}(V_\delta)$ and a vector w in W. Since 1_λ is a $\Delta(H)$ (in fact $\Delta(G)$ invariant) invariant vector both v and w are also $\Delta(H)$ invariant vectors. Therefore, w is a linear combination of $(T_H \times T_H)_1$ weight vectors of $(T_H \times T_H)_1$ weights of the form $(\nu, -\nu)$, where ν is T_H weight. Since w (being an element of $\mathrm{End}(V_\lambda)$) is a linear combination of $T_G \times T_G$ weight vectors of the form $(\lambda - \sum_{i=1}^{l_G} n_i\beta_i, -(\lambda - \sum_{i=1}^{l_G} n_i'\beta_i))$, where n_i, n_i''s are nonnegative integers and β_i's are simple roots of G with respect to B_G, any such T_H weight ν as above is the restriction of T_G weight of the form $\lambda - \sum_i n_i\beta_i$,

where n_i's are nonnegative integers. But, by hypothesis, the restrictions $\phi^*(\beta_i)$'s must lie in $\mathbb{Z}_{\geq 0}$ span of Γ_{B_H}. Therefore, any such T_H weight ν as above must be of the form $\delta - \sum_{i=1}^{l_H} m_i\alpha_i$.

Thus, we have proved (d).

We now complete the proof of (2)$\Longrightarrow$(1) :

By Lemma 4.1 [pp [16–17] [D-P]], the canonical rational map from the projective space $\mathbb{P}(\mathrm{End}(V_\lambda))$ into the projective space $\mathbb{P}(\mathrm{End}(V_\delta))$ (which exists by (c)) is defined on the scheme theoretic closure Y of the $H \times H$ orbit of the homogeneous point of $\mathbb{P}(\mathrm{End}(V_\lambda))$ defined by the vector $1_\lambda = v + w$ and is an isomorphism onto the scheme theoretic closure of the $H \times H$ orbit of the homogeneous point defined by the vector v in $\mathbb{P}(\mathrm{End}(V_\delta))$, which is actually the wonderful compactification $\overline{H}$ of H.

Thus, we have proved the theorem. □

Remark 4.8. (1) Let H denote a semisimple algebraic group of adjoint type over $\mathbb{C}$ that is not isomorphic to $\mathbb{P}GL_2$. Consider the adjoint representation of H; $\phi : H \hookrightarrow G := \mathbb{P}GL(\mathrm{Lie}(H))$. For this ϕ, the statement (2) of Theorem 4.7 is not satisfied.

(2) Let H denote the adjoint group associated to the symplectic group $\mathrm{Sp}(2n)$ over $\mathbb{C}$. For the natural inclusion $\phi : H \hookrightarrow G := \mathbb{P}GL_{2n}$, the statement (2) of Theorem 4.7 is satisfied.

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Seiberg–Witten invariants – An expository account

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Abstract. The Seiberg–Witten monopole equations, and a new invariant for 4-manifolds which results from these equations, are introduced in this paper.

Keywords. Seiberg–Witten invariants; monopoles; 4-manifolds; Spin_c structures; Dirac operator.

1. Introduction

In this paper, we will introduce the Seiberg–Witten moduli space, which is a moduli space to a system of equations known as the 'monopole equations' (taken modulo a gauge group). This leads to invariants for 4-manifolds, which are more manageable than the earlier $SU(2)$ instanton invariants, and have had some notable new consequences, such as the proof of Thom's conjecture due to Kronheimer and Mrowka (see the accompanying paper [P]).

The break-up of this paper is as follows. In § 2, the linear algebra related to Spin and Spin_c groups is introduced. Section 3 discusses Spin_c structures on 4-manifolds, and a proof of the theorem of Hirzebruch–Hopf, i.e. that every compact orientable Riemannian 4-manifold admits a Spin_c structure is presented. Thereafter, in § 4, Spin_c connections, the Dirac operator and monopole equations, and the relevant gauge group is introduced, and on quotienting out by its action, the moduli space for the space of solutions (the Seiberg–Witten moduli space) is constructed, and its virtual dimension calculated from Fredholm considerations. Section 5 contains a proof of the Weitzenbock formula for the Spin_c-Dirac operator, and is included for the sake of completeness (even though it is standard). Section 6 contains the proof of the crucial properness result which results in compactness of the moduli space constructed, and the dependence upto cobordism of the moduli space on the Riemannian metric on the 4-manifold is proved (for a more detailed discussion of this matter, see [P]). The last § 7 contains calculations for the case of a Kahler manifold.

For a rapid primer on Fredholm theory, the reader may look at the appendix of the accompanying paper [P].

2. Spin groups

We recall some constructions of spin groups in low dimensions.

2.1 *Dimension* 3

Let W be a vector space of dimension 2. Consider the representation of $GL(W)$ on $\mathrm{End}^0(W)$, the space of all traceless endomorphisms of W. There is a natural non-degenerate form $\{\,,\}$

on $\mathrm{End}^0(W)$ given by

$$\{f, g\} = \mathrm{Trace}_W f \circ g.$$

Moreover, we have a sequence of isomorphisms of representations of $GL(W)$,

$$\overset{3}{\wedge}\mathrm{End}^0(W) = \overset{4}{\wedge}\mathrm{End}(W) = \overset{4}{\wedge}(W^* \otimes W) = (\overset{2}{\wedge}W^*)^{\otimes 2} \otimes (\overset{2}{\wedge}W)^{\otimes 2} = \mathbf{1},$$

where $\mathbf{1}$ denotes the trivial representation. Thus we obtain a natural homomorphism $GL(W) \to SO(\mathrm{End}^0(W))$. Over the complex numbers this identifies $GL(2)$ with the 'Cspin' group of $SO(3)$. The subgroup $SL(2)$ is identified with the spin group.

2.2 *Dimension* 4

Let W_+ and W_- be two vector spaces of dimension 2 and let $\phi : \overset{2}{\wedge}W_- \to \overset{2}{\wedge}W_+$ be an isomorphism. Then the vector space $U = \mathrm{Hom}(W_+, W_-)$ is isomorphic to its dual via a map $B : U \to U^* = \mathrm{Hom}(W_-, W_+)$ defined by the identity

$$\phi(f(w_+) \wedge w_-) = w_+ \wedge B(f)(w_-).$$

Thus we have a non-degenerate pairing

$$(f, g) = \mathrm{Trace}_{W_+} B(f) \circ g = \mathrm{Trace}_{W_-} g \circ B(f)$$

which can be seen to be a symmetric form. The group of automorphisms of the triple (W_+, W_-, ϕ) is

$$S(GL(W_+) \times GL(W_-)) = \{(g, h) \mid \det(g) = \det(h)\}.$$

We have a sequence of isomorphisms of representations of this group

$$\overset{4}{\wedge}(U) = \overset{4}{\wedge}(W_+^* \otimes W_-) = (\overset{2}{\wedge}W_+^*)^{\otimes 2} \otimes (\overset{2}{\wedge}W_-)^{\otimes 2} \xrightarrow{\phi \otimes \phi} \mathbf{1}.$$

Thus we obtain a morphism

$$S(GL(W_+) \times GL(W_-)) \to SO(\mathrm{Hom}(W_+, W_-)).$$

Over the complex numbers this identifies the group $S(GL(2) \times GL(2))$ with the 'Cspin' group of $SO(4)$ and the subgroup $SL(2) \times SL(2)$ is identified with the spin group of $SO(4)$.

2.3 *Dimension* 6

Let U be a four dimensional vector space and let $\psi : \overset{4}{\wedge}U \to \mathbf{1}$ be a chosen isomorphism so that the group of automorphisms of the pair (U, ψ) is $SL(U)$. Consider the pairing $\langle , \rangle$ on $\overset{2}{\wedge}U$ given by the composite

$$\overset{2}{\wedge}U \otimes \overset{2}{\wedge}U \xrightarrow{\wedge} \overset{4}{\wedge}U \xrightarrow{\psi} \mathbf{1}.$$

This is symmetric and non-degenerate. Moreover, we have a natural sequence of isomorphisms of representations of $SL(U)$

$$\overset{6}{\wedge}(\overset{2}{\wedge}U) = (\overset{4}{\wedge}U)^{\otimes 6} \xrightarrow{\psi^{\otimes 6}} \mathbf{1}.$$

Thus we have a representation of $SL(U)$ in $SO(\overset{2}{\wedge}U)$. Over the complex numbers this identifies $SL(4)$ with the spin group of $SO(6)$.

2.4 *Combination of the above*

Now consider the situation of (2.3) where $U = \mathrm{Hom}(W_+, W_-)$. In this situation U carries a non-degenerate pairing $(,)$ as described above and hence there is an induced pairing on $\overset{2}{\wedge} U$ which we also denote by $(,)$. We then have an automorphism $*$ on $\overset{2}{\wedge} U$ defined by the identity $(\alpha, \beta) = \langle \alpha, *\beta \rangle$. Now the fact that $\psi = \phi \otimes \phi$ satisfies $(\psi, \psi) = 1$ implies that $*^2 = 1$. Moreover, one can see that the positive (resp. negative) eigenspace Λ^+ (resp. Λ^-) of $*$ is of dimension 3. Thus the combined representation

$$S(GL(W_+) \times GL(W_-)) \to SO(U) \hookrightarrow SL(U) \to SO(\overset{2}{\wedge} U)$$

gives a morphism into $SO(\Lambda^+) \times SO(\Lambda^-)$. Now we have natural maps

$$S(GL(W_+) \times GL(W_-)) \to GL(W_\pm)$$

and hence we have representations of $S(GL(W_+) \times GL(W_-))$ into $SO(\mathrm{End}^0(W_\pm))$. Consider the homomorphisms of representations of $S(GL(W_+) \times GL(W_-))$

$$\overset{2}{\wedge} U \to \mathrm{End}^0(W_+) \text{ where } f \wedge g \mapsto B(f) \circ g - B(g) \circ f$$

and similarly

$$\overset{2}{\wedge} U \to \mathrm{End}^0(W_-) \text{ where } f \wedge g \mapsto f \circ B(g) - g \circ B(f).$$

These induce isomorphisms of $\mathrm{End}^0(W_\pm)$ with $\Lambda^\pm$.

2.5 *Compact forms*

Let us fix hermitian structures $h_\pm$ on $W_\pm$ so that ϕ is an isometry. The group of automorphisms then becomes $S(U(W_+) \times U(W_-))$. We define a $\mathbb{C}$-anti-linear automorphism $f \mapsto f^\dagger$ defined by the identity

$$h_+(f^\dagger(w), w') = h_+(w, B(f)(w')).$$

One sees that $f^{\dagger\dagger} = f$. Thus we obtain a real vector space T so that $U = T + \iota T$. Moreover, one sees that the form $(,)$ restricts to a positive definite form on T; hence we obtain a representation $S(U(W_+) \times U(W_-)) \to SO(T)$. The above discussion then gives us a decomposition of $\overset{2}{\wedge} T$ into $\Lambda^\pm_{\mathbb{R}}$.

We have a $\mathbb{C}$-anti-linear endomorphism $f \mapsto f^\dagger$ of $\mathrm{End}^0(W_\pm)$ given by

$$h_\pm(f^\dagger w, w') = h_\pm(w, f(w')).$$

One shows that under the isomorphism between $\mathrm{End}^0(W_\pm)$ and $\Lambda^\pm = \Lambda^\pm_{\mathbb{R}} + \iota \Lambda^\pm_{\mathbb{R}}$, we obtain identifications of $\Lambda^\pm_{\mathbb{R}}$ with the spaces $\mathrm{End}^0(W_\pm)^{ah}$ consisting of $f = -f^\dagger$.

We note that for any pair of elements Φ, Ψ of W_+ we have an element $\sigma(\Phi, \Psi)$ of $\mathrm{End}^0(W_+)$ given by

$$w \mapsto i(h_+(w, \Psi) \cdot \Phi - \tfrac{1}{2} h_+(\Phi, \Psi) \cdot w).$$

When $\Phi = \Psi$ this is an element of $\mathrm{End}^0(W_+)^{ah}$. We identify this with an element of $\Lambda^+_{\mathbb{R}}$.

2.6 *Unitary group case*

We now further specialise to the case when $W_+ = \mathbf{1} \oplus \det W_-$. For ease of notation we use W for W_-. In this case, we have a natural sequence of identifications

$$\mathrm{Hom}_{\mathbb{C}}(W_+, W_-) = W \oplus W^* = W \oplus \overline{W} = W \otimes_{\mathbb{R}} \mathbb{C}.$$

Thus we can identify the special orthogonal representation T with the underlying real vector space of W. Now let $\overset{(2,0)}{\wedge} T$ denote the underlying real vector space to $\overset{2}{\wedge}_{\mathbb{C}} W$ and let $\overset{(1,1)}{\wedge} T$ the real vector space such that $W \otimes \overline{W} = \overset{(1,1)}{\wedge} T \otimes_{\mathbb{R}} \mathbb{C}$. We have a natural decomposition

$$\overset{2}{\wedge} T = \overset{(2,0)}{\wedge} T \oplus \overset{(1,1)}{\wedge} T.$$

The imaginary part of the hermitian metric on W gives a natural element ω of the latter space. One then computes that

$$\lambda_{\mathbb{R}}^{+} = \overset{(2,0)}{\wedge} T \oplus \mathbb{R} \cdot \omega \text{ and } \lambda_{\mathbb{R}}^{-} = \omega^{\perp} \cap \overset{(1,1)}{\wedge} T.$$

Moreover, under the identification between $\Lambda_{\mathbb{R}}^{+}$ and $\mathrm{End}(W_{+})^{ah}$ we obtain identifications

$$\overset{(2,0)}{\wedge} T = \mathrm{Hom}_{\mathbb{C}}(\mathbf{1}, \det W) = \det W \text{ and } \mathbb{R} = \mathbb{R} \cdot \omega = \mathbb{R} i \cdot \mathbf{1}_{\det W}.$$

3. Spin structures on four manifolds

Let X be a compact oriented four manifold. For any metric g on X we have the principal $SO(4)$ bundle P on X which consists of oriented orthonormal frames. This corresponds to a class $[P]$ in $H^1(X, SO(4))$. Using the exact sequence

$$1 \to U(1) \to \mathrm{Spin}_c(4) \to SO(4) \to 1$$

we see that we have an exact sequence

$$H^1(X, U(1)) \to H^1(X, \mathrm{Spin}_c(4)) \to H^1(X, SO(4)) \to H^2(X, U(1)).$$

We see that the obstruction to giving a reduction of structure group from $SO(4)$ to $\mathrm{Spin}_c(4)$ is given by a class in $H^2(X, U(1))$. Moreover, from the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(4) \to SO(4) \to 1$$

we see that the obstruction to giving a spin structure lies in $H^2(X, \mathbb{Z}/2\mathbb{Z})$. Under the natural inclusion of $\mathbb{Z}/2\mathbb{Z}$ in $U(1)$, the obstruction for spin maps to the obstruction for Spin_c. In fact consider the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Spin}(4) & \to & SO(4) & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & U(1) & \to & \mathrm{Spin}_c(4) & \to & SO(4) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & U(1) & = & U(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

By the associated diagram of cohomologies

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & & \downarrow \\
\to & H^1(X,\mathbb{Z}/2\mathbb{Z}) & \to & H^1(X,\mathrm{Spin}(4)) & \to & H^1(X,SO(4)) & \to & H^2(X,\mathbb{Z}/2\mathbb{Z}) \\
 & \downarrow & & \downarrow & & \| & & \downarrow \\
\to & H^1(X,U(1)) & \to & H^1(X,\mathrm{Spin}_c(4)) & \to & H^1(X,SO(4)) & \to & H^2(X,U(1)) \\
 & \downarrow & & \downarrow & & & & \downarrow \\
 & H^1(X,U(1)) & = & H^1(X,U(1)) & & & & H^2(X,U(1)) \\
 & \downarrow & & \downarrow & & & &
\end{array}$$

we see that the distinct lifts of a given $SO(4)$ bundle to a $\mathrm{Spin}_c(4)$ bundle correspond exactly to the different lifts of the $\mathrm{Spin}(4)$ obstruction class in $H^2(X,\mathbb{Z}/2\mathbb{Z})$ to a class in $H^1(X,U(1))$. We note that the latter is the group of metrised complex line bundles.

Now we have a natural exact sequence (the exponential sequence) of sheaves

$$0 \to \mathbb{Z} \to C^\infty \to U(1) \to 1$$

which gives the natural isomorphisms $H^{i+1}(X,\mathbb{Z}) = H^i(X,U(1))$. Moreover, under these isomorphisms the exact sequence

$$\to H^1(X,U(1)) \to H^1(X,U(1)) \to H^2(X,\mathbb{Z}/2\mathbb{Z}) \to H^2(X,U(1)) \to H^2(X,U(1))$$

is the same as the exact sequence

$$\to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}/2\mathbb{Z}) \to H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{Z}).$$

To summarise, the obstruction to giving a $\mathrm{Spin}_c(4)$ structure is the image in $H^3(X,\mathbb{Z})$ of the obstruction to a $\mathrm{Spin}(4)$ which lies in $H^2(X,\mathbb{Z}/2\mathbb{Z})$. If the former is zero then the different $\mathrm{Spin}_c(4)$ structures correspond to the different lifts of the $\mathrm{Spin}(4)$ obstruction class to $H^2(X,\mathbb{Z})$.

In the case when the principal bundle is the one associated with the metrised tangent bundle as above, we have the result that the obstruction to having a spin structure is given by $w_2(X)$ in $H^2(X,\mathbb{Z}/2\mathbb{Z})$; the second Stiefel–Whitney class of X. Then we have Wu's formula which implies that for any y in $H^2(X,\mathbb{Z}/2\mathbb{Z})$ we have $w_2(X)\cap y = y\cap y$. Now consider the image w of $w_2(X)$ in $H^3(X,\mathbb{Z})$; this is a 2-torsion class. Let $H^2(X,\mathbb{Z})_T$ denote the group of torsion elements in $H^2(X,\mathbb{Z})$. There is a natural duality between the 2-torsion in $H^3(X,\mathbb{Z})$ and the group $H^2(X,\mathbb{Z})_T \otimes \mathbb{Z}/2\mathbb{Z}$; this duality is given as follows. Let $a \in H^2(X,\mathbb{Z})_T$ be a torsion class and let $b \in H^3(X,\mathbb{Z})$ be a 2-torsion class. Let b' be a class in $H^2(X,\mathbb{Z}/2\mathbb{Z})$ whose image is b. Let a' be the image of a in $H^2(X,\mathbb{Z}/2\mathbb{Z})$, then $\langle a,b\rangle = (a',b')$. By this identification we have

$$\langle a,w\rangle = (a',w_2(X)) = \mathrm{Trace}(a'\cap w_2(X)) = \mathrm{Trace}(a'\cap a') = 0$$

for all a in $H^2(X,\mathbb{Z})_T$. But then by the duality we see that w is 0. Hence, in this case we obtain that $w_2(X)$ is the reduction modulo 2 of an integral cohomology class; in other words *an oriented compact Riemannian four manifold always has a* $\mathrm{Spin}_c(4)$ *structure.*

4. Monopole equations and their moduli space

In this section we describe the monopole moduli spaces and compute the expected dimension.

4.1 *Connections for Spin structures*

Let (X, g, c) be a compact oriented Riemannian four manifold with a Spin_c structure (denoted by c). Let Q denote the corresponding principal $\mathrm{Spin}_c(4)$ bundle over X. Then the principal bundle of oriented orthonormal frames on X is given by $P = Q/U(1)$. We have a natural torsion free connection on this bundle called the Riemannian connection. The pull-back of this to Q gives us a 1-form on Q with values in $\mathrm{Lie}(SO(4))$ which is invariant for the action of $\mathrm{Spin}_c(4)$. Now consider the principal $U(1)$ bundle $Q/\mathrm{Spin}(4)$ associated with Q which is just the space of all unit vectors in the line bundle $L = \det(W_\pm)$. Let A be connection on this line bundle. We can pull this back to a form on Q. Adding the above two forms together we obtain a connection on Q which we shall denote by ∇_A since the Riemannian connection is unique whereas A can be varied.

4.2 *Dirac equation*

Fixing A for the time being we have the differential operator $\nabla_A : W_+ \longrightarrow W_+ \otimes T^*X$ induced by the connection as above. On the one hand the Riemannian structure gives us a natural (flat) identification between T^*X and TX and on the other we have seen that TX can be thought of as a subspace of $\mathrm{Hom}_{\mathbb{C}}(W_+, W_-)$; moreover, this identification is also invariant under the connection (flat). Thus by contraction we obtain the composite differential operator of order 1

$$D_A = D_{A,+} : W_+ \longrightarrow W_+ \otimes T^*X \longrightarrow W_+ \otimes TX \longrightarrow W_-.$$

This is called the Dirac operator. As seen earlier we have a natural identification

$$\mathrm{Hom}_{\mathbb{C}}(W_+, W_-) = \mathrm{Hom}_{\mathbb{C}}(W_-, W_+).$$

Thus we also obtain an operator $D_A^* = -D_{A,-} : W_- \longrightarrow W_+$. We have the identity (see § 5)

$$\int_X h_-(-D_{A,+}\Phi, \Psi) = \int_X h_+(\Phi, D_{A,-}\Psi)$$

so that we see that D_A^* is the adjoint of D_A. This justifies the notation.

The first monopole equation is the Dirac equation $D_A(\Phi) = 0$.

4.3 *Second monopole equation*

Consider the curvature F_A of the connection A on L. This gives a two form with values in the Lie algebra of $U(1)$ which is just $\mathbb{R}$. Let F_A^+ denote the projection into Λ^+. We have also defined the map $\sigma : W_+ \otimes \overline{W_+} \longrightarrow \mathrm{End}(W_+)^{ah}$. Moreover, we have obtained an identification of Λ^+ with the space of skew-Hermitian endomorphisms of W_+. The second monopole equation is

$$F_A^+ = \sigma(\Phi, \Phi).$$

4.4 *Gauge group*

Let $\mathcal{G} = \mathrm{Map}(X, U(1))$ and consider the action of this group on the space $\mathcal{N}$ of pairs (A, Φ) where A is a connection on the line bundle L and Φ a section of W_+ given by

$$g \cdot (A, \Phi) = (g \cdot A, g \cdot \Phi) = (A - (1/2\pi i)g^{-1}dg, g\Phi).$$

We see easily that if (A, Φ) satisfies the monopole equation then so does $g \cdot (A, \Phi)$. In fact we have

$$D_{g \cdot A}(g \cdot \Phi) = g \cdot D_A\Phi \text{ and } F_{g \cdot A} = F_A \text{ and } \sigma(g\Phi, g\Psi) = \sigma(\Phi, \Psi).$$

Thus we may consider the 'moduli space' of monopoles

$$M = M_c = \{(A, \Phi) \mid D_A\Phi = 0 \text{ and } F_A^+ = \sigma(\Phi, \Phi)\}/\mathcal{G}.$$

We will show that for 'good' metrics this is a compact orientable manifold. We shall also find out how it depends on this choice of metric.

Now let $\mathcal{W}_\pm$ be the spaces of sections of $W_\pm$. We have a map

$$\nu : \mathcal{N} \to \mathcal{W}_- \text{ given by } (A, \Phi) \mapsto (D_A\Phi).$$

Let $\mathcal{M}$ denote the inverse image $\nu^{-1}(0)$ and let $\mathcal{M}^*$ denote the open subset consisting of pairs (A, Φ) where $\Phi \neq 0$. The differential of the map $(A, \Phi) \to D_A\Phi$ is given by

$$(a, \phi) \mapsto D_A\phi + 2\pi i a \circ \Phi.$$

Suppose Ψ is orthogonal to the image. Then we obtain the equations

$$D_A^*\Psi = 0 \text{ and } \Phi \otimes \Psi = 0$$

by orthogonality with the image of vectors of the form $(a, 0)$ and $(0, \phi)$ respectively. Now a solution of an elliptic operator vanishes on an open set only if it is identically 0. Thus we see that $\Phi = 0$; in other words ν is a submersion when restricted to the space $\mathcal{N}^*$ consisting of pairs (A, Φ) where $\Phi \neq 0$. Thus $\mathcal{M}^*$ is a manifold (albeit of infinite dimension).

The group $\mathcal{G}$ acts freely on $\mathcal{M}^*$ since a solution of an elliptic operator cannot vanish on an open set unless it is 0. Consider the space Ω^{2+} consisting of 2-forms invariant under $*$ with the trivial action of $\mathcal{G}$. The map $\mathcal{M} \to \Omega^{2+}$ given by $F_A^+ - \sigma(\Phi, \Phi)$ factors through the quotient $\mathcal{M}/\mathcal{G}$. We thus obtain a 'complex' $\mathcal{G} \to \mathcal{M}^* \to \Omega^{2+}$. The moduli space can be thought of as being its 'cohomology'.

4.5 *Virtual dimension of the moduli space*

To compute the dimension of the moduli space we need to compute the cohomology of the complex of differentials of the complex $\mathcal{G} \to \mathcal{M}^* \to \Omega^{2+}$. The tangent space to $\mathcal{G}$ at identity can be identified with Ω^0 the space of functions and the tangent space to Ω^{2+} can be identified with itself since it is a vector space. We have an exact sequence

$$0 \to T\mathcal{M}^* \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \to 0,$$

where we have identified the tangent space of $\mathcal{A}$ with Ω^1 the space of 1-forms. Thus the complex of differentials

$$T\mathcal{G} \to T\mathcal{M}^* \to \Omega^{2+}$$

is quasi-isomorphic to the complex

$$\Omega^0 \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \oplus \mathcal{W}_-,$$

where the maps are

$$h \mapsto (-dh, 2\pi i h\Phi) \text{ and } (a, \psi) \mapsto (d^+a - \Im\sigma(\Phi, \phi), D_A\phi + 2\pi i a \circ \Phi)$$

for $h \in \Omega^0$, $a \in \Omega^1$ and $\phi \in \mathcal{W}_+$. Here d^+ denotes the exterior derivative combined with the projection to Ω^{2+} and $\Im\sigma(\Phi, \phi)$ denotes the skew-Hermitian part of $\sigma(\Phi, \phi)$. This complex is homotopic to the complex where the first map is $h \mapsto (-dh, 0)$ and the second is $(a, \phi) \mapsto (d^+a, D_A\phi)$ since the difference between these two complexes is given by compact operators. Thus the index of our complex of differentials is the index of the

complex

$$\Omega^0 \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \oplus \mathcal{W}_-,$$

where the maps are

$$h \mapsto (-dh, 0) \text{ and } (a, \psi) \mapsto (d^+ a, D_A \phi).$$

This is a topological invariant for the pair (X, c) by the Atiyah-Singer Index theorem; we call this the virtual dimension of the moduli space. In case we can find a point $\delta \in \Omega^{2+}$ which is a regular value for the map $\mathcal{M}^* \to \Omega^{2+}$ we see that this index will be the dimension of

$$M_{c,\delta} = \{(A, \Phi) \mid D_A \Phi = 0 \text{ and } F_A^+ = \sigma(\Phi, \Phi) + \delta\}/cG.$$

We call this the perturbed moduli space. We will show that such a value of δ exists and that $M_{c,\delta}$ is a compact orientable manifold whose dimension is the virtual dimension.

5. Differential calculus

We derive various identities among differential operators in the context of Spin_c connections.

5.1 *The adjoint of the Dirac operator*

We have defined the Dirac operator as the composite

$$D_{A,+} = D_A : W_+ \xrightarrow{\nabla_A} T^*X \otimes W_+ \to W_-,$$

where the latter map is the contraction under the identification of T^*X with $TX \subset \mathrm{Hom}_{\mathbb{C}}(W_+, W_-)$. We have similarly the Dirac operator $D_{A,-} : W_- \to W_+$ since we have an identification of $\mathrm{Hom}_{\mathbb{C}}(W_+, W_-)$ with its dual space $\mathrm{Hom}_{\mathbb{C}}(W_-, W_+)$. In terms of an orthonormal frame of tangent vectors $\{e_i\}$ we obtain a sequence of identities:

$$(D_A\Phi, \Psi) = \sum_i (e_i \circ \nabla_{e_i}\Phi, \Psi)$$

and since $(f \circ \Phi, \Psi) = (f^\dagger \circ \Phi, \Psi) = (\Phi, f \circ \Psi)$ for all f in TX,

$$(D_A\Phi, \Psi) = \sum_i (\nabla_{e_i}\Phi, e_i \circ \Psi).$$

Now the fact that ∇ is a metric connection means that

$$(\nabla_{e_i}\Phi, e_i \circ \Psi) = e_i(\Phi, e_i \circ \Psi) - (\Phi, \nabla_{e_i}(e_i \circ \Psi)).$$

Let $d\tau$ denote the volume form. Then for any function f and any vector field v we have,

$$v(f)d\tau = d(f(v \lrcorner d\tau)) - fd(v \lrcorner d\tau).$$

Thus we obtain

$$(D_A\Phi, \Psi)d\tau = \sum_i d((\Phi, e_i \circ \Psi)e_i \lrcorner d\tau) - \sum_i (\Phi, e_i \circ \Psi)d(e_i \lrcorner d\tau)$$
$$- \sum_i (\Phi, \nabla_{e_i}(e_i \circ \Psi)).$$

For any vector field v we have the identity

$$d(v \lrcorner d\tau) = \sum_j (e_j, \nabla_{e_j} v) d\tau.$$

Moreover, since $(e_j, e_j) = \delta_{i,j}$ is a constant we have

$$\begin{aligned}\sum_i (\Phi, e_i \circ \Psi) d(e_i \lrcorner d\tau) &= \sum_{i,j} (\Phi, e_i \circ \Psi)(e_j, \nabla_{e_j} e_i) d\tau \\ &= -\sum_{i,j} (\Phi, e_i \circ \Psi)(\nabla_{e_j} e_j, e_i) d\tau.\end{aligned}$$

The other term can be written as follows

$$\nabla_{e_i}(e_i \circ \Psi) = (\nabla_{e_i} e_i) \circ \Psi + e_i \circ \nabla_{e_i} \Psi.$$

and

$$(\nabla_{e_i} e_i) \circ \Psi = \sum_j (\nabla_{e_i} e_i, e_j) e_j \circ \Psi.$$

Combining the above identities we obtain

$$(D_A \Phi, \Psi) = \sum_i d((\Phi, e_i \circ \Psi) e_i \lrcorner d\tau) - \sum_i (\Phi, e_i \circ (\nabla_{e_i} \Psi)) d\tau.$$

Hence

$$\int_X (-D_{A,+}\Phi, \Psi) = \int_X (\Phi, D_{A,-}\Psi)$$

and $-D_{A,-}$ is the adjoint operator of $D_{A,+}$.

By an entirely similar chain of reasoning we show that the adjoint $\nabla^* : TX \otimes W_- \to W_+$ of ∇ on W_+ is given by

$$\nabla(v \otimes \Phi) = -\left(\sum_i (e_i, \nabla_{e_i} v)\Phi + \nabla_v \Phi \right).$$

In invariant terms, we can describe this as the composite

$$TX \otimes W_+ \xrightarrow{-\nabla} T^*X \otimes TX \otimes W_+ \xrightarrow{\text{Trace}\otimes 1} W_+.$$

5.2 *The Weitzenbock formula*

We now compute the composite $D_A^* D_A \Phi$. As before we choose a local orthonormal frame $\{e_i\}$ for X. We then have

$$D_A^* D_A \Phi = \sum_i -D_A(e_i \circ \nabla_{e_i} \Phi) = -\sum_{i,j} e_j \circ \nabla_{e_j}(e_i \circ \nabla_{e_i} \Phi)).$$

We expand the summand to obtain

$$e_j \circ \nabla_{e_j} e_i \circ \nabla_{e_i} \Phi + e_j \circ e_i \nabla_{e_j} \nabla_{e_i} \Phi.$$

As above the first term can be expanded again as

$$\sum_k (e_k, \nabla_{e_j} e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi = -\sum_k (\nabla_{e_j} e_k, e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi.$$

We obtain the formula

$$D_A^* D_A \Phi = \sum_{i,j,k} (\nabla_{e_j} e_k, e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi - \sum_{i,j} e_j \circ e_i \circ \nabla_{e_j} \nabla_{e_i} \Phi.$$

Now defining $\nabla^2_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}$,

$$D_A^* D_A \Phi = -\sum_{i,j} e_j \circ e_i \circ \nabla^2_{e_j, e_i} \Phi.$$

Similar calculations yield the formula

$$\nabla_A^* \nabla_A \Phi = -\sum_i \nabla^2_{e_i, e_i} \Phi.$$

Now the difference gives us

$$D_A^* D_A \Phi - \nabla_A^* \nabla_A \Phi = -\sum_{i \neq j} e_j \circ e_i \circ \nabla^2_{e_j, e_i} \Phi.$$

From the definition of $\nabla^2_{\cdot,\cdot}$ we have

$$\nabla^2_{V,W} - \nabla^2_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V W} - \nabla_W \nabla_V - \nabla_{\nabla_W V} = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}$$

using the fact that the connection is torsion free. Since we have an orthonormal basis we have $e_i \circ e_j = -e_j \circ e_i$ so that we obtain

$$D_A^* D_A \Phi - \nabla_A^* \nabla_A \Phi == -\sum_{i<j} e_j \circ e_i \circ R(e_j, e_i) \Phi,$$

where $R(V, W) = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}$ is the curvature tensor.

5.3 *The curvature tensors*

The Spin_c connection has been expressed as a sum of the Riemannian connection and the $U(1)$ connection A on L. Thus the curvature tensor R is also the sum of the Riemann curvature tensor S and the curvature of A. The former can be expressed as

$$S(V, W) = \sum_{k,l} (S(V, W) e_l, e_k) e_k \circ e_l.$$

Thus we obtain

$$\sum_{i,j} e_j \circ e_i \circ S(e_j, e_i) = \sum_{i,j,k,l} (S(e_j, e_i) e_l, e_k) e_j \circ e_i \circ e_k \circ e_l.$$

By the orthonormality of e_i's we easily resolve the latter to obtain $\sum_{i,j} (S(e_j, e_i) e_i, e_j)$ which is the negative of the scalar curvature s. The curvature of A considered as an operator on W_+ acts as $2\pi i F_A^+$. Thus the final (Weitzenbock) formula reads

$$D_A^* D_A - \nabla_A^* \nabla_A = s - 2\pi i F_A^+.$$

˙.4 *Extrema*

et x be a point of our manifold where (Φ, Φ) attains a maximum. Then for any vector v : x we have $v((\Phi, \Phi))(x) = 0$. Thus consider the following identity (where $\Re$ denotes the

real part)

$$\Re(\nabla_A^*\nabla_A\Phi,\Phi) = -\sum_i \frac{1}{2}(e_ie_i(\Phi,\Phi) - (\nabla_{e_i}\Phi,\nabla_{e_i}\Phi)) + \sum_{i,j}(e_j,\nabla_{e_i}e_i)\Re(\nabla_{e_i}\Phi,\Phi).$$

Since $e_i(\Phi,\Phi)(x) = 0$ the last term vanishes at x. Moreover, since x is a local maximum for (Φ,Φ) the term $e_ie_i(\Phi,\Phi)(x)$ is negative. Thus we see that $\Re(\nabla_A^*\nabla_A\Phi,\Phi)$ is positive at x.

6. The Seiberg–Witten invariants

In this section we construct the Seiberg–Witten invariants. First of all we fix a four manifold X, a Riemannian metric g and a Spin_c structure c. At the end of the section we will discuss the independence of the invariants on the metric considered.

6.1 *Statement of the basic construction*

Let M_δ denote the fibre of $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ over the point δ. We wish to show that there is a δ such that this is a compact manifold. To show this we need to show

1. There are regular values for $\mathcal{M}^*/\mathcal{G} \to \Omega^{2+}$.
2. There are regular values as above such that the fibre of $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ is contained in $\mathcal{M}^*/\mathcal{G}$.
3. The map $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ is proper.

6.2 *Properness*

Let δ_i be a convergent sequence of elements in Ω^{2+}. This means that the sequence converges in the L_k^2 Sobolev norm for every k. Let (A_i,Φ_i) be such that $D_{A_i}\Phi_i = 0$ and $F_{A_i}^+ - \sigma(\Phi_i,\Phi_i) = \delta_i$. To show properness we need to find a convergent subsequence of (A_i,Φ_i); for which it is enough to show that this sequence is bounded in the L_k^2 Sobolev norm for every k.

Let B be a fixed smooth connection on L; we express $A_i = B + a_i$ where a_i are 1-forms. Consider the function $h_i = G * d * a_i$ and let $g_i^0 = \exp(2\pi i h_i)$. Then $g_i^0 \cdot A_i = B + a_i - dh_i$ and we obtain $* d * (a_i - dh_i) = 0$. Now we can choose g_i so that the harmonic part of a_i lies in the fundamental domain for $H^1(X,\mathbb{Z})$ in $H^1(X,\mathbb{R})$. Thus upto gauge invariance we can replace A_i by another so that $* d * a_i = 0$ and the harmonic part α_i of a_i is bounded. Let $b_i = a_i - \alpha_i$. The second monopole equation becomes

$$d^+b_i = \sigma(\Phi_i,\Phi_i) - F_B^+ + \delta_i$$

so that an L_k^2 bound on Φ_i will give an L_k^2 bound on d^+b_i. But now $b_i = G * d * d^+b_i$ by the above construction of b_i; here G is the Green's operator. Thus we obtain a bound on the L_{k+1}^2 norm of b_i since G is 2-smoothing.

Let us write $\Phi_i = \Psi_i + \phi_i$ where $D_B\Psi_i = 0$ and ϕ_i is orthogonal to the space of solutions of D_B. Hence $\phi = GD_B^*D_B\Phi$ and the first monopole equation becomes

$$D_B\Phi_i = -(b_i + \alpha_i) \circ \Phi_i$$

so that an L_k^2 bound on Φ_i and b_i gives us an L_2^{k+1} bound on ϕ_i. We also need to find a way to uniformly bound Ψ_i. We do this by finding a uniform bound for Φ_i.

Let x_i be a point where (Φ_i, Φ_i) attains a supremum. Applying the Weitzenbock formula we see that at x_i we have

$$0 \geq \Re(\nabla_{A_i}^* \nabla_{A_i} \Phi_i, \Phi_i)(x_i) = -\Re(s\Phi_i, \Phi_i)(x_i) + 2\pi \Im(F_{A_i}^+ \Phi_i, \Phi_i)(x_i).$$

Note that $F_{A_i}^+$ is a skew-Hermitian endomorphism of W_+ and thus

$$\Im(F_{A_i}^+ \Phi_i, \Phi_i) = (F_{A_i}^+ \Phi_i, \Phi_i).$$

The second monopole equation gives us

$$F_{A_i}^+ \Phi_i = \sigma(\Phi_i, \Phi_i)\Phi_i + \delta_i \Phi_i$$

and the expression for σ gives us

$$\sigma(\Phi_i, \Phi_i)\Phi_i = \frac{i}{2}(\Phi_i, \Phi_i)\Phi.$$

Combining the above we obtain

$$(\Phi_i, \Phi_i)(x_i) \leq \max\{0, -s + \| \delta_i \|\}.$$

Thus we uniformly bound Φ_i in the C^0-norm. This gives us uniform bounds for Ψ_i and ϕ_i in the C^0-norm. Now Ψ_i are solutions of the Dirac equation $D_B \Psi_i = 0$. Thus the set of C^0-bounded solutions is a compact set; in particular, we obtain L_k^2 bounds on Ψ_i for all i.

The above arguments applied inductively gives the required result. We note that the above arguments also prove that the solutions of the monopole equations are smooth since any solution which is bounded in L_k^2 norm for some k is actually bounded in all $L_{k'}^2$ norms as above.

6.3 *Regular values*

Now consider the compact space M_0 of solutions of the unperturbed monopole equations. For each point (A, Φ) of M_0 we have a neighbourhood U in $\mathcal{N}$ of (A, Φ) and a finite dimensional linear space $H \subset \Omega^{2+}$ such that the composite

$$U \to \mathcal{N} \to \mathcal{W}_- \times \Omega^{2+} \to H^\perp$$

is a submersion. By compactness we can find a common H and a saturated (for $\mathcal{G}$) open set U in $\mathcal{N}$ containing the inverse image of M_0 such that the above composite is a submersion. Since the derivative is a Fredholm map, the fibre over 0 is a finite dimensional manifold N. We now consider the map of finite dimensional manifolds $N \to H$. By Sard's theorem we have a dense subset of H which consists of regular values.

Now assume that b_2^+ which is the codimension in Ω^{2+} of the δ's of the form $F_B^+ + d^+ b$ is greater than zero. Then the collection of those δ for which the fibre is contained in $\mathcal{M}^*$ is a non-empty open set. If $b_2^+ > 1$ then this open set is even path-wise connected. Thus in this situation the cobordism class of the fibre is independent of the regular value chosen.

6.4 *Dependence on the metric*

Let C denote the space of all metrics g on X under which the fixed volume form $d\tau$ has norm one. We have a natural map $C \to G$ where G denotes the Grassmannian of rank b_2^+ quotients of $H^2(X, \mathbb{R})$. The corresponding tangent level map is

$$\mathrm{Hom}(\Lambda^-, \Lambda^+) = TC \to TG = \mathrm{Hom}(H_g^{2-}, H_g^{2+}),$$

where a map $f : \Lambda^- \to \Lambda^+$ goes to its harmonic projection.

For any class $c = c_1(L)$ in $H^2(X, \mathbb{R})$ let S_c denote the subvariety of G where the class c goes to zero in H^{2+}. At a point of S_c the tangent space to S_c is given by the kernel of the evaluation map $g \mapsto g(c)$. Consider the composite map

$$\mathrm{Hom}(\Lambda^-, \Lambda^+) \to \mathrm{Hom}(H_g^{2-}, H_g^{2+}) \to H_g^{2+}.$$

If we show that this map is surjective, then the space of all metrics under which the class c becomes $*$-anti-invariant will be of codimension b_2^+. The argument of the previous section will apply to show that the Seiberg-Witten invariant is independent of the metric when $b_2^+ > 1$.

To show that the above map is surjective suppose that d is perpendicular to the image. We will then obtain that $c \otimes d$ is identically zero. But now if $c \neq 0$ then it is represented by a harmonic form which cannot vanish on an open set. Thus d must vanish on an open set. But we represent d by a harmonic form too. Thus $d = 0$ as required.

7. The case of Kähler manifolds

We now specialize to the case of Kähler surfaces.

7.1 *Spin structures*

For any four manifold with almost complex structure and (hermitian) metric we have a natural Spin_c structure given by taking $W_+^0 = \overset{2}{\Lambda}_{\mathbb{C}} TX \oplus \mathbf{1}$ and $W_-^0 = TX$. The inclusion of TX in $\mathrm{Hom}_{\mathbb{C}}(W_+^0, W_-^0)$ is the natural one as discussed at the end of § 2. Thus any Spin_c structure on X is given by $W_+ = M \otimes_{\mathbb{C}} \overset{2}{\Lambda}_{\mathbb{C}} TX \oplus M$ and $W_- = TX \otimes_{\mathbb{C}} M$. For ease of notation we adopt the standard convention $\overset{2}{\Lambda}\, T^*X = K_X$.

7.2 *Spin_c connections*

Any $U(2)$ connection on TX gives a connection on all associated bundles. In particular we obtain connections on $W_\pm^0$. However, in order that these be Spin_c connections it is necessary that the induced connection on TX be the Riemannian (torsion-free) connection. This can only happen if the (almost) complex structure is parallel with respect to the Riemannian connection; thus in this case the manifold must be Kähler.

To give a connection in the general Spin_c structure we need in addition to give a $U(1)$ connection on M.

7.3 *The first monopole equation*

Consider a Spin_c connection as above. We then obtain a Dirac operator on $M \oplus M \otimes K_X^{-1}$. By the above discussion we note that the restriction of this to M is the composite

$$M \to M \otimes_{\mathbb{R}} T^*X = M \otimes_{\mathbb{C}} T^*X \oplus M \otimes_{\mathbb{C}} \overline{T^*X} \to M \otimes_{\mathbb{C}} TX.$$

Here we have used the identification of T^*X with $\overline{TX}$ given by the hermitian structure. The first map in the above composite is the $U(1)$ connection on M. Thus we see that the restriction of the Dirac operator to M is $\nabla^{(0,1)}$. We similarly show that the restriction of the Dirac operator to $M \otimes K_X^{-1}$ is also $\nabla^{(0,1)}$ for the induced $U(1)$ connection on this line bundle.

7.4 *The second monopole equation*

Following § 2 we compute that the $(2,0)$ part of $\sigma(\Phi, \Phi)$ for $\Phi = (\alpha, \beta)$ is $\overline{\alpha}\beta$ and the $(1,1)$ part of it is $\frac{1}{2}(\|\beta\|^2 - \|\alpha\|^2)$. Thus the second monopole equation becomes

$$(F_A^+)^{(2,0)} = \overline{\alpha}\beta \text{ and } (F_A^+)^{(1,1)} = \frac{1}{2}(\|\beta\|^2 - \|\alpha\|^2)\omega.$$

7.5 The Weitzenbock formula

We next apply the Weitzenbock formula for any pair (A, Φ)

$$D_A^* D_A \Phi = \nabla_A^* \nabla_A \phi + s\Phi - 2\pi i F_A^+ \Phi$$

to obtain an equality of global inner products

$$(D_A\Phi, D_A\Phi)_X = (\nabla_A\Phi, \nabla_A\Phi)_X + (s\Phi, \Phi)_X + 2\pi\Im(F_A^+\Phi, \Phi)_X.$$

On the other hand we compute the global norm of $F_A^+ - \sigma(\Phi, \Phi)$ as follows

$$\| F_A^+ - \sigma(\Phi, \Phi) \|_X^2 = \| F_A^+ \|_X^2 + \| \sigma(\Phi, \Phi) \|_X^2 - 2\Re(F_A^+, \sigma(\Phi, \Phi))_X.$$

The last term is computed by the integral of the function

$$\Re\, \mathrm{Trace}_{W_+}(F_A^+ \circ \sigma(\Phi, \Phi)) = -\Im(F_A^+\Phi, \Phi) + \tfrac{1}{2}\mathrm{Trace}(F_A^+)\|\Phi\|^2.$$

Now $\mathrm{Trace}(F_A^+)$ is 0. Thus adding the above two identities we obtain

$$\|D_A\Phi\|_X^2 + \|F_A^+ - \sigma(\Phi, \Phi)\|_X^2 = \|\nabla_A\Phi\|_X^2 + (s\Phi, \Phi)_X + 2\pi(\|F_A^+\|_X^2 + \|\sigma(\Phi, \Phi)\|_X^2).$$

We note that the right hand side is equal to

$$\|\nabla_A\alpha\|_X^2 + \|\nabla_A\beta\|_X^2 + (s\alpha, \alpha)_X + (s\beta, \beta)_X + 2\pi\|F_A^+\|_X^2 + 2\pi(\|\alpha\|_X^2 + \|\beta\|_X^2)^2$$

which is invariant under a change of sign for α or β.

Now suppose that (A, Φ) solve the monopole equations and consider the pair (A, Φ_1) where $\Phi_1 = (\alpha, -\beta)$. By the above discussion we see that (A, Φ_1) is also a solution for the monopole equations. But then we must have

$$(F_A)^{(2,0)} = \overline{\alpha}\beta = -\overline{\alpha}\beta = 0.$$

Thus we obtain the fact that F_A is a holomorphic connection on $M^{\otimes 2} \otimes K_X$. Moreover, by ellipticity of the Dirac operator (and its components) we must have that either α or β is zero according as $(F_A^+)^{(1,1)}$ is a positive or negative multiple of ω. By the first monopole equation it then follows that α and β are holomorphic sections of the corresponding line bundles.

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The theorem of Kronheimer–Mrowka

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Abstract. A proof of the conjecture of Thom (algebraic curves in the complex projective plane minimize genus within their homology class) due to Kronheimer–Mrowka is presented. The proof uses the mod-2 Seiberg–Witten invariants.

Keywords. Thom conjecture; Seiberg–Witten invariants; 4-manifolds.

1. Some transversality results

1.1 *Introduction*

This note is an exposition of the proof of Thom's conjecture, namely that algebraic curves minimize genus within their homology class in $\mathbb{C}P^2$, due to Kronheimer–Mrowka (see [KM]). New invariants for 4-manifolds, called Seiberg–Witten invariants, are used in the proof.

For the sake of completeness we have included an appendix at the end on Fredholm theory. As background material, the reader may wish to consult the references [D], [PP], [W].

1.2 *Metrics*

Let X be a connected, compact, oriented 4-manifold. Fix a reference metric g_0 on X, which defines a volume form dV_{g_0} on X, and hence a trivialization of the determinant bundle $\wedge^4(T^*X)$ of the real cotangent bundle. This trivialization of $\wedge^4$ will be fixed throughout, and will be denoted dV without any ambiguity.

Thus the structure group for X is reduced to $SL(4, \mathbb{R})$, and one lets $P_X SL(4)$ denote the principal $SL(4, \mathbb{R})$ bundle on X consisting of all frames in TX on which dV yields the constant function 1 on X.

The corresponding ad-bundle, denoted $\mathrm{ad}\,\underline{sl}_4$, the vector bundle on X whose fibre over $x \in X$ is the vector space of traceless endomorphisms $\mathrm{End}^0(T_x X)$ of $T_x X$ splits into the direct sum of $\mathrm{ad}\,\underline{so}_4$ and $\mathrm{ad}\mathcal{P}$ (abuse of notation since $\mathcal{P}$ is not a Lie algebra) corresponding to the Cartan decomposition: $\underline{sl}_4 = \underline{so}_4 \oplus \mathcal{P}$. Here, the bundle $\mathrm{ad}\,\underline{so}_4$ is the adjoint bundle corresponding to the principal $SO(4, \mathbb{R})$-bundle $P_X SO(4)$ consisting of g_0-orthonormal oriented frames. It has fibre consisting of traceless g_0-skew-symmetric endos of $T_x X$ over x. $\mathrm{ad}\,\mathcal{P}$ has fibre consisting of traceless g_0-symmetric endos of $T_x X$ over x. Both are real rank-3 bundles.

Clearly, if g is *another* Riemannian metric on X with $dV_g = dV$, pointwise, then $g(u, v) = g_0((\exp h)u, (\exp h)v)$ for some section h of $\mathrm{ad}\,\mathcal{P}$, and thus the space of all C^r Riemannian metrics g whose volume form coincides with the prescribed one dV is precisely the space of C^r-sections $\Gamma^r(\mathrm{ad}\,\mathcal{P})$. This space clearly contains exactly one representative from each

conformal class of C^r-metric on X, and may consequently be thought of as the space of equivalence classes of conformal C^r-metrics on X.

The space $\mathcal{C} := \Gamma^r(\operatorname{ad}\mathcal{P})$ maybe given a Banach space structure (the reference metric g_0 is the origin, corresponds to the zero section) via the norm

$$\|s\| := \sup\{\|\nabla^i_{g_0,v}s\|_{x,g_0} : 0 \le i \le r, v \in \otimes^i TX, \|v\|_{x,g_0} = 1\}$$

which is the topology of uniform convergence of all covariant derivatives up to order r. We will fix r to be suitably large later on.

1.3 $*_g$ *and self-duality*

In whatever follows, all metrics g will be elements of $\mathcal{C}$, unless indicated otherwise.

Let $\Omega^i(X)$ denote the space of smooth sections of $\wedge^i(T^*X)$. The bundle $\wedge^i(T^*X)$ will be denoted simply as $\wedge^i(X)$ in future. The bundle $\wedge^4(X)$ is identified with $\wedge^0(X)$, the trivial bundle on X, via the trivialization defined in the last section, i.e., the volume form $\mathrm{d}V = \mathrm{d}V_g$ goes to the constant function 1 under this identification.

The Hodge star-operator $*_g$ is the pointwise operator which makes the following diagram commute:

$$\begin{array}{ccccc} \wedge^2(X) & \otimes & \wedge^2(X) & \xrightarrow{\wedge} & \wedge^0(X) \\ \| & & \uparrow *_g & & \| \\ \wedge^2(X) & \otimes & \wedge^2(X) & \xrightarrow{(\,,\,)_g} & \wedge^0(X) \end{array} \tag{1}$$

where $(\,,\,)_g$ denotes the pointwise g inner product on 2-forms. One easily checks that $*_g \circ *_g = \mathrm{id}$, and the fact that $*_g$ is an isometry with respect to the global g-inner product defined on $\Omega^2(X)$ by

$$\langle\omega,\tau\rangle_g = \int_X (\omega,\tau)_g \mathrm{d}V = \int_X \omega \wedge *_g\tau.$$

Remark 1.3.1. We remark here that on *2-forms* the $*$-operator is a conformal invariant of the metric. This follows because if $g' = \lambda g$, where λ is a C^∞ function, then $(\omega,\tau)_{g'} = \lambda^{-2}(\omega,\tau)_g$ whereas $\mathrm{d}V_{g'} = \lambda^2\mathrm{d}V_g$, so that $(\omega,\tau)_{g'}\mathrm{d}V_{g'} = (\omega,\tau)_g\mathrm{d}V_g$. This shows that the $*$-operator and the global inner product $\langle\,,\,\rangle_g$ are both invariant under conformal changes of the metric.

We then have the eigenbundle decomposition with respect to the involution $*_g$, namely

$$\wedge^2(X) = \wedge^{2+}_g \oplus \wedge^{2-}_g,$$

where $\wedge^{2\pm}_g$ denotes the ± 1-eigenspaces of $*_g$ in $\wedge^2(X)$. The projections $\pi^{\pm}_g(\omega)$ are $\frac{1}{2}(\omega \pm *_g\omega)$, and g-orthogonal projections with respect to the pointwise g-inner product. The space $\Omega^2(X)$ then decomposes correspondingly to $\Omega^{2\pm}_g(X)$, and this decomposition is orthogonal with respect to the global inner-product $\langle\,,\,\rangle_g$.

Lemma 1.3.2 *(Decomposition for Ω^{2+}_g). There is a $\langle\,,\,\rangle_g$-orthogonal decomposition:*

$$\Omega^{2+}_g = \operatorname{Im} d^+_g \oplus \mathcal{H}^{2+}_g$$

where $d^+_g := \pi^+_g \circ d$ and $\mathcal{H}^{2+}_g$ denotes the Δ_g-harmonic forms in Ω^{2+}_g.

Proof. By the Hodge decomposition theorem, if $\alpha \in \Omega_g^{2+}$, we may write

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3,$$

where $\alpha_1 \in \mathcal{H}_g^2$, $\alpha_2 \in \operatorname{Im} d$, and $\alpha_3 \in \operatorname{Im} \delta_g$, where $\delta_g = *_g d *_g$ is the adjoint of d with respect to the global L^2-inner product $\langle\, , \rangle_g$, and the decomposition is orthogonal with respect thereto. Since $*_g$ and Δ_g commute, it follows that $*_g$ maps $\mathcal{H}_g^2$ to itself, and interchanges $\operatorname{Im} \delta_g$ and $\operatorname{Im} d_g$. Thus $\alpha = *_g\alpha$ implies that $\alpha_1 = *_g\alpha_1$, $\alpha_2 = *_g\alpha_3$, and $\alpha_3 = *_g\alpha_2$. Writing $\alpha_2 = d\gamma$, we have $\alpha = \alpha_1 + d\gamma + *_g d\gamma$, and since $d\gamma + *_g d\gamma = 2\pi_g^+ d\gamma = 2d_g^+\gamma$, the lemma follows. □

In future, it will be our convention to identify $H^2(X, \mathbb{R})$ with the space of g_0-harmonic 2-forms $\mathcal{H}_{g_0}^2$, which will henceforth be simply written $\mathcal{H}^2$. Similarly, the symbols $*$, $\pi^\pm$, $\Omega^\pm$, $\mathcal{H}^{2\pm}$, and $d^\pm$ without subscripts will mean that the reference metric g_0 is understood. The commutative square (1) above implies that $< \alpha, \beta >_g = \int_X \alpha \wedge *_g\beta = [\alpha] \cup *_g[\beta]$ for $\alpha, \beta \in \mathcal{H}_g^2$ and all Riemannian metrics g. Box brackets around a closed form α will always denote cohomology class ($= \Delta_{g_0}$-harmonic component).

Remark 1.3.3. If $\alpha \in \mathcal{H}_g^{2+}$, and $\alpha \neq 0$, then $[\alpha] \cup [\alpha] = \langle \alpha, *_g\alpha \rangle_g = \|\alpha\|_g^2 > 0$, and similarly $\alpha \in \mathcal{H}_g^{2-}$ implies $[\alpha] \cup [\alpha] < 0$. Thus if α is self (resp. anti-selfdual) with respect to any metric g, its cup product with itself is positive (resp. negative). So, for any metric g, the cup product pairing is positive definite (resp. negative definite) on $\mathcal{H}_g^{2+}$ (resp. $\mathcal{H}_g^{2-}$). Hence the numbers $b_2^\pm = \dim \mathcal{H}_g^{2\pm}$ give the signature type of the cup product pairing $\cup$ for any metric g.

PROPOSITION 1.3.4

Let $\dim H^2(X, \mathbb{R}) = r$, $b_2^+ = 1$, *and* $b_2^- = r - 1$. *Let the positive cone of the* $\cup$*-product pairing be denoted by*

$$C := \{\alpha \in H^2(X, \mathbb{R}) : \alpha \cup \alpha > 0\}$$

and let C_+ *and* C_- *denote the two components of* C *determined by the sign of* $\pi^+\alpha$. *Then for any metric* $g \in C$, *there exists a unique* g*-harmonic* 2*-form* $\omega_g \in \mathcal{H}_g^{2+}$ *satisfying* (i) $[\omega_g] \in C_+$, (ii) $[\omega_g] \cup [\omega_g] = \langle \omega_g, *_g\omega_g \rangle = 1$. *Finally,* (iii) $[\omega_g] = [\omega_{g'}]$ *if* g *and* g' *are conformally equivalent.*

Proof. For any Riemannian metric g, there is the composite map:

$$\mathcal{H}^2 \hookrightarrow \Omega^2 \to \mathcal{H}_g^2$$

which we call ψ_g. For all g, this is an isomorphism, taking a cohomology class ($= \Delta_{g_0}$-harmonic form) to its Δ_g-harmonic representative. Clearly $\psi_g^{-1}(\alpha) = [\alpha]$ for $\alpha \in \mathcal{H}_g^2$. Via ψ_g^{-1}, $*_g$ maybe be regarded as an involution of $H^2(X, \mathbb{R}) = \mathcal{H}^2$. Choose a $\langle\, , \rangle_g$ unit length element of the $+1$-eigenspace of $*_g$ (which is one dimensional by assumption) in $H^2(X, \mathbb{R})$. Multiplying this element by (-1) if necessary, one can ensure that it lies in C_+. This cohomology class has unique g-harmonic representative, denoted by ω_g, and is the required element, proving (i) and (ii). Finally, (iii) follows from the Remark 1.3.1. □

1.4 *The map* ρ_g

We recall the bundle isometry $\rho_g : \Lambda_g^{2+} \to \underline{su}(W_+)$ of § 2.4 in [PP], where $W_\pm$ are the Hermitian rank 2 bundles arising from the Spin_c structure on X compatible with the

Riemannian metric g. The metrics on the bundles $h_\pm$ are to be fixed once and for all, and thus the corresponding (Hilbert–Schmidt) inner product on the bundles $\underline{su}(W_+)$ are also fixed once and for all. Let us make the isometric identification

$$\rho_0 : \Lambda^{2+}_{g_0} := \Lambda^{2+} \to \underline{su}(W_+)$$

where $\rho_0 := \rho_{g_0}$, with respect to the reference metric g_0 once and for all. Then we can view ρ_g as a bundle isometry

$$\rho_g : \Lambda^{2+}_g \to \Lambda^{2+}$$

with $\rho_0 = \mathrm{Id}$. This map can be explicitly described as follows. Write

$$g(v,w) = g_0(\exp(h)v, \exp(h)w),$$

where h is a smooth section of the bundle ad $\mathcal{P}$ of § 1.2 above. Then for $\omega, \tau \in \Lambda^2(T^*_x(X))$, one has

$$(\omega, \tau)_g = (\wedge^2(\exp(-h))\omega, \wedge^2(\exp(-h))\tau)_0.$$

Thus $\rho_g = \wedge^2(\exp(-h))$.

Note that $\wedge^4(\exp(-h)) = \wedge^2(\rho_g) = \mathrm{Id}$, since $\mathrm{d}V_g = \mathrm{d}V_{g_0} = \mathrm{d}V$, i.e. h is a traceless endomorphism of the tangent bundle. Now one can calculate the Hodge star operator $*_g$ in terms of $* := *_{g_0}$.

Lemma 1.4.1. *The Hodge star operators $*_g$ and $*$ are related by the formula:*

$$\rho_g *_g = * \rho_g.$$

Consequently, $\rho_g \pi^+_g = \pi^+ \rho_g$, and the bundle map $\rho_g : \wedge^2 \to \wedge^2$ is an automorphism of the bundle $\wedge^2$, effecting an isometry between the metrics $(\,,\,)_g$ and $(\,,\,)_0$, carrying $\wedge^{2+}_g$ onto $\wedge^{2+}$ and the g-self-dual 2-forms Ω^{2+}_g to the g_0-self-dual 2-forms Ω^{2+}.

Proof. We have

$$\begin{aligned} \omega \wedge_{*_g} \tau = (\omega, \tau)_g \mathrm{d}V_g &= (\rho_g \omega, \rho_g \tau)_0 \mathrm{d}V \\ &= \rho_g \omega \wedge * \rho_g \tau = \wedge^2(\rho_g)(\omega \wedge \rho_g^{-1} * \rho_g \tau) \\ &= \omega \wedge \rho_g^{-1} * \rho_g \tau, \end{aligned}$$

since $\wedge^2(\rho_g) = \mathrm{Id}$. Thus $*_g = \rho_g^{-1} * \rho_g$, and the result follows. □

1.5 *Transversality*

Notation 1.5.1. Let $b_2^+ \geq 1$. Let C be as in the subsection 1.2, and, as per our convention, Ω^{2+}, $\mathcal{H}^{2+}$ be the $+1$ eigenspaces with respect to $*_{g_0}$ where g_0 is the reference metric. Let $\pi_{\mathcal{H}}$ denote the orthogonal (with respect to $\langle\,,\,\rangle_0$) projection from Ω^{2+} to $\mathcal{H}^{2+}$ from Lemma 1.3.2. Let $\delta \in \Omega^{2+}$, and let $c = c_1(L)$ denote a fixed element of $H^2(X, \mathbb{R}) = \mathcal{H}^2 \subset \Omega^2$ such that $c \cup c < 0$.

Let G denote the Grassmanian of b_2^--dimensional subspaces of $\mathcal{H}^2$. There is a natural rank b_2^+ bundle γ^+ on G whose fibre over $P \in G$ is $\mathcal{H}^2/P$. One may regard this as the quotient bundle of $G \times \mathcal{H}^2$ by the tautological subbundle on G. By definition, there is the natural quotient map $\tau : G \times \mathcal{H}^2 \to \gamma^+$ of bundles on G, viz. $\tau(P, \alpha) = \alpha (\operatorname{mod} P)$.

If $c \in \mathcal{H}^2$ is a cohomology class, then c defines the constant section of $G \times \mathcal{H}^2$, also denoted by c. Let its image $\tau(c)$ be denoted by s_c, a section of γ^+. The zero locus of this

section is

$$S_c = \{P \in G : c \in P\}.$$

If $g \in \mathcal{C}$ is a Riemannian metric, we have the g-self dual projector $\pi_g^+ : \mathcal{H}_g^2 \to \mathcal{H}_g^{2+}$. Thus one has a natural map

$$P : \mathcal{C} \to G$$
$$g \mapsto \operatorname{Ker}(\pi_g^+ \circ \psi_g : \mathcal{H}^2 \to \mathcal{H}_g^{2+}),$$

where ψ_g is the isomorphism from $\mathcal{H}^2$ to $\mathcal{H}_g^2$ introduced in 1.3.4. This map is easily seen to be C^r.

Note that if $P(g) \in S_c$, we have that $\psi_g(c) \in \mathcal{H}_g^{2,-}$, which implies $[\psi_g(c)] \cup [\psi_g(c)] = c \cup c < 0$, by 1.3.3. So $\operatorname{Im} P \cap S_c = \phi$ if $c \cup c > 0$. When $c \cup c < 0$, we have the following lemma, proved in § 7.4 of [PP] (see also [DK], § 4.3.14).

Lemma. Let $c \in \mathcal{H}$ satisfy $c \cup c < 0$. The map $P : \mathcal{C} \to G$ defined above is transverse to S_c, and its inverse image (the submanifold of 'c-bad metrics') $B_c = P^{-1}(S_c)$ is therefore a C^r submanifold of $\mathcal{C}$ of codimension b_2^+.

Now let $\delta \in \Omega^{2+}$. Then $\rho_g^{-1}(\delta) \in \Omega_g^{2+}$. Its g-harmonic component $\rho_g^{-1}(\delta)_{\mathcal{H}_g}$ is in $\mathcal{H}_g^{2+}$, and thus defines a cohomology class $[\rho_g^{-1}(\delta)_{\mathcal{H}_g}]$ in $H^2(X, \mathbb{R}) = \mathcal{H}^{2+}$. Consider the element $c(g, \delta) = c - (1/2\pi)[(\rho_g^{-1}\delta)_{\mathcal{H}_g}] \in \mathcal{H} = H^2(X, \mathbb{R})$. (Box brackets denote cohomology class.) Note $c(g, 0) \equiv c$. We then have the section $(g, c(g, \delta))$ of the trivial bundle $\mathcal{C} \times \mathcal{H}^2$. Let $\sigma_{c,\delta}$ denote the image of this section under the bundle map $P^*\tau : \mathcal{C} \times \mathcal{H} \to P^*(\gamma^+)$. Note that $\sigma_{c,0}$ is just the section $P^* s_c$ where s_c is defined above. Let us denote $\sigma_{c,0}$ by σ_c.

Let $B_{c,\delta}$ denote the zero locus of this section. Note $B_{c,0}$ is just the submanifold B_c defined above.

Let $\mathcal{I}$ be a C^r-embedded arc, parametrized by $[-1, 1]$ in $\mathcal{C}$, meeting B_c transversely. We will denote a typical element of $\mathcal{I}$ by g_t. We will also use the subscript t wherever a subscript g_t occurs, e.g. ω_t for ω_{g_t} etc., for notational convenience. We now have a couple of transversality lemmas, for the two separate cases $b_2^+ \geq 2$ and $b_2^+ = 1$.

PROPOSITION 1.5.2

Let $b_2^+ \geq 2$, and $\mathcal{I}$ be transverse to B_c as above. In this case of $b_2^+ \geq 2$, this means $\mathcal{I} \cap B_c = \phi$. Then, for all δ in an ϵ-ball $U = B(0, \epsilon)$ of the origin in Ω^{2+}, the intersection $\mathcal{I} \cap B_{c,\delta}$ is empty.

Proof. We are given that $\mathcal{I} \cap B_c = \phi$. This means that $\mathcal{I}$ does not meet the zero locus of the section $\sigma_c = \sigma_{c,0}$ defined above, i.e. that $\sigma_{c,0}(g_t) \neq 0$ for all $t \in [-1, 1]$. If $\| \ \|$ is some bundle metric on $P^*(\gamma^+)$, let $a := \min(\|\sigma_{c,0}(g_t)\| : t \in [-1, 1])$, so that $a > 0$. Since $\sigma_{c,\delta}(g_t)$ varies continuously with δ, if one chooses ϵ small enough, then one can arrange that $\|\sigma_{c,0}(g_t) - \sigma_{c,\delta}(g_t)\| < a/2$ for all $t \in \mathcal{I}$ and all $\delta \in B(0, \epsilon)$. It will then follow that $\sigma_{c,\delta}(g_t)$ is non-zero for all $t \in [-1, 1]$, i.e. that $\mathcal{I} \cap B_{c,\delta} = \phi$, proving the proposition. □

PROPOSITION 1.5.3

Let $b_2^+ = 1$, and let $\mathcal{I}$ meet the submanifold B_c transversally at a single point, say $\{g_a\}$. Then, for all δ in an ϵ-ball $U = B(0, \epsilon)$ of the origin in Ω^{2+}, the map

$$f_\delta : \mathcal{I} \to \mathbb{R}$$

$$g_t \mapsto \left(c - \frac{1}{2\pi} [(\rho_g^{-1}\delta)_{\mathcal{H}_g}] \right) \cup [\omega_g] = c(g,\delta) \cup [\omega_g],$$

where $[\omega_g] = \psi_g^{-1}(\omega_g)$ *is the cohomology class of* $\omega_g \in \mathcal{H}_g^{2+}$, *has a unique zero, and this zero is a regular value.*

Proof. Let $\sigma_c(g_t)$ denote the section σ_c restricted to the arc $\mathcal{I}$. Let $p : P^*(\gamma^+) \to \mathcal{C}$ denote the bundle projection, and Z the zero section of this bundle. Since $b_2^+ = 1$, this a real line bundle, and may be regarded as the line subbundle of $\mathcal{C} \times \Omega^2$ whose fibre over g is $\mathcal{H}_g^{2+}$. This bundle has a trivialization over all of $\mathcal{C}$, defined by $g \mapsto \omega_g$, where $\omega_g \in \mathcal{H}_g^{2+} \subset \Omega_g^{2+}$ is the form defined in 1.3.4. By the fact that the zero section of γ^+ and the section $s_c(G)$ meet transversely inside γ^+, and the transversality of P to S_c, it easily follows that $\sigma_c(\mathcal{C})$ and Z meet transversely in $P^*(\gamma^+)$. By assumption, the intersection

$$\mathcal{I} \cap B_c = p\sigma_c(\mathcal{I}) \cap p(\sigma_c(\mathcal{C}) \cap Z)$$

is a transverse intersection at the single point $\{g_a\}$. Since $p : \sigma_c(\mathcal{C}) \to \mathcal{C}$ is a diffeomorphism, which carries $\sigma_c(\mathcal{I})$ and $\sigma_c(\mathcal{C}) \cap Z$ diffeomorphically to $\mathcal{I}$ and B_c respectively, it follows that inside the manifold $\sigma_c(\mathcal{C})$, $\sigma_c(\mathcal{I})$ meets $\sigma_c(\mathcal{C}) \cap Z$ transversely at the single point $\sigma_c(g_a)$. This means that the tangent space $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{I}))$, which is a one dimensional subspace of $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}))$ is linearly independent of the subspace $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}) \cap Z) = T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C})) \cap T_{\sigma_c(g_a)}(Z)$, which is of codimension one in $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}))$. This means that the tangent space $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{I}))$ is not contained in $T_{\sigma_c(g_a)}(Z)$. Since this last space is of codimension one in the tangent space $T_{\sigma_c(g_a)}(P^*\gamma^+)$, one has that the curve $\sigma_c(\mathcal{I})$ meets the zero section Z transversally in the singleton $\{\sigma_c(g_a)\}$. Now let $\theta : P^*(\gamma^+) \to \mathbb{R}$ be the map $\theta(\alpha) = [\alpha] \cup [\omega_g] = \int_X \alpha \wedge \omega_g$, where $\alpha \in \mathcal{H}_g^+$, coming from the trivialization of the line bundle $P^*(\gamma^+)$ described above. Then θ is a submersion, and $Z = \theta^{-1}(0)$. So saying that the curve $\sigma_c(\mathcal{I})$ meets the zero section Z transversally in the singleton $\{\sigma_c(g_a)\}$ is equivalent to the statement that $f_0 = \theta \circ \sigma_c : \mathcal{I} \to \mathbb{R}$ has a unique zero at g_a, and $f_0'(a) := f_0'(g_a)$ is non-zero. Say $f_0'(a) > 0$. Then,

(i) $f_0'(t)$ will be strictly positive on a closed neighborhood V of g_a, and also
(ii) $\min\{|f_0(g_t)| : g_t \in \mathcal{I} - V^\circ\} = b > 0$.

Since $\sigma_{c,\delta}$ varies smoothly with δ, (i) and (ii) above continue to hold good for $f_\delta := \theta \circ \sigma_{c,\delta}$ for all $\delta \in U$, where U is an ϵ-ball around 0 for ϵ suitably small. This proves the proposition. □

2. The parametrized Seiberg–Witten moduli space

2.1 *The gauge group action*

In the sequel, we assume $b_2^+(X) \geq 1$. Let L, W_+, W_- be a Spin_c structure on X. c will always denote $c_1(L) \in H^2(X, \mathbb{R}) = \mathcal{H}^{2+}$. Let g_0 be the reference metric as before. Let $k \geq 6$. We need to define various spaces

$\mathcal{A} :=$ the completion of $C^\infty U(1)$-connections on L (an affine space modelled on $\Omega^1(X)$), with respect to Sobolev k-norm L_k^2. By abuse of language, we shall denote the L_k^2-completion of Ω^1 as Ω^1.

$\Gamma(W_+) :=$ L_k^2 completion of C^∞ complex-valued sections of W_+.

$\Gamma(W_-) :=$ L^2_{k-1} completion of C^∞ complex-valued sections of W_-.
$\mathcal{G} :=$ $\mathrm{Map}(X, S^1)$. A Hilbert manifold whose Lie algebra is the L^2_{k+1}-completion of $\Omega^0(X)$.
$\Omega^{2+} :=$ L^2_{k-1} completion of real valued $*_{g_0}$ self-dual 2-forms, again denoted by same symbol by abuse of language.
$\mathcal{N} :=$ $\mathcal{A} \times \Gamma(W_+)$.
$\mathcal{N}^* :=$ $\mathcal{A} \times (\Gamma(W_+) - \{0\})$.

The choice of $k \geq 6$ implies, by Sobolev's Lemma, that elements of each of the Sobolev spaces defined above are at least twice continuously differentiable.

$\mathcal{G}$ acts on $\mathcal{N}$ by the action:

$$g \cdot (A, \Phi) = \left(A - \left(\frac{1}{2\pi i}\right) g^{-1}\mathrm{d}g,\ g\Phi\right) = (gA, g\Phi)$$

and the choice of norms above makes this a smooth action. Note that $g.(A,0) = (A,0)$ for all $g \in S^1 \subset \mathcal{G}$, so that the gauge group action of $\mathcal{G}$ on $\mathcal{N}$ is free on $\mathcal{N}^*$, and has (as can be checked easily) isotropy S^1 on $(A,0)$.

Remark 2.1.1. The decomposition lemma 1.3.2 for $\Omega^{2+} = \Omega^{2+}_{g_0}$ continues to hold for the Sobolev completions defined above because $d : \Omega^1_{L^2_k} \longrightarrow \Omega^2_{L^2_{k-1}}$ has closed range and $\mathcal{H}^{2+}_{L^2_{k-1}}$ consists of smooth forms by elliptic regularity.

Let $\mathcal{I}$ denote a compact arc in $\mathcal{C}$, meeting B_c as in the hypotheses of Propositions 1.5.2 and 1.5.3. A typical point on $\mathcal{I}$ will be denoted by g_t, and a subscript t anywhere will mean that the metric g_t is being used. Absence of subscript, as always, will mean that the reference metric g_0 is understood. We also recall the map $f_\delta : \mathcal{I} \longrightarrow \mathbb{R}$ in the setting of $b_2^+ = 1$, that was defined in Proposition 1.5.3. For simplicity let us assume that $f_\delta^{-1}(0)$ is the reference metric g_0, and 0 is a regular value as stated there. We recall the notation and definitions of § 4, 5 and 6 of [PP]. For a metric g_t on X, (A, Φ) is called a 'monopole' if it satisfies the following equations:

$$\begin{aligned} &D_{A,t}\Phi = 0, \\ &\rho_t(F_A^{+,t}) - \sigma(\Phi, \Phi) = 0, \end{aligned} \tag{2}$$

where $F_A^{+,t} = \pi_t^+(F_A)$, and we have, once and for all, identified $\Gamma(\underline{su}(W_+))$ with $\Omega_0^{2+} = \Omega^{2+}$ via ρ_0, (see 1.4), and the pairing $\sigma : W_+ \otimes \overline{W}_+ \longrightarrow \Omega^{2+} \simeq \Gamma(\underline{su}(W_+))$ is the pairing defined in [PP], § 2.5. $\rho_t = \rho_{g_t}$ is the isomorphism identifying Ω_t^{2+} with $\Gamma(\underline{su}(W_+)) = \Omega^{2+}$ as defined in 2.4 (see also § 1.4 of [PP]).

Let $\mathcal{G}$ act trivially on Ω^{2+} and $\mathcal{I}$, and consider the G-equivariant map:

$$\begin{aligned} Q : \mathcal{A} \times \Gamma(W_+) \times \mathcal{I} &\longrightarrow \Gamma(W_-) \times \Omega^{2+} \\ (A, \Phi, g_t) &\mapsto (D_{A,t}\Phi, \rho_t(F_A^{+,t}) - \sigma(\Phi, \Phi)). \end{aligned}$$

Thus the solutions to the Seiberg–Witten equations (2) are precisely the elements of $Q^{-1}(0,0)$. Since $(0,0)$ will not in general be a regular value for Q, we need to consider the δ-perturbed Seiberg–Witten equations, viz. $Q^{-1}(0,\delta)$, where δ is a suitably small element of Ω^{2+}. Since the map Q is $\mathcal{G}$ equivariant, and $(0,\delta)$ is fixed by $\mathcal{G}$, it is natural to quotient out $Q^{-1}(0,\delta)$ by the $\mathcal{G}$-action. We shall proceed to do this in detail, in the sequel.

2.2 *The derivative of Q*

Lemma 2.2.1. *The derivative of Q is as follows:*

(i) $DQ_{(A,\Phi,g_t)}(a,\phi,0) = (D_{A,t}\phi + 2\pi i a \circ \Phi, \rho_t(d_t^+ a) - 2\Im\sigma(\Phi,\phi))$
where $\Im$ *denotes imaginary part, and* $\circ$ *Clifford multiplication.*

(ii) *In the case when* $b_2^+ = 1$, *let* $\delta := Q(A,0,g_0) = F_A^+$, *and* $\mathcal{I}$, f_δ *be as in Proposition* 1.5.3. *Then*

$$DQ_{A,0,g_0}(0,0,g'(0)) = (0, d^+\alpha + 2\pi f_\delta'(0)\omega_0)$$

where α *is some* 1*-form and* $\omega_0 := \omega_{g_0}$ *is as in Proposition* 1.3.4, *and* g_0 *is the unique zero of the function* f_δ, *as in* 1.5.3.

Proof. Using the fact that $D_A = \sum c(e_i) \circ \nabla_{A,e_i}$ and that $\nabla_{A+sa,e_i} = \nabla_{A,e_i} + 2\pi i s a(e_i)$, we obtain that $D_{A+sa} = D_A + 2\pi i s a \circ (-)$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}s_{|s=0}}(D_{A+sa,t}(\Phi + s\phi)) = D_{A,t}\phi + 2\pi i a \circ \Phi.$$

Finally, since $F_{A+sa} = F_A + s\,\mathrm{d}a$, we have $F_{A+sa}^{+,t} = F_A^{+,t} + s\, d_t^+ a$, so that

$$\frac{\mathrm{d}}{\mathrm{d}s_{|s=0}}(\rho_t(F_{A+sa}^{+,t})) = \rho_t(d_t^+ a).$$

Skew-sesquilinearity of σ implies that

$$\frac{\mathrm{d}}{\mathrm{d}s_{|s=0}}\sigma(\Phi + s\phi, \Phi + s\phi) = \sigma(\Phi,\phi) + \sigma(\phi,\Phi) = 2\Im\sigma(\Phi,\phi)$$

and we have (i).

To see (ii), since we are computing the derivative at $\Phi = 0$, in the direction of $\phi = 0$, we have $D_{A,t}\Phi = 0$ and $\sigma(\Phi,\Phi) = 0$ for all t, so the first coordinate of the right hand side of the equation in the statement is clearly zero. Now one just needs to compute $\mathrm{d}/\mathrm{d}t_{|t=0}$ $(\rho_t(F_A^{+,t}))$. By 1.3.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}(\rho_t(F_A^{+,t})) = \left\langle \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}(\rho_t(F_A^{+,t})), \omega_0 \right\rangle_0 \omega_0 + d^+\alpha$$

for some 1-form α, where $\langle\, ,\rangle_0$ is the global inner product on Ω^2 with respect to g_0. Recall from § 1.4 the facts that $\rho_0 = \mathrm{Id}$ and $(\rho_t(-), \rho_t(-))_0 = (\, ,\,)_t$, for the pointwise inner product, and so we have the same formula for the global inner product, viz. $\langle \rho_t(-), \rho_t(-)\rangle_0 = \langle\, ,\rangle_t$. We compute the first term:

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}(\rho_t(F_A^{+,t})), \omega_0 \right\rangle_0 &= \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\langle \rho_t(F_A^{+,t}), \rho_t(\omega_t)\rangle_0 - \left\langle F_A^{+,0}, \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\rho_t(\omega_t)\right\rangle_0 \\
&= \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\langle F_A^{+,t}, \omega_t\rangle_t - \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\langle \delta, \rho_t(\omega_t)\rangle_0 = \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}(\langle F_A, \omega_t\rangle_t - \langle \rho_t^{-1}\delta, \omega_t\rangle_t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}(2\pi[c] \cup [\omega_t] - \langle (\rho_t^{-1}\delta)_{\mathcal{H}_t}, \omega_t\rangle_t) \\
&= 2\pi \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\left([c] \cup [\omega_t] - \frac{1}{2\pi}[(\rho_t^{-1}(\delta))_{\mathcal{H}_t}] \cup [\omega_t]\right)
\end{aligned}$$

$$= 2\pi \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}} \left(c - \frac{1}{2\pi}[(\rho_g^{-1}\delta)_{\mathcal{H}_g}] \right) \cup [\omega_t] = 2\pi f_\delta'(0),$$

because for $\alpha \in \mathcal{H}_t^2$, the $*_t$ self duality of ω_t implies $[\alpha] \cup [\omega_t] = \langle \alpha, \omega_t \rangle_t$. This proves the proposition. □

2.3 *Moduli spaces*

PROPOSITION 2.3.1

There is a Baire subset U of an ϵ-ball around 0 in Ω^{2+} for which both of the following hold:

(i) $(0, \delta)$ *is a regular value for* $Q_{|\mathcal{N}^* \times \mathcal{I}}$.
(ii) *The conclusions of Proposition* 1.5.2, *when* $b_2^+ \geq 2$, *and Proposition* 1.5.3, *when* $b_2^+ = 1$ *are satisfied by* δ.

Proof. Since Q is *not* a Fredholm map, one cannot directly apply Proposition 5.1.2 or Corollary 5.1.3 of the Appendix to it. However, consider the space:

$$\mathcal{M}^*(\mathcal{I}) := \{(A, \Phi, g_t) : D_{A,t}\Phi = 0, \ \Phi \neq 0, \ g_t \in \mathcal{I}\}.$$

Since Q is $\mathcal{G}$-equivariant, and $\mathcal{G}$ acts trivially on $\mathcal{I}$ and Ω^{2+}, it follows that the $\mathcal{G}$ action on $\mathcal{M}^*(\mathcal{I})$ is free. Also, $\mathcal{M}^*(\mathcal{I})$ is fibred over $\mathcal{I}$ via projection to the last coordinate. Note that $\mathcal{M}^*(\mathcal{I}) = Q_1^{-1}(0)$ where

$$Q_1 = \pi_{\Gamma(W_-)} \circ Q : \mathcal{N}^* \times \mathcal{I} \to \Gamma(W_-).$$

By § 4.4 of [PP], we know that 0 is a regular value of $Q_{1,t} : \mathcal{N}^* \times \{g_t\} \to \Gamma(W_-)$. Thus it is a regular value of Q_1, so that $\mathcal{M}^*(\mathcal{I})$ is an (infinite dimensional) submanifold of $\mathcal{N}^* \times \mathcal{I}$, whose fibre over $g_t \in \mathcal{I}$ is $\mathcal{M}^*(g_t)$ (as defined in § 4.4 of [PP]). Now, by Lemma 5.1.6 of the Appendix, regular values $(0, \delta)$ will exist for $Q_{|\mathcal{N}^* \times \mathcal{I}}$ whenever regular values exist for the map:

$$Q_2 = \pi_{\Omega^{2+}} \circ Q : \mathcal{M}^*(\mathcal{I}) \to \Omega^{2+}$$
$$(A, \Phi, g_t) \mapsto (\rho_t(F_A^{+,t}) - \sigma(\Phi, \Phi)).$$

Again, Q_2 is *not* a Fredholm map. On the other hand, it is constant along $\mathcal{G}$-orbits.

Claim 1. Q_2 descends to a map $\mathcal{M}^*(\mathcal{I})/\mathcal{G} \to \Omega^{2+}$ (which we also denote by Q_2), and this map is Fredholm of index $d(L) + 1$ where

$$d(L) := \tfrac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2\chi(X)).$$

($\sigma(X)$ is the signature, and $\chi(X)$ the Euler characteristic of X.)

Proof of Claim 1. Since $\rho_t(F_A^{+,t}) - \sigma(\Phi, \Phi) = \rho_t(F_{gA}^{+,t}) - \sigma(g\Phi, g\Phi)$ for all $g \in \mathcal{G}$, it is clear that Q_2 descends to the quotient space $\mathcal{M}^*(\mathcal{I})/\mathcal{G}$.

Now, the inclusion of tangent spaces: $T_x(\mathcal{M}^*(g_t)/\mathcal{G}) \hookrightarrow T_x(\mathcal{M}^*(\mathcal{I})/\mathcal{G})$ has codimension one at each x (= dimension of the tangent space to $\mathcal{I}$), it is enough to prove that

$$Q_{2,t} : \mathcal{M}^*(g_t)/\mathcal{G} \to \Omega^{2+}$$

is a Fredholm map of index $d(L)$. The diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & T_e(\mathcal{G}) & \longrightarrow & T_{(A,\Phi,g_t)}(\mathcal{M}^*(g_t)) & \longrightarrow & T_{([A,\Phi],g_t)}(\mathcal{M}^*(g_t)/\mathcal{G}) & \rightarrow & 0 \\
 & & \downarrow_{d^*d=\Delta} & & \downarrow_{\chi} & & \downarrow_{DQ_{2,t}} & & \\
0 & \rightarrow & \Omega^0 & \hookrightarrow & \Omega^{2+}\oplus\Omega^0 & \longrightarrow & \Omega^{2+} & \longrightarrow & 0
\end{array}$$

shows that $DQ_{2,t}$ on the right is Fredholm of index $d(L)$ iff the middle map χ given by $\chi(a,\phi,0) = (DQ_{2,t}(a,\phi), d^*a)$ is Fredholm of index $d(L)$, since the laplacian Δ has one-dimensional kernel and cokernel, X being connected.

However, the diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & T_{(A,\Phi,g_t)}(\mathcal{M}^*(g_t)) & \rightarrow & T_{(A,\Phi,g_t)}(\mathcal{N}^*\times\{g_t\}) & \xrightarrow{DQ_{1,t}} & \Gamma(W_-) & \rightarrow & 0 \\
 & & \downarrow_{\chi} & & \downarrow_{\chi_1=(DQ_t,d^*)} & & \| & & \\
0 & \rightarrow & \Omega^{2+}\oplus\Omega^0 & \hookrightarrow & \Gamma(W_-)\oplus\Omega^{2+}\oplus\Omega^0 & \longrightarrow & \Gamma(W_-) & \rightarrow & 0
\end{array}$$

where $\chi_1(a,\phi,0) = (DQ_t(a,\phi), d^*a)$, implies that χ is a Fredholm operator iff χ_1 is a Fredholm operator, and of the same index since the vertical map on the right is an equality. Now, note that by the Proposition 2.2.1, part (i), we have

$$\chi_1(a,\phi,0) = (D_{A,t}\phi + 2\pi i a\circ\Phi, \rho_t(d_t^+a) - 2\Im(\sigma(\Phi,\phi)), d^*a).$$

Now, $a\in L_k^2$, $\Phi\in L_k^2$ implies that $a\circ\Phi\in L_k^2$, (Leibnitz rule for kth derivative of a product and Schwartz inequality), and since we have L_{k-1}^2 Sobolev norm on $\Gamma(W_-)$, Rellich's Lemma implies that the map $a\mapsto a\circ\Phi$ is a compact operator from $\Omega^1_{L_k^2}$ into $\Gamma(W_-)$. Similar considerations apply to $\Im\sigma(\Phi,\phi)$. Thus χ_1 is a compact perturbation of the map:

$$\begin{aligned}
&\Gamma(W_+)\times\Omega^1 \rightarrow \Gamma(W_-)\times\Omega^{2+}\times\Omega^0 \\
&(a,\phi)\mapsto (D_{A,t}\phi, \rho_t(d_t^+a), d^*a)
\end{aligned}$$

whose index is clearly $\operatorname{index}(D_A) + \operatorname{index}(d^+, d^*)$. The index of the second map is the negative of the Euler characteristic of the complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}$$

which is $(-\dim H^{2+} - \dim H^0 + \dim H^1)$, which is $-1/2(\sigma(X)+\chi(X))$. The Atiyah–Singer index theorem for the Dirac operator implies:

$$\operatorname{index} D_A = -\tfrac{1}{4}(\sigma(X) - c_1(L)^2).$$

Combining these, we have the index of $Q_2 : \mathcal{M}^*(\mathcal{I})/\mathcal{G} \rightarrow \Omega^{2+}$ to be

$$d(L) := \tfrac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2\chi(X))$$

which proves Claim 1. □

We now return to the proof of our Proposition. By 5.1.3 and 5.1.6, we have a Baire subset of a neighbourhood of 0 in Ω^{2+} such that for δ in this subset, $(0,\delta)$ is a regular value for $Q_{|\mathcal{N}^*\times\mathcal{I}}$. This proves (i). To get (ii), one just intersects this Baire subset with the ϵ-ball U that is guaranteed by Propositions 1.5.2 and 1.5.3. □

Notation 2.3.2. We now fix a δ so that both (i) and (ii) of the last Proposition 2.3.1 are satisfied. In the case $b_2^+ = 1$, we assume that the function $f_\delta : \mathcal{I}\rightarrow\mathbb{R}$ has its unique zero

at g_0, and 0 is a regular value for it. For notational simplicity we modify Q to Q_δ, a translate of Q, by the formula

$$Q_\delta(A, \Phi, g_t) = Q(A, \Phi, g_t) - (0, \delta).$$

We thus have the following consequence to the propositions and Corollaries 5.1.2, 5.1.3, 5.1.6, 5.1.6 and 2.3.1.

PROPOSITION 2.3.3

We have the following facts about Q_δ:

(i) *The map*

$$Q_\delta : \mathcal{N} \times \mathcal{I} \to \Gamma(W_-) \times \Omega^{2+}$$

is a $\mathcal{G}$-equivariant map with $(0,0)$ as a regular value for $Q_{\delta|\mathcal{N}^* \times \mathcal{I}}$, so that $Q_\delta^{-1}(0, 0) \cap \mathcal{N}^* \times \mathcal{I}$ is a Banach manifold, which we will denote as $\mathcal{M}_\delta^*(\mathcal{I})$. It is fibred over $\mathcal{I}$ with fibre $\mathcal{M}_\delta^*(g_t)$ on g_t.

(ii) *When $b_2^+ \geq 2$, $Q_\delta^{-1}(0, 0) \subset \mathcal{N}^* \times \mathcal{I}$. In this case the group of gauge transformations $\mathcal{G}$ acts freely on all of $Q_\delta^{-1}(0, 0)$, and consequently the quotient space $Q_\delta^{-1}(0, 0)/\mathcal{G} = \mathcal{M}_\delta^*(\mathcal{I})/\mathcal{G} := M_{c,\delta}(\mathcal{I})$ is a manifold of dimension $d(L) + 1$. This manifold may also be regarded as $Q_{2,\delta}^{-1}(0)$, where $Q_{2,\delta} = \pi_{\Omega^{2+}} \circ Q_\delta$ is as in Claim 1 in the proof of Proposition 2.3.1. Its fibre over g_t, $M_{c,\delta}(g_t)$ is compact, and of dimension $d(L)$ whenever it is a manifold. Thus $M_{c,\delta}(\mathcal{I})$ is also compact. In particular, when $d(L) = 0$, it is a finite union of arcs, and the cardinality $\#(M_{c,\delta}(g_t))$ (mod 2) is independent of t, and consequently an invariant of X (with its given* Spin$_c$ *structure).*

(iii) *If $b_2^+ = 1$, and $H^1(X, \mathbb{R}) = 0$, the space*

$$(Q_\delta^{-1}(0, 0) \cap (\mathcal{A} \times \{0\} \times \mathcal{I}))/\mathcal{G}$$

is just a single point (called a reducible solution), say $([A_0, 0], g_0)$. A neighbourhood of this point in $Q_\delta^{-1}(0, 0)/\mathcal{G}$ is homeomorphic to $\phi^{-1}(0, 0)/S^1$ where:

$$\phi : \operatorname{Ker} D_{A_0,0} \to \operatorname{Coker} D_{A_0,0}$$

is a smooth map which is (a) *S^1-equivariant, and* (b) *has 0 as a regular value in a small deleted neighbourhood $U - \{0\}$ of 0 in $\operatorname{Ker} D_{A_0,0}$. For metrics g_t such that $t \neq 0$ the moduli space $M_{c,\delta}(g_t)$ is a finite set of points if $d(L) = 0$.*

Proof. (i) follows immediately from Proposition 2.3.1. For (ii), note that $Q_\delta(A, \Phi, g_t) = (0, 0)$ implies that $D_{A,t}\Phi = 0$, and $\rho_t(F_A^{+,t}) = \sigma(\Phi, \Phi) + \delta$. This implies

$$\frac{1}{2\pi}(\rho_t^{-1}\sigma(\Phi, \Phi))_{\mathcal{H}_t} = \frac{1}{2\pi}(F_A^{+,t} - \rho_t^{-1}(\delta))_{\mathcal{H}_t} = \pi_t^+ \circ \psi_t(c(g_t, \delta)) = \sigma_{c,\delta}(g_t)$$

in the notation of 1.5.1. But $\mathcal{I} \cap B_{c,\delta} = \phi$, i.e. the section $\sigma_{c,\delta}$ is nonvanishing on $\mathcal{I}$ by our choice of $\mathcal{I}$ and δ, so $\sigma(\Phi, \Phi) \neq 0$, and so $\Phi \neq 0$. Thus there are no reducible solutions, and $Q_\delta^{-1}(0,0) \subset \mathcal{N}^* \times \mathcal{I}$. Hence $Q_\delta^{-1}(0,0)/\mathcal{G}$ is a smooth manifold, since the $\mathcal{G}$ action is free on $\mathcal{N}^* \times \mathcal{I}$, and since each fibre $M_{c,\delta}(g_t)$ (notation of § 4.5 of [PP]) is compact by § 6.2 of [PP]. Since $\mathcal{I}$ is compact, so is $M_{c,\delta}(\mathcal{I})$. In the case when $d(L) = 0$, $M_{c,\delta}(g_t)$ is then a finite set of points, and $M_{c,\delta}(g_a)$ and $M_{c,\delta}(g_b)$ are *cobordant*, and so have the same cardinality (modulo 2). In particular, $M_{c,\delta}(g_1)$ and $M_{c,\delta}(g_{-1})$ have the same cardinality

modulo 2. Thus we have (ii), since the rest of it follows from Proposition 2.3.1 and Lemma 5.1.6 of the Appendix.

To get (iii), note that when $b_2^+ = 1$, $Q_{2,\delta}(A, 0, g_t) = (0, 0)$ implies that $\sigma_{c,\delta}(g_t) = 0$. This happens (see Proposition 1.5.3) iff $f_\delta(g_t) = 0$. By our choice of $\mathcal{I}$ and δ, this happens only at $g_t = g_0$. Now choose a point $(A_0, 0, g_0) \in Q_\delta^{-1}(0, 0) \cap (\mathcal{A} \times \{0\} \times \mathcal{I})$. Then $Q_\delta(A_0 + a, 0, g_0) = (0, 0)$ implies $\rho_0(d_0^+ a) = d^+ a = 0$. Let $g \in \mathcal{G}$. The gauge action takes A_0 to $A_0 + g^*\omega$, where $\omega \in H^1(S^1, \mathbb{R})$ is the generating 'angle' 1-form of S^1. But $g^*\omega$ is thus a closed 1-form on X. Since $H^1(X, \mathbb{R}) = 0$ by assumption, $g^*\omega = d\alpha$ for some function $\alpha \in \Omega^0$. Conversely, given an $\alpha \in \Omega^0$, the map $g : X \to S^1$ defined by $g(x) = e^{2\pi i \alpha(x)}$ satisfies $g^*(\omega) = d\alpha$. Thus $(Q_\delta^{-1}(0, 0) \cap (\mathcal{A} \times \{0\} \times \mathcal{I}))/\mathcal{G} \simeq \operatorname{Ker} d^+/\operatorname{Im} d = H^1(X, \mathbb{R}) = 0$. So it consists of a single point $[(A_0, 0, g_0)]$.

Now we need to get a model for a neighbourhood of this point. It is well known that for a smooth action, the neighbourhood of a point in the orbit space corresponding to an orbit with isotropy G is homeomorphic to a neighbourhood in the orthogonal slice of that orbit divided by the isotropy G. A slice in $\Omega^1 \times \Gamma(W_+) \times \mathcal{I}$ orthogonal to the $\mathcal{G}$-orbit of $(A_0, 0, g_0)$, which we will take as $(0, 0, g_0)$ by setting the origin at A_0, is clearly $(\operatorname{Im} d)^\perp \times \Gamma(W_+) \times \mathbb{R}$ because the $\mathcal{G}$-orbit has tangent space $\operatorname{Im} d \times 0 \times 0$ at $(0, 0, g_0)$. Also $DQ_{\delta; A_0, 0, g_0}$ is an S^1-equivariant map, (because Q_δ is $\mathcal{G}$ equivariant), where S^1 is the isotropy group of the point $(A_0, 0, g_0)$, and of course, this S^1-action is orthogonal and linear (it is the derivative of the S^1-action on the space $\mathcal{A} \times \Gamma(W_+) \times \mathcal{C}$). By Lemma 2.2.1, we have

$$DQ_{\delta; A_0, 0, g_0}(a, \phi, 0) = (D_{A_0}\phi, d^+ a)$$
$$DQ_{\delta; A_0, 0, g_0}(0, 0, \lambda g'(0)) = (0, \lambda(2\pi f_\delta'(0)\omega_{g_0} + d^+\alpha)).$$

We now claim that:

$$DQ_{\delta; A_0, 0, g_0} : (\operatorname{Im} d)^\perp \times \Gamma(W_+) \times \mathbb{R} \to \Gamma(W_-) \times \Omega^{2+}$$

is a Fredholm operator, whose kernel is $\operatorname{Ker} D_{A_0} \subset \Gamma(W_+)$, and cokernel is $\operatorname{Coker} D_{A_0} \subset \Gamma(W_-)$.

From the above formulae it follows that

$$\begin{aligned}\operatorname{Ker}(DQ_{\delta; A_0, 0, g_0}) = \{(a, \phi, \lambda g'(0)) : a \perp \operatorname{Im} d,\ D_{A_0}\phi = 0,\ d^+ a \\ + \lambda(d^+\alpha + 2\pi f_\delta'(0)\omega_{g_0}) = 0\}.\end{aligned}$$

Now, $\omega_{g_0} \perp \operatorname{Im} d^+$, so on the right hand side we must have $\lambda 2\pi f_\delta'(0)\omega_{g_0} = 0$. But $f_\delta'(0) \neq 0$, by our choice of $\mathcal{I}$, δ and Proposition 1.5.3, so $\lambda = 0$. But this implies that $D_{A_0}\phi = 0$ and $d^+ a = 0$. Thus

$$\operatorname{Ker}(DQ_{\delta; A_0, 0, g_0}) = ((\operatorname{Im} d)^\perp \cap \operatorname{Ker} d^+) \times \operatorname{Ker} D_{A_0}.$$

Since $(\operatorname{Im} d)^\perp \cap \operatorname{Ker} d^+ \simeq \operatorname{Ker} d^+/\operatorname{Im} d \subset H^1(X, \mathbb{R}) = 0$, this is just $\operatorname{Ker} D_{A_0}$. It is finite dimensional by ellipticity of the Dirac operator.

To show that

$$DQ_{\delta; A_0, 0, g_0}((\operatorname{Im} d)^\perp \times \Gamma(W_+) \times \mathbb{R})$$

is closed, it is enough to show that $DQ_{\delta; A_0, 0, g_0}((\operatorname{Im} d)^\perp \times \Gamma(W_+) \times 0)$ is closed. But this is just $\operatorname{Im} D_{A_0} \times d^+(\operatorname{Im} d)^\perp$. Now $(\operatorname{Im} d)^\perp = \operatorname{Ker} d^* = \mathcal{H}^1 \oplus d^*\Omega^2$, so $d^+(\operatorname{Im} d)^\perp = d^+ d^*\Omega^2$. Since $\Omega^1 = \mathcal{H}^1 \oplus d^*\Omega^2 \oplus d\Omega^0$, $d^+\Omega^1 = d^+ d^*\Omega^2$ as well. Thus $d^+(\operatorname{Im} d)^\perp = d^+\Omega^1$. This is clearly closed by the decomposition of Lemma 1.3.2. Since D_{A_0}, the Dirac operator, is

elliptic, its range is closed too, so the range of

$$DQ_{\delta;A_0,0,g_0}:(\operatorname{Im} d)^{\perp}\times\Gamma(W_+)\times\mathbb{R}\to\Gamma(W_-)\times\Omega^{2+}$$

is closed. Its cokernel is

$$\{(\psi,\tau):\psi\perp\operatorname{Im}D_{A_0};\tau\perp d^+a+\lambda(d^+\alpha+2\pi f_\delta'(0)\omega_0\,\forall\,\lambda\in\mathbb{R},\ a\in(\operatorname{Im} d)^{\perp}\}.$$

This implies $\psi\in\operatorname{Coker}D_{A_0}$. Also, setting $\lambda=0$, one finds that $\tau\perp d^+((\operatorname{Im} d)^{\perp})$, i.e. $\tau\perp d^+\Omega^1$. Since $\tau\in\Omega^{2+}$, the Lemma 1.3.2 implies that $\tau\in\mathcal{H}^{2+}$. However, by setting $a=0$ and $\lambda=1$, τ is also orthogonal to $2\pi f_\delta'(0)\omega_0+d^+\alpha$, and since we already have $\tau\perp d^+\Omega^1$, it follows that $\tau\perp\omega_0$, since $f_\delta'(0)\neq 0$ by our choice of $\mathcal{I}$ (from 1.5.3). We now note that ω_0 was chosen as the basis element of $\mathcal{H}^{2+}$, so $\tau=0$. Thus the cokernel of our map is just $\operatorname{Coker}D_{A_0}$. This proves our Fredholm-ness assertion, and the identifications of kernel and cokernel.

Thus, Proposition 5.1.5 applies, and a neighbourhood of $(A,0,g_0)$ in $Q_\delta^{-1}(0,0)/\mathcal{G}$ is homeomorphic to a neighbourhood of 0 in $\phi^{-1}(0)/S^1$, where the S^1-equivariant map:

$$\phi:\operatorname{Ker}D_{A_0}\to\operatorname{Coker}D_{A_0}$$

has 0 as a regular value when restricted to $\phi^{-1}(0)-\{0\}$. The last statement follows from the fact that if $t\neq 0$ then $f_\delta(g_t)\neq 0$, and $\mathcal{M}_\delta^*(g_t)=\mathcal{M}_\delta(g_t)$ is a Banach manifold (by part (i)), since there are no reducible solutions, and its quotient by $\mathcal{G}$ has dimension $d(L)=0$ by part (ii) above. This proves (iii), and the proposition. □

3. Computations

3.1 *The case of* $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$

Let $X=\mathbb{C}\mathrm{P}^2\#n\overline{\mathbb{C}\mathrm{P}^2}$, which is just the blow-up of $\mathbb{C}\mathrm{P}^2$ at n points. The case $n=0$ is just $\mathbb{C}\mathrm{P}^2$, which is a special case of the ensuing discussion. Since X is a complex manifold, we have a canonical Spin$_c$ structure on X (as in § 7.1 of [PP]), with $L=K_X^{-1}$, $W_+=K_X^{-1}\oplus 1_{\mathbb{C}}$, $W_-=T_{\mathrm{hol}}(X)$. Also

(i) $H^1(X,\mathbb{R})=0$, and

(ii) $H^2(X,\mathbb{R})=\mathbb{R}H\oplus\sum_{i=1}^n\mathbb{R}E_i$, where H is the pullback of the hyperplane class in $\mathbb{C}\mathrm{P}^2$ via the blow up map π and will, from here on, be called the hyperplane class by abuse of language. E_i is the generating $\overline{\mathbb{C}\mathrm{P}}^1$ in the ith copy of $\overline{\mathbb{C}\mathrm{P}}^2$. The cup products between these classes are:

$$H\cup H=1,\quad E_i\cup E_j=-\delta_{ij},\quad H\cup E_i=0\,\forall\,1\le i,j\le n.$$

Thus $b_2^+=1$, $b_2^-=n$ and the cup pairing is of type $(1,n)$.

(iii) From the formula $K_X=\pi^*(K_{\mathbb{C}\mathrm{P}^2})\otimes[E]$, where $E:=\sum_{i=1}^n E_i$ is the exceptional divisor, and that $c_1(K_{\mathbb{C}\mathrm{P}^2}^{-1})=c_1(\mathbb{C}\mathrm{P}^2)=3H_{\mathbb{C}\mathrm{P}^2}$, it follows that $c_1(L)=c_1(K_X^{-1})=3H-E$. Thus $c_1(L)^2=9-n$, and this is <0 whenever $n>9$. Since $\chi(X)=n+3$, and $\sigma(X)=1-n$, we have $d(L)=1/4(c_1(L)^2-3\sigma-2\chi)=0$ (see Proposition 2.3.1 for the definition of $d(L)$).

PROPOSITION 3.1.1

For $X=\mathbb{C}\mathrm{P}^2\#n\overline{\mathbb{C}\mathrm{P}^2}$, *the Seiberg–Witten moduli space* $M_{c,\delta}(g_t)$ *consists of finitely many points whenever* $f_\delta(g_t)\neq 0$ (*i.e. when* $t\neq 0$) *where* f_δ *is the function defined in* 1.5.3.

Proof. Follows immediately from (iii) of Proposition 2.3.3. □

Of course, this proposition does not tell us what the cardinality (mod 2) of the moduli space $M_{c,\delta}(g_t)$ might be. To show that there exist metrics for which this cardinality is non-zero will be our next goal.

PROPOSITION 3.1.2

Let $X = \mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$, as above, with $n > 9$. Let $\mathcal{I}$ be an arc in $\mathcal{C}$ chosen in accordance with the Proposition 2.3.3 (i.e. $(0,0)$ is a regular value of $Q_{\delta|\mathcal{N}^*\times\mathcal{I}}$, and the function $f_\delta : \mathcal{I} \to \mathbb{R}$ satisfies the conclusion of Proposition 1.5.3). Then

$$\#M_{c,\delta}(g_{-1}) - \#M_{c,\delta}(g_1) = 1 \pmod 2.$$

Proof. By (iii) of Proposition 2.3.3, a neighbourhood of $(A_0, 0, g_0)$, the unique reducible point in $M_{c,\delta}(\mathcal{I})$ is homeomorphic to a neighbourhood of 0 in $\phi^{-1}(0)/S^1$, where $\phi : \operatorname{Ker} D_{A_0} \to \operatorname{Coker} D_{A_0}$ is a smooth S^1-equivariant map, and ϕ has 0 as a regular value in a deleted neighbourhood of 0 in $\operatorname{Ker} D_{A_0}$. We need to show that an odd number of arcs emerge from 0 in this neighbourhood.

Now

$$\begin{aligned}\mathrm{index}_{\mathbb{C}}(D_{A_0}) &= \tfrac{1}{2}\mathrm{index}_{\mathbb{R}}(D_{A_0}) = \tfrac{1}{8}(c_1(L)^2 - \sigma(X))\\ &= \tfrac{1}{8}(9 - n - 1 + n) = 1.\end{aligned}$$

Thus if $\dim_{\mathbb{C}}(\operatorname{Coker} D_{A_0}) = r$, then $\dim_{\mathbb{C}}(\ker D_{A_0}) = r+1$. So our ϕ is an S^1-equivariant smooth map (with S^1 acting as scalar multiplication on both sides) from $\mathbb{C}^{r+1}$ to $\mathbb{C}^r$, with 0 a regular value for $\phi_{|\mathbb{C}^{r+1}-0}$. Let $\mathcal{O}(-1)$ denote the tautological bundle on $\mathbb{C}\mathrm{P}^r$, and let

$$\pi : (\mathbb{C}^{r+1} - 0)/S^1 \simeq \mathbb{C}\mathrm{P}^r \times \mathbb{R}_+ \to \mathbb{C}\mathrm{P}^r$$

denote the projection into the first factor. If we then denote by $\langle z_0, \ldots, z_r\rangle$ the S^1-equivalence class of $(z_0, \ldots, z_r)$ in the orbit space $(\mathbb{C}^{r+1} - 0)/S^1$, we get a natural section s of the bundle $\pi^* \hom(\mathcal{O}(-1), \mathbb{C}^r) \simeq \pi^*(\mathbb{C}^r \otimes \mathcal{O}(1))$ by setting

$$s(\langle z_0, \ldots, z_r\rangle)(u) = \phi(z_0, \ldots, z_r),$$

where u is the unit vector $(z_0, \ldots, z_r)/\|(z_0, \ldots, z_r)\|$. The S^1-equivariance of ϕ implies that this map s is well defined, and that 0 is a regular value for ϕ on a deleted neighbourhood means that s is transverse to the zero-section. Now, $s^{-1}(0)$ is clearly $\phi^{-1}(0,0) - \{0\}/S^1$. Thus, modulo 2, the number of arcs going to 0 is precisely the number of points in $s^{-1}(0) \cap (\mathbb{C}\mathrm{P}^r \times \{\epsilon\})$ modulo 2, for generic ϵ. But this is just the Euler number of the bundle $\mathbb{C}^r \otimes \mathcal{O}(1)$, which is 1. Thus

$$\#M_{c,\delta}(g_{-1}) - \#M_{c,\delta}(g_1) = 1 \pmod 2$$

and we are done. □

PROPOSITION 3.1.3 (Hitchin)

There exist metrics g on $X = \mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$, such that

(i) *g is Kähler.*
(ii) *$[\omega_g] \cup c_1(L)$ has the same sign as the scalar curvature s_g, which is positive.*

(iii) *For any metric g' conformally equivalent to a Kähler metric g satisfying* (i) *and* (ii) *above,* $[\omega_{g'}] \cup c_1(L) > 0$.

Proof. See reference number [4] in [KM] for the proof of (i) and (ii). For (iii), note that $[\omega_{g'}] = [\omega_g]$ when g and g' are conformally equivalent, by (iii) of 1.3.4. $\square$

Clearly, for a metric as in Proposition 3.1.3 above, the moduli space $M_{c,\delta}(g)$ is empty, by § 6.2 of [PP]. Consequently, we have the following corollary to Proposition 3.1.2.

COROLLARY 3.1.4

If g is a metric on $X = \mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}^2}$ such that $c_1(L) \cup [\omega_g] < 0$, then the moduli space $M_{c,\delta}(g) \neq \phi$.

Proof. Assuming there is such a metric, join it by an arc $\mathcal{I}$ in C meeting B_c transversely, at one point, and apply Propositions 1.5.3, 3.1.2. $\square$

3.2 *The tubing construction*

Let X be a compact, connected oriented Riemannian 4-manifold, and let Y be a compact 3-manifold, also oriented (so that its normal bundle in X is trivial) such that

$$X - Y = X_+ \cup X_-$$

as two disjoint components. Assume further that there exists a Riemannian metric g on X such that $g_{|\nu(\epsilon)} = \mathrm{d}t^2 \times g_Y$, where $\nu(\epsilon)$ is an ϵ-tubular neighbourhood of Y in X, and g_Y is a smooth Riemannian metric on Y. Let us denote $g_{|X_\pm} := g_\pm$.

DEFINITION 3.2.1

Define the *R-tubing of* X, denoted (X_R, g_R) to be the manifold.

$$X_R = (X_- \cup ([-R, 0] \times Y)) \cup (([0, R] \times Y) \cup X_+)$$

with the metric g_R being defined by $g_{R|X_\pm} = g_\pm$ and $g_{|[-R,R]\times Y} = \mathrm{d}t^2 \times g_Y$ on the piece $[-R, R] \times Y$. Note the ends $\{\pm R\} \times Y$ are identified with $\partial X_\pm$.

Clearly, since X_R is diffeomorphic to X, one may view g_R as a new metric on X. Of course, its volume form $\mathrm{d}V_{g_R}$ will change, but by a global conformal change one may restore the old volume form. Note that this conformal change does not affect ω_{g_R}. Now let us go back to $X = \mathbb{C}\mathrm{P}^2 \# d^2 \overline{\mathbb{C}\mathrm{P}^2}$, where $d > 3$ (so that $n = d^2 > 9$ and $c_1(L)^2 = 9 - n < 0$ from the opening discussion of this section).

Let Σ be an oriented smoothly embedded surface in $\mathbb{C}\mathrm{P}^2$ whose homology class is Poincare dual to $\mathrm{d}H$ (i.e., it is of degree d) in $H^2(\mathbb{C}\mathrm{P}^2)$. Thus the cap (or Kronecker) product $H.[\Sigma] = d$.

Let S_i be a sphere $\mathbb{C}\mathrm{P}^1$ in the ith copy of $\overline{\mathbb{C}\mathrm{P}^2}$ which is dual to $(-E_i)$, so that $E_j.S_i = -\delta_{ij}$. Let us define the internal connected sum $\tilde{\Sigma} = \Sigma \# S_1 \# S_2 \ldots \# S_{d^2}$. Clearly the genus $g(\tilde{\Sigma})$ is the same as that of Σ, i.e. $g(\Sigma)$, since Σ and $\tilde{\Sigma}$ are homeomorphic. Also $[\tilde{\Sigma}]$ is the homology class Poincare dual to $\mathrm{d}H - E$.

Lemma 3.2.2. *$\tilde{\Sigma}$ has trivial normal bundle in X.*

Proof. Let $\nu = \nu(\tilde{\Sigma})$ be the normal bundle of $\tilde{\Sigma}$ in X. Then we know that the Euler number of this bundle is precisely the self-intersection number of $\tilde{\Sigma}$ in X, i.e.

$(dH - E) \cdot (dH - E) = d^2H^2 + E^2 = d^2 + \sum E_i^2 = d^2 - d^2 = 0$. Thus $\nu(\tilde{\Sigma})$ has a nowhere vanishing section, i.e. has a trivial line bundle as a summand. Since it is orientable, it follows that it is trivial. $\square$

Thus if $\nu_\epsilon(\tilde{\Sigma})$ is an ϵ-tubular neighbourhood of $\tilde{\Sigma}$, we have $\nu_\epsilon(\tilde{\Sigma}) \simeq D_\epsilon \times \tilde{\Sigma}$, and its boundary $\partial\nu_\epsilon(\tilde{\Sigma}) \simeq S^1 \times \tilde{\Sigma}$. Call this last space Y. Now apply the tubing construction of 3.2.1 to X, Y, with $X_- = X - \nu_\epsilon(\tilde{\Sigma})$, $X_+ = \nu_\epsilon(\tilde{\Sigma})$, $Y = \partial\nu_\epsilon(\tilde{\Sigma})$. Note that an ϵ-neighborhood of Y is diffeomorphic to $(-\epsilon, \epsilon) \times Y$.

PROPOSITION 3.2.3

Assume there exists a metric g_0 on X such that $g_{0|\nu_\epsilon(Y)}$ is a product metric, and let $(X(R), g_R)$ denote the R-tubing of g as defined above. Then, for R sufficiently large, $c_1(L) \cup [\omega_{g_R}] < 0$.

Proof. For notational ease we shall denote ω_{g_R} by ω_R. Let R_i be a sequence of positive numbers tending to ∞. We know by Proposition 1.3.4 that in $H^2(X, \mathbb{R}) = \mathcal{H}$, $[\omega_{R_i}] \cup [\omega_{R_i}] = 1$, and $[\omega_{R_i}] \in C_+$ where C_+ is the preferred component of the positive cone of the cup-product form, which contains the class H defined above. So $[\omega_{R_i}] \cup H > 0$ for all i. Let us normalise and define ω_i to be the unique Δ_0 harmonic 2-form representing $([\omega_{R_i}] \cup H)^{-1} [\omega_{R_i}]$. It is clearly enough to show that $c_1(L) \cup \omega_i < 0$ for i large enough. Since $\omega_i \cup H = 1$ for all i, and hence ω_i lie in the bounded region $C_+ \cap \{\beta : \beta \cup H = 1\}$, so have a (with respect to the global inner product $\langle\, ,\rangle_{g_0}$) L^2-convergent subsequence, which we continue to call ω_i. Since ω_i are Δ_{g_0}-harmonic, the Garding inequality says that they converge in all Sobolev L_k^2-norms (with respect to the g_0-metric on X), and hence uniformly on compact subsets $K \subset X_+ \cup (0, \epsilon] \times Y \subset X$, in particular. Now, there is a diffeomorphism $\phi_i : X(R_i) \to X$, which takes the piece $X_+ \cup (0, R_i] \times Y$ to the piece $X_+ \cup (0, \epsilon] \times Y$, taking the metric $g_{R_i} := g_i$ on $X(R_i)$ to the metric g_i on X. Note that ϕ_i are identity on X_+, and just a scaling of ϵ/R_i of the t variable on $(0, R_i] \times Y$ and identity on the Y variable. So, on the piece $(0, \epsilon] \times Y$, the 1-form ds has length ϵ/R_i with respect to g_i, but length 1 with respect to g_0. The 1-forms on Y have the same length with respect to g_i and g_0. It is now more convenient to change the variable t to $R - t$, which replaces $X_+ \cup (0, R] \times Y$ with the isometric manifold $X_+ \cup [0, R) \times Y$, where ∂X_+ is glued to $\{0\} \times Y$, and similarly s to $\epsilon - s$ taking $(0, \epsilon]$ to $[0, \epsilon)$. Let us define $X_{i+} = X_+ \cup [0, R_i) \times Y$, and ϕ_i is the composite diffeomorphism $\phi_i : X_{i+} \to X_+ \cup [0, \epsilon) \times Y$. Also, since ϕ_i and ϕ_j are both identity on Y, and $\phi_i^* ds = \epsilon/R_i dt$, $\phi_j^* ds = (\epsilon/R_j)\, dt$ for $j \geq i$, where $s \in [0, \epsilon)$, it follows that for $j \geq i$ we have, for any i-form ω on $[0, \epsilon) \times Y$, the inequalities:

$$\|\phi_i^*\omega\|_{g_i, X_{i+}} \leq \|\omega\|_{g_0, X_+ \cup [0,\epsilon) \times Y}$$

$$\|\phi_i^*\omega - \phi_j^*\omega\|_{g_j, X_{j+}} \leq \epsilon\left(\frac{1}{R_i} - \frac{1}{R_j}\right)\|\omega\|_{g_0, X_+ \cup [0,\epsilon) \times Y}. \tag{3}$$

Note that all the X_{i+} are isometrically embedded in the non-compact manifold with infinite end $X_\infty := X_+ \cup [0, \infty) \times Y$, where $\{0\} \times Y$ is glued to ∂X_+, with the complete metric defined by g_+ on X_+ and $dt^2 \times g_Y$ on the infinite tube $[0, \infty) \times Y$ (call this metric g_∞), so that $g_{\infty|X_i} = g_i$. Extending $\phi_i^*\omega_i$ on X_{i+} by 0 to all of X_∞ defines an $L^2(g_\infty)$ form on X_∞, which we continue to denote by the same symbol. Now let $\tilde{\omega}_i$ be the Δ_{g_∞}-harmonic part of $\phi_i^*\omega_i$. This is possible in view of the Kodaira decomposition:

$$L^2 = \mathcal{H}_\infty^2 \oplus \overline{d\Lambda_c} \oplus \overline{\delta\Lambda_c}$$

which is always true for a *complete* Riemannian metric (see [Ko]). (For such a complete metric g, $\operatorname{Ker}\Delta_g$ is the same as the space of closed and co-closed forms.) Now let K be a compact subset of X_{i+}, and hence X_∞. Since for $j \geq i$, we have $X_{i+} \subset X_{j+} \subset X_\infty$ and $g_j = g_i$ on X_{i+}, we have the inequalities of sup (C^0) norms:

$$\begin{aligned}
\|\tilde{\omega}_i - \tilde{\omega}_j\|_{\infty,K,g_\infty} &\leq \|\phi_i^*\omega_i - \phi_j^*\omega_j\|_{\infty,K,g_\infty} \\
&= \|\phi_i^*\omega_i - \phi_j^*\omega_j\|_{\infty,K,g_j} \\
&\leq \|\phi_i^*\omega_i - \phi_j^*\omega_i\|_{\infty,K,g_j} + \|\phi_j^*\omega_i - \phi_j^*\omega_j\|_{\infty,K,g_j} \\
&\leq \epsilon\left(\frac{1}{R_i} - \frac{1}{R_j}\right)\|\omega_i\|_{\infty,\phi_j(K),g_0} + \|\omega_i - \omega_j\|_{\infty,\phi_j(K),g_j} \\
&\leq \epsilon\left(\frac{1}{R_i} - \frac{1}{R_j}\right)\|\omega_i\|_{\infty,\phi_j(K),g_0} + \|\omega_i - \omega_j\|_{\infty,\phi_j(K),g_0}
\end{aligned}$$

by using the two inequalities (3) above. This shows that the Δ_{g_∞}-harmonic forms $\tilde{\omega}_i$ are uniformly Cauchy on compact subsets of X_∞, and hence converge uniformly on compact sets to some Δ_{g_∞}-harmonic form $\tilde{\omega}$. Also,

$$\begin{aligned}
\|\tilde{\omega}\|_{g_\infty,X_\infty} &= \lim_{i\to\infty} \|\tilde{\omega}_i\|_{g_i,X_i} \\
&\leq \lim_{i\to\infty} \|\phi_i^*\omega_i\|_{g_i,X_i} \\
&\leq \lim_{i\to\infty} \|\omega_i\|_{g_0,X_+\cup[0,\epsilon)\times Y} < \infty
\end{aligned}$$

shows that $\tilde{\omega}$ is an L^2 2-form with respect to g_∞ on X_∞, so it is in $\mathcal{H}^2_\infty$. However, the Kodaira decomposition above shows that this space is precisely the image of $H^2_c(X_\infty)$ in $H^2(X_\infty)$. However, from the definition of X_+, this is precisely the image of $H^2(X_+, \partial X_+)$ in $H^2(X_+)$, which is the kernel of the restriction map $H^2(\tilde{\Sigma} \times D^2) \to H^2(\tilde{\Sigma} \times S^1)$, which is zero (e.g. by Kunneth formula). Thus $\tilde{\omega} = 0$.

Now, since $\nu_\epsilon(\tilde{\Sigma})$ is a trivial bundle, we may find a copy of $\tilde{\Sigma}$, call it $\tilde{\Sigma}_1$ which lies in $\partial X_+ = \partial\nu_\epsilon(\tilde{\Sigma})$, and is homologous to $\tilde{\Sigma}$ in X. Note that $\tilde{\Sigma}_1$ therefore lies in X_{i+} for all i, and in X_∞, satisfying $\phi_i^*(\tilde{\Sigma}_1) = \tilde{\Sigma}_1$ for all i. Now,

$$\begin{aligned}
\lim_{i\to\infty}(\omega_i \cup (\mathrm{d}H - E)) &= \lim_{i\to\infty}\int_{\tilde{\Sigma}} \omega_i = \lim_{i\to\infty}\int_{\tilde{\Sigma}_1} \omega_i \\
&= \lim_{i\to\infty}\int_{\tilde{\Sigma}_1} \phi_i^*\omega_i = \lim_{i\to\infty}\int_{\tilde{\Sigma}_1} \tilde{\omega}_i \\
&= \int_{\tilde{\Sigma}_1} \tilde{\omega} = 0.
\end{aligned}$$

Consequently, since $c_1(L) = 3H - E$, we have

$$\begin{aligned}
\lim_{i\to\infty}(\omega_i \cup c_1(L)) &= \lim_{i\to\infty} \omega_i \cup (dH - E) - (d-3)(\omega_i \cup H) \\
&= 0 - (d-3) < 0
\end{aligned}$$

since $d > 3$ by assumption. This proves the proposition. □

COROLLARY 3.2.4

There exists a metric g on X, which is a product in a tubular neighbourhood of Y, such that $\#M_{c,\delta}(g_R) \neq 0$ for R large enough.

Proof. Since Y has a product neighbourhood, one can put a product metric on an ϵ-neighbourhood of Y, and extend it to all of X by using a partition of unity. The rest follows from the Corollary 3.1.4 and the proposition above. $\square$

3.3 *Temporal gauge solutions*

We go back to the setting of the tubing construction of Def. 3.2.1, and do some analysis on the tube portion $[-R,R]\times Y$. Look at the restrictions of the $U(2)$ bundles $W_\pm$ coming from the chosen Spin$_c$ structure on X to $[-R,R]\times Y$, viz. $W_{\pm|[-R,R]\times Y}$. These are both isomorphic via $c(\mathrm{d}t)$, Clifford multiplication by the (unit length) 1-form $\mathrm{d}t$, and hence may both be regarded as pullbacks via the projection map $[-R,R]\times Y \to Y$ of a $U(2)$-bundle W_3 on Y. If A is a connection on $[-R,R]\times Y$, then $A_{|\{t\}\times Y} := A(t)$ is a connection for $L_{|Y}$ for each t, and maybe regarded as a path in $\mathcal{A}_Y$, the affine space of $U(1)$ connections on $L_{|Y}$. Similarly, if $\Phi\in\Gamma(W_+)$, one may regard the restriction to the t-slice $\Phi(t)$ as a path of sections of $\Gamma(W_3)$. Note that the metric on $[-R,R]\times Y$ is a product, and hence induces the same metric on each slice $\{t\}\times Y$.

DEFINITION 3.3.1

We say a connection A on $\mathbb{R}\times Y$ is in *temporal gauge* if it has no $\mathrm{d}t$ component. (We are of course fixing a reference connection A_0, as always). Say it is *translation invariant in a temporal gauge* if it is the pullback of a connection on $L_{|Y}$. Similar definitions make sense for $[0,R)\times Y$, and $[-R,R]\times Y$. Finally, a solution (A,Φ) to the Seiberg–Witten equations on $(X_R = X,\ g_R)$ will be called a *temporal gauge solution* if A is in temporal gauge on the tube $T=[-R,R]\times Y$, and similarly for translation invariant temporal gauge solutions.

Clearly, if a connection A on $L_{|\mathbb{R}\times Y}$ is in temporal gauge, it can be recovered from the path $A(t)$, since it has no $\mathrm{d}t$ component.

Remark 3.3.2. If $A=A_R$ is any connection on X_R (see 3.2.1), there exists a gauge transformation $g(R)$ in the connected component of id_X in the gauge group $\mathcal{G}$ such that $g(R)A$ is in temporal gauge on $[-R,R]\times Y$. Further, if $A_{R|X_+}$ is a fixed connection A_+ on $L_{|X_+}$ independent of R, we may choose $g(R)$ to be the identity map id_{X_+} on the piece X_+ for all R. Similarly for X_-.

Proof of remark. Let $A_0(y,t)\mathrm{d}t$ be the $\mathrm{d}t$ component of A on $[-R,R]\times Y$. There is a C^∞ function $h_R(y,t)$ on $[-R,R]\times Y$ which satisfies: $A_0(t,y)=\partial h_R(t,y)/\partial t$. Choose a C^∞-function f extending h_R to $X=X_R$. If $A_R(R,y)=A_+(R,y)$ is independent of R, we may choose $h_R(t,y)$ to be such that $h(R,y)=0$ for all R, and choose f to be identically 0 on X_+ for all R. Now take the gauge transformation $g(R)=\exp(2\pi i)f(t,y)$. Now, since

$$g(R)^{-1}\mathrm{d}g(R) = 2\pi i\left(\frac{\partial h_R(t,y)}{\partial t}\right)\mathrm{d}t = 2\pi i A_0(y,t)\mathrm{d}t$$

on $[-R,R]\times Y$, it follows that $gA = A-(1/2\pi i)g^{-1}\mathrm{d}g$ has no $\mathrm{d}t$ component on $[-R,R]\times Y$. $\square$

Let $\partial_{A(t)}:\Gamma(W_3)\to\Gamma(W_3)$ be the induced Dirac operator on W_3, with respect to the connection $A(t)$. In view of Remark 3.3.2 above, any solution (A,Φ) to the Seiberg–Witten equations on (X_R,g_R) can be assumed to be a temporal gauge representative in its gauge equivalence class. Our first goal is to consider the restriction of such a temporal

gauge solution to the tube $T = [-R,R] \times Y$, and view it as a time-dependent solution $(A(t), \Phi(t))$ of some equations on Y involving $\partial_{A(t)}$ etc.

Lemma 3.3.3. *Let us denote by $*_Y$ the star operator on Y defined by the metric g_Y (induced from g_R on X as in* 3.2.1), *and let a 2-form ω on X be expressed as $\omega = \mathrm{d}t \wedge \phi + \psi$ on the tube $T = [-R,R] \times Y$ where ϕ, ψ are devoid of* dt. *Then*

$$*\omega = *_Y\phi + \mathrm{d}t \wedge *_Y\psi,$$

where $*$ is the star operator of g_R on X.

Proof. Is a straightforward exercise, since $g_R = \mathrm{d}t^2 \times g_Y$ and so $\mathrm{d}V_{g_R} = \mathrm{d}t\,\mathrm{d}V_Y$, and the star operator is characterised by the diagram in the subsection 1.3 (self-duality) of § 1. □

COROLLARY 3.3.4

Let A be a connection on $X = X_R$, and let F_A be its curvature. Let T denote the tube $[-R,R] \times Y$, with the product metric $g_R = \mathrm{d}t^2 \times g_Y$. Then, if A is translation invariant in a temporal gauge, we have the equality of pointwise norms

$$(F_A^+(y), F_A^+(y))_{y,g_R} = (F_A^-(y), F_A^-(y))_{y,g_R}.$$

Proof. Write $F_A = \mathrm{d}t \wedge (\mathrm{d}A/\mathrm{d}t) + F_{A,Y}$. By the Lemma above,

$$F_A^+ = \frac{1}{2}(F_A + *F_A) = \frac{1}{2}\left[\mathrm{d}t \wedge \left(\frac{\mathrm{d}A}{\mathrm{d}t} + *_Y F_{A,Y}\right) + \left(F_{A,Y} + *_Y \frac{\mathrm{d}A}{\mathrm{d}t}\right)\right],$$
$$F_A^- = \frac{1}{2}(F_A + *F_A) = \frac{1}{2}\left[\mathrm{d}t \wedge \left(\frac{\mathrm{d}A}{\mathrm{d}t} - *_Y F_{A,Y}\right) + \left(F_{A,Y} - *_Y \frac{\mathrm{d}A}{\mathrm{d}t}\right)\right].$$

Thus, since $\mathrm{d}A/\mathrm{d}t = 0$ for A translation-invariant and in temporal gauge, we have the result. □

The proof above also shows that the spaces of $*$-self dual and antiself dual 2-forms on T which are t-translation invariant are both isomorphic to $\Omega^1(Y)$. The self-dual one is given as $1/2(\mathrm{d}t \wedge *_Y\omega(y) + \omega(y))$ and the anti-selfdual one as $1/2(-\mathrm{d}t \wedge *_Y\omega(y) + \omega(y))$.

For notational convenience, we denote the isomorphism $\omega \mapsto 1/2(-\mathrm{d}t \wedge *_Y\omega + \omega)$ by $\theta : \Omega^1(Y) \to \Omega^{2+}_{\mathrm{inv}}(T)$, where the right hand side is the space of translation invariant self-dual forms on T.

The Clifford structure map $\gamma : \Lambda^1(T) \otimes \mathbb{C} \to \mathrm{Hom}(W_+, W_-)$ restricts to the Clifford isometry $\tilde{\gamma} : \Lambda^1(Y) \otimes \mathbb{C} \to \mathrm{End}^0(W_3)$. It is easily checked that $\rho \circ \theta = \tilde{\gamma}$. Finally, we recall the pairing σ defined by $\sigma(\Phi, \Psi) = i(h_+(-, \Psi)\Phi - (1/2)h_+(\Phi, \Psi)\mathrm{Id})$ over the tube T:

$$\sigma : W_+ \otimes \overline{W}_+ \to \mathrm{End}^0(W_+) \simeq \Lambda^{2+} \otimes \mathbb{C}.$$

Since $W_+ = \pi^*(W_3)$, we have the pairing:

$$\tau : W_3 \times \overline{W}_3 \to \Lambda^1(Y) \otimes \mathbb{C},$$

where $\tilde{\gamma} \circ \tau = \sigma$. So, if we regard a section $\Phi \in \Gamma(W_{+|T})$ as a section in $\Gamma(\pi^* W_3)$, which is the same as a path of sections $\Phi(t)$ in $\Gamma(W_3)$, then by definition,

$$\sigma(\Phi(t,y), \Phi(t,y)) = \tilde{\gamma}(\tau(\Phi(t)(y), \Phi(t)(y))).$$

Now we look at the Dirac operator on T. The covariant derivative with respect to A and compatible with the Levi-Civita connection of g_R is given by

$$\nabla_{A,\mathrm{d}t} = i(A \vdash \mathrm{d}t) \otimes (-) + \frac{\partial}{\partial t},$$

where $\vdash$ denotes contraction. Since A is in temporal gauge, $A \vdash \mathrm{d}t = 0$, and so $\nabla_{A,\mathrm{d}t} = \partial/\partial t$. Of course, viewing $\Phi(t,-)$ as a path of sections $\Phi(t)$ of W_3, we have

$$\frac{\partial\Phi(t,-)}{\partial t} = \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}.$$

Also, the Clifford multiplication $c(\mathrm{d}t)$ is what makes W_+ isomorphic to $W_- \simeq \pi^*(W_3)$. So the Dirac operator for the bundle W on the tube T with induced Spin_c structure on T reads as

$$D_A = \frac{\mathrm{d}}{\mathrm{d}t} + \partial_{A(t),Y},$$

where $\partial_{A(t),Y}$ is the Dirac operator for the induced Spin_c structure on Y. It follows that time-dependent Seiberg–Witten equations read on Y as (assuming, as usual, that A is in temporal gauge):

$$\left(\frac{\mathrm{d}A}{\mathrm{d}t} + *_Y F_{A,Y}\right) - \tau(\Phi,\Phi) - *_Y\frac{\delta}{2} = 0$$
$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -\partial_{A(t),Y}\Phi(t), \tag{4}$$

where $\delta \in \Omega^2(Y)$. Hence we have the following:

PROPOSITION 3.3.5

If (A,Φ) is a temporal gauge solution to the Seiberg–Witten equations on the tube $T = \mathbb{R} \times Y$ or $[-R,R] \times Y$, then $(A(t),\Phi(t))$ is a path in $\mathcal{A}_Y \times \Gamma(W_3)$ which is a trajectory of eq. (4).

We now note that the eq. (4) above are the gradient-flow equations of a functional defined on $\mathcal{A}_Y \times \Gamma(W_3)$.

PROPOSITION 3.3.6

The equations (4) *are the gradient flow equations for the functional defined on $\mathcal{A}_Y \times \Gamma(W_3)$ by*

$$C_\delta(A,\Phi) = -\frac{1}{2}\left(\int_Y (A-B)\wedge(F_A-\delta) + \int_Y (\Phi,\partial_A\Phi)_{W_{3,y}}\mathrm{d}V_Y\right),$$

where B is a reference connection on $L_{|Y}$ and the integrand on the extreme right is the inner product on the fibre $W_{3,y}$, and δ is a 2-form on Y. Hence, $(A(t),\Phi(t))$ satisfying (4) *is the gradient flow for this functional, and C_δ is monotonically increasing along this trajectory.*

Proof. For simplicity, denote $A-B$ as A, where B is the reference connection. Let $\{e_i\}_{i=1}^2$ be a $(-,-)_{W_3}$ unitary frame for W_3. We recall that $\sigma(\Phi,\Psi) = i(h_+(-,\Psi)\Phi - (1/2)\,h_+(\Phi,\Psi)\mathrm{Id})$. The foregoing definitions lead to the following identity (since $\tilde{\gamma}$ maps into

traceless endos):

$$\begin{aligned}\langle\tau(\Phi,\Phi),\omega\rangle &= \langle\sigma(\Phi,\Phi),\tilde{\gamma}(\omega)\rangle = \frac{1}{2}\mathrm{Tr}((-,\Phi)\Phi\circ\tilde{\gamma}(\omega)^t)\\ &= \frac{i}{2}\sum_{j=1}^{2}((\tilde{\gamma}(\omega)^t e_j,\Phi)\Phi,e_j) = \frac{i}{2}(\Phi,\tilde{\gamma}(\omega)\Phi),\end{aligned}$$

where the round brackets denote the hermitian inner product $(-,-)_{W_{3,y}}$ on the fibre $W_{3,y}$ and the angular brackets denote the Riemannian inner product on Y. In view of the calculation above, and that $\partial_A = \sum_{j=1}^{3}\tilde{\gamma}(\omega_j)\nabla_{A,\omega_j}$ the inner product $(\Phi,\ \partial_A\Phi)_{W_3}$ satisfies:

$$\begin{aligned}(\Phi,\partial_A\Phi)_{W_3} - (\Phi,\partial_B\Phi)_{W_3} &= -i\sum_{j=i}^{3}(\Phi,\langle A,\omega_j\rangle\tilde{\gamma}(\omega_j)\Phi)_{W_3}\\ &= -i\sum_{j=1}^{3}\langle A,\omega_j\rangle(\Phi,\tilde{\gamma}(\omega_j))_{W_3} = -2\langle\tau(\Phi,\Phi),A\rangle,\end{aligned}$$

where $\{\omega_j\}_{j=1}^3$ is a local orthonormal frame for $\Omega^1(Y)$.

Thus the integrand of C_δ becomes

$$Q(A,\Phi) = -[\tfrac{1}{2}\langle A,*_Y(F_A-\delta)\rangle + \tfrac{1}{2}(\Phi,\partial_B\Phi)_{W_{3,y}} - \langle\tau(\Phi,\Phi),A\rangle].$$

Thus,

$$\begin{aligned}\frac{\partial Q}{\partial A} &= -\left(\left\langle -,*_Y\left(F_A-\frac{1}{2}\delta\right)\right\rangle - \langle -,\tau(\Phi,\Phi)\rangle\right)\\ \frac{\partial Q}{\partial\Phi} &= -\frac{1}{2}((-,\partial_A\Phi)_{W_{3,y}} + (\Phi,\partial_A -)_{W_{3,y}}\\ &= -(-,\partial_A\Phi)_{W_{3,y}}\end{aligned}$$

by using Stokes formula, $F_A = \mathrm{d}A$ and the self-adjointness of ∂_A on Y. Thus the gradient flow equations for the functional, viz.,

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\partial Q}{\partial A},\\ \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \frac{\partial Q}{\partial\Phi},\end{aligned}$$

lead, respectively, to the required eq. (4), and the proposition is proved. □

We next investigate what happens to C_δ under the gauge group action. We shall drop the subscript δ from C_δ for notational convenience.

PROPOSITION 3.3.7

Under a gauge transformation $g \in \mathrm{Map}(Y,S^1)$, we have the transformation formula

$$C(gA,g\Phi) = C(A,\Phi) + 2\pi^2[g]\cup\left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right),$$

where $[g]$ denotes the cohomology class of g in $H^1(Y,\mathbb{R})$. Since $[g]$ is an integral cohomology class, this shows that on a gauge orbit, C is well defined in $\mathbb{R}/2\pi^2\mathbb{Z}\cup(c_1(L)-(1/2\pi)[\delta_{\mathcal{H}}])$.

Proof. We apply the formulas $gA = A - g^{-1}dg$, $\partial_{gA}(g\Phi) = g\partial_A\Phi$, and $F_{gA} = F_A$ to compute:

$$\begin{aligned} C(gA, g\Phi) &= C(A,\Phi) + \frac{1}{2}\int_Y g^{-1}dg \wedge (F_A - \delta) \\ &= C(A,\Phi) + \pi[g] \cup 2\pi\left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \\ &= C(A,\Phi) + 2\pi^2[g] \cup \left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \end{aligned}$$

proving the proposition. □

Now we are ready for the main proposition of this section.

PROPOSITION 3.3.8

In the notation of Definition 3.2.1, *assume that* $M_{c,\delta}(g_R)$ *is non-empty for* R *large enough. Then there exists a solution* (A, Φ) *to the Seiberg–Witten equations on* $\mathbb{R} \times Y$ *which is translation invariant in a temporal gauge.*

Proof. Let (A_R, Φ_R) be a solution to the Seiberg–Witten equations on (X_R, g_R) for sufficiently large R. By gauge transforming if necessary (see Remark 3.3.2), let us assume that all the A_R are in temporal gauge. Take reference connections B_R on the bundle $L \to X_R$ such that the restrictions $B_{R|X_\pm} := B_\pm$ are fixed, independent of R.

Let us denote the change in the functional C along the tube $T = [-R, R] \times Y \subset X_R$ by

$$l_{A,\Phi}(R) = C(A_R(R), \Phi_R(R)) - C(A_R(-R), \Phi_R(-R)).$$

By the Proposition 3.3.6, we have that $C(A_R(t), \Phi_R(t))$ are monotonic functions of t. Now, (A_R, Φ_R) will restrict to solutions on $X^\pm$. By construction, the scalar curvature of g_R has the same infimum on X_R for all R. Hence, by § 6.2 of [PP], there is a uniform C^0-bound for all Φ_R on X_R independent of R. The (compactness) argument of § 6.2 of [PP] shows that there exist gauge transformations $h_R^\pm$ of $X_\pm$ such that

$$h_R^\pm A_R - B_R^\pm = h_R^\pm A_R - B^\pm$$

are both bounded in Sobolev L_k^2-norm (for k suitably large) uniformly for all R. By Sobolev's Lemma, this implies uniform C^0 bounds on zeroth and first derivatives of A_R, Φ_R for all R. Since $C(h_R^\pm A_R(\pm R), \Phi_R(\pm R))$ are the evaluations of C (which only involves derivatives upto first order) to the ends $\{\pm R\} \times Y$ of the tube $[-R, R] \times Y$, we have uniform bounds for $C(h_R^\pm A_R(\pm R), \Phi_R(\pm R))$ independent of R. Now let γ be a 1-cycle Poincare-dual to $c_1(L) - (1/2\pi)[\delta_{\mathcal{H}}]$ in Y. It is the intersection with Y of the 2-cycle Γ which is Poincare-dual to $c_1(L) - (1/2\pi)[\delta_{\mathcal{H}}]$ in X.

Let $i_\pm : Y \hookrightarrow X_\pm$ denote the inclusions (as ends), and $[h_R^\pm]$ denote the cohomology classes of $h_R^\pm$ in $H^1(Y)$, or $H^1(X_\pm)$. Then,

$$\begin{aligned} &\left\langle \left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \cup [h_R^\pm], [Y] \right\rangle = \left\langle i_\pm^*\left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \cup [h_R^\pm], [Y] \right\rangle \\ &= \left\langle i_\pm^*\left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \cup [h_R^\pm], \partial_\pm[X_\pm] \right\rangle \\ &= \left\langle \delta_\pm i_\pm^*\left(c_1(L) - \frac{1}{2\pi}[\delta_{\mathcal{H}}]\right) \cup [h_R^\pm], [X_\pm] \right\rangle = 0, \end{aligned}$$

where $[Y]$, $[X_\pm]$ denote orientation classes, and $\partial_\pm : H_4(X_\pm, Y) \to H_3(Y)$ and $\delta_\pm : H^3(Y) \to H^4(X_\pm, Y)$ denote respectively the connecting homomorphisms in the long exact homology and cohomology sequences of the pair $(X_\pm, Y)$. Thus, by Proposition 3.3.7 we have

$$C(h_R^\pm A_R(\pm R), h_R^\pm \Phi_R(\pm R)) = C(A_R(\pm R), \Phi_R(\pm R)).$$

Thus we have a uniform bound M on l_{A_R,Φ_R} independent of R.

Now let R be a positive integer, say $R = N$, and denote by Δ_i the change in C across $[i-1, i] \times Y$, viz.

$$\Delta_i = C(A_N(i), \Phi_N(i)) - C(A_N(i-1), \Phi_N(i-1)).$$

We saw in Proposition 3.3.5 that (A_N, Φ_N), being solutions to Seiberg–Witten equations and A_N being in temporal gauge implied that they were (time-dependent) solutions to the equations (4), and the Proposition 3.3.6 then implied that C was monotonic increasing in time for these solutions. Thus all the Δ_i are non-negative. Let $\Delta_{\min,N} = \min_i \Delta_i$. Hence we have

$$2N\Delta_{\min,N} \le \sum_{i=-N}^{+N} \Delta_i = C(A_N(N), \Phi_N(N)) - C(A_N(-N), \Phi_N(-N)) = l_{A_N,\Phi_N} \le M$$

for all N.

Hence $\lim_{N\to\infty} \Delta_{\min,N} = 0$. Denote by $(A_{(N)}, \Phi_{(N)})$ the restriction of (A_N, Φ_N) to the interval $[i-1, i] \times Y$ on which $\Delta_{\min,N} = |\Delta_i|$. This may be viewed as a solution on $[0, 1] \times Y$, denoted by the same symbol $(A_{(N)}, \Phi_{(N)})$. As we saw above, we have

$$C(A_{(N)}(1), \Phi_{(N)}(1)) - C(A_{(N)}(0), \Phi_{(N)}(0)) \le \frac{M}{N}$$

which goes to 0 as $N \to \infty$. The uniform C^0 bound on (A_N, Φ_N), and hence $(A_{(N)}, \Phi_{(N)})$ gives a solution (on passing to a subsequence) (A, Φ) on $[0, 1] \times Y$ for which

$$\begin{aligned}(C(A(1), \Phi(1)) - C(A(0), \Phi(0))) = \lim(C(A_{(N)}(1), \Phi_{(N)}(1)) \\ - C(A_{(N)}(0), \Phi_{(N)}(0))) = 0.\end{aligned}$$

The monotonicity of C across $[0, 1] \times Y$ implies that (A, Φ) is constant along $[0, 1]$. This solution is clearly therefore a translation invariant solution on $[0, 1] \times Y$, which extends to all of $\mathbb{R} \times Y$ by time-translating for all times. □

4. Proof of Thom's conjecture

We first need a lemma.

Lemma 4.1.9. *Let $Y = S^1 \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is a Riemannian 2-manifold of constant scalar curvature s and genus $g \ge 1$. Assume the metric on $\tilde{\Sigma}$ is normalized so that its volume is* 1 *(and thus $s = 2\pi\chi(\tilde{\Sigma}) = 2\pi(2-2g)$). Let Y have a metric g_Y extending this metric on $\tilde{\Sigma}$, and let the infinite tube $T = \mathbb{R} \times Y$ have a product metric* $\mathrm{d}t^2 \times g_Y$*, and L be the line bundle associated with a compatible* Spin_c *structure. Suppose there is a solution to the Seiberg–Witten equations on (T, g_Y) which is translation invariant in a temporal gauge. Then*

$$\left|\frac{1}{2\pi}\int_{\tilde{\Sigma}} F_A\right| \le 2g - 2.$$

Proof. From the C^0 bound (see § 6.2, [PP]), the sup norm satisfies $|\Phi|^2_\infty \leq 2\pi(2g-2)+ ||\delta||_\infty$. Since $|\sigma(\Phi,\Phi)|^2 = \frac{1}{2}|\Phi|^4$, we have

$$|\sigma(\Phi,\Phi)|_\infty \leq \frac{1}{\sqrt{2}}((2\pi(2g-2)) + ||\delta||_\infty).$$

Thus, from the Seiberg–Witten equations:

$$||F_A^+||_{\infty,Y} \leq \frac{1}{\sqrt{2}}(2\pi(2g-2)) + 2||\delta||_\infty.$$

However, by Corollary 3.3.4, we have the pointwise norm equality $||F_A^+|| = ||F_A^-||$ because our solution is translation invariant in a temporal gauge. Therefore,

$$||F_A|| \leq \sqrt{2}||F_A^+|| \leq 2\pi(2g-2) + O(||\delta||_\infty).$$

Thus

$$\left|\frac{1}{2\pi}\int_{\tilde{\Sigma}} F_A\right| \leq \frac{1}{2\pi}\left[2\pi(2g-2) + O\left(\int_{\tilde{\Sigma}} ||\delta||_\infty\right)\right].$$

This proves the lemma, since δ is arbitarily small. □

Now we can prove the main theorem.

Theorem 4.1.10 (Kronheimer–Mrowka). *If Σ is an oriented 2-manifold smoothly embedded in $\mathbb{C}P^2$ so as to represent an algebraic curve of degree d, then the genus $g(\Sigma)$ of Σ satisfies*

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}.$$

Proof. The cases of $d=1,2$ are trivial, and $d=3$ is due to Kervaire–Milnor (see ref. [6] in [KM]). So we will assume $d>3$ in the sequel. By Proposition 3.2.3, there exists a metric g_R on $X = \mathbb{C}P^2\#d^2\overline{\mathbb{C}P}^2$ such that $c_1(L)\cup[\omega_{g_R}] < 0$. By Corollary 3.2.4, the moduli space $M_{c,\delta}(g_R) \neq \phi$. (We just need to ensure that the reference metric we started with on X is a product metric in a tubular neighbourhood of $Y = S^1\times\tilde{\Sigma}$.) By Proposition 3.3.8, there is a solution on $\mathbb{R}\times Y$ which is translation invariant in a temporal gauge. By the Lemma 4.1.9 above, $|c_1(L).[\tilde{\Sigma}]| \leq 2g-2$, which implies $c_1(L).\tilde{\Sigma} \geq 2-2g$. By the opening discussion of § 3.1 we have $c_1(L) = 3H-E$, and by construction $[\tilde{\Sigma}] = \mathrm{d}H - E$. Thus $(3H-E).(\mathrm{d}H-E) \geq 2-2g$. Applying $H.E = 0$, $H.H = 1$, $E.E = -d^2$, we get $3d - d^2 \geq 2-2g$, i.e. $g \geq ((d-1)(d-2))/2$, proving the theorem. □

5. Appendix: Fredholm theory

5.1 *Preliminaries*

All Hilbert manifolds in the sequel are assumed to be second countable and paracompact. We recall that a bounded operator $T:\mathcal{H}_1\to\mathcal{H}_2$ is said to be Fredholm if $\mathrm{Ker}\,T$ and $\mathrm{Coker}\,T$ are finite dimensional and the range $\mathrm{Im}\,T$ is closed.

DEFINITION 5.1.1

We call a smooth map:

$$f:\mathcal{M}\to\mathcal{N}$$

a *Fredholm map* if the derivative $Df(x) : T_x\mathcal{M} \longrightarrow T_x\mathcal{N}$ is a Fredholm operator for each $x \in \mathcal{M}$.

It is necessary to extend results like the implicit function theorem in the finite dimensional case to the infinite dimensional case, so as to make manifolds out of inverse images of regular values etc. The key to doing it is the following proposition, which enables one to construct a 'standard local model' of a smooth map whose derivative is given to be Fredholm at a point p, in a neighbourhood of p. Its main utility is to decompose a (non-linear) smooth map with infinite-dimensional range into a *linear map* (with infinite-dimensional range) and a *non-linear map* with *finite dimensional* range, in a small neighbourhood of p.

PROPOSITION 5.1.2

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and let $f : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a smooth map between them, such that $f(0) = 0$. Assume that the derivative $T = Df(0)$ is a Fredholm operator from $T_0\mathcal{H}_1 = \mathcal{H}_1$ to $T_0\mathcal{H}_2 = \mathcal{H}_2$. Note that we are only requiring the derivative to be Fredholm at a point, not that f necessarily be a Fredholm map. Then, with the orthogonal decomposition $\mathcal{H}_1 = \operatorname{Ker} T \oplus V_1$, $\mathcal{H}_2 = \operatorname{Im} T \oplus V_2$, there exists a (non-linear) map $\phi : \mathcal{H}_1 \longrightarrow V_2 = \operatorname{Coker} T$ and a diffeomorphism $h : U \longrightarrow h(U)$ for U a neighbourhood of 0 such that:

(i) $f \circ h(n, v_1) = (Tv_1, \phi(n, v_1))$ *for* $n \in \operatorname{Ker} T$, $v_1 \in V_1$, $(n, v_1) \in U$. *So ϕ is smooth with finite dimensional range $V_2 = \operatorname{Coker} T$. In particular,*
(ii) *f is a Fredholm map in the neighbourhood U of* 0.
(iii) $\phi(0) = 0$, $D\phi(0) = 0$.
(iv) *If G is a group acting via an orthogonal linear action on $\mathcal{H}_1$ and $\mathcal{H}_2$, and f is G-equivariant, then V_1, V_2 are G-invariant, and h and ϕ are also G-equivariant.*

Proof. Clearly if f is G-equivariant, $T = Df(0)$ is also G-equivariant, and the orthogonality of the action implies that both $\operatorname{Ker} T$ and $\operatorname{Im} T$ being G-invariant, their orthogonal complements V_1 and $V_2 = \operatorname{Coker} T$ are G-invariant.

Note that $T_{|V_1} : V_1 \longrightarrow \operatorname{Im} T$ is a bounded bijective linear operator, so has a bounded inverse $T^{-1} : \operatorname{Im} T \longrightarrow V_1$ by the open mapping theorem. Let $\pi : \mathcal{H}_2 \longrightarrow \operatorname{Im} T$ be the orthogonal projection onto $\operatorname{Im} T$. Let $\tilde{T}$ denote the composite $T^{-1} \circ \pi : \mathcal{H}_2 \longrightarrow V_1$, and $\theta : \mathcal{H}_1 \longrightarrow \operatorname{Ker} T$ be the orthogonal projection onto $\operatorname{Ker} T$. Consider the map

$$\begin{aligned} \chi : \mathcal{H}_1 &\longrightarrow \mathcal{H}_1 \\ x &\longmapsto (\theta(x), \tilde{T}(f(x))). \end{aligned}$$

Then, since $\theta(n) = n$ for $n \in \ker T$, we compute:

$$\begin{aligned} D\chi(0)(n, v) &= (n, \tilde{T} \circ Df(0)(n, v)) = (n, \tilde{T} \circ T(n, v)) \\ &= (n, T^{-1} \circ \pi \circ T(n, v)) = (n, T^{-1} \circ T(n, v)) = (n, v). \end{aligned}$$

Note also that in the G-setting, χ is G-equivariant. Now $D\chi(0) = \mathrm{id}$ implies by the (infinite-dimensional) inverse function theorem that there exists a ball $V = B(0, \delta)$ around the origin in $\mathcal{H}_1$ on which χ is a diffeomorphism onto its image. Let $U_1 = B(0, \epsilon) \subset \chi(V)$, and for $y \in U_1$, define $\phi_1 : U_1 \longrightarrow V_2$ by $\phi_1 = \pi_{V_2} \circ f \circ \chi^{-1}(y)$. Note that again, in the G-setting, ϕ_1 is G-equivariant since it is a composite of G-equivariant maps. Now for $y \in U_1 = B(0, \epsilon)$, we have, using $T(\theta(u), v) = T(v)$ and the definitions above:

$$f\circ\chi^{-1}(y)=(\pi\circ f\circ\chi^{-1}(y),\pi_{V_2}\circ f\circ\chi^{-1}(y))=(T\circ T^{-1}\circ\pi\circ f\circ\chi^{-1}(y),\phi_1(y))$$
$$=(T\circ\tilde{T}\circ f(\chi^{-1}(y),\phi_1(y))=(T(\theta(\chi^{-1}(y)),\tilde{T}f(\chi^{-1}(y))),\phi_1(y))$$
$$=(T\circ\chi\circ\chi^{-1}(y),\phi_1(y))=(T(y),\phi_1(y)).$$

All we need to do now is extend $\phi_1 : U_1 \to V_2$ to $\phi : \mathcal{H}_1 \to V_2$ in a G-equivariant manner, and this is easily done by using the map

$$\rho : \mathcal{H}_1 \to \mathcal{H}_1$$
$$x \mapsto \frac{\epsilon x}{\epsilon + \psi(\|x\|)\|x\|},$$

where $\psi : \mathbb{R} \to [0,1]$ is a C^∞-map which is identically zero for $|t| < \epsilon/2$ and identically $= 1$ for $|t| > \epsilon$. Then ρ is G-equivariant since G preserves $\|\ \|$; and maps $\mathcal{H}_1$ into $B(0,\epsilon)$ and is equal to the identity map on $B(0,(\epsilon/2))$. Thus the map $\phi = \phi_1 \circ \rho$ is a G-equivariant map agreeing with ϕ_1 on $B(0,(\epsilon/2))$. Now take $U = B(0,(\epsilon/2))$ and $h = \chi^{-1}$. Clearly, $\phi(0) = 0$ by construction, and since $Df(0) = T$, we have $D\phi(0) = 0$, and hence (i), (iii), and (iv) follow. To see (ii) note that on U, we have f equivalent to the map (T,ϕ) via the local diffeomorphism h applied on the domain. Since T is a linear isomorphism from V_1 to $\operatorname{Im} T$, and ϕ is smooth with finite dimensional range, it is easy to check that (T,ϕ) is Fredholm on U. Thus f is also Fredholm on U, proving (ii). □

COROLLARY 5.1.3

Let f, ϕ, U, $\mathcal{H}_1$, and $\mathcal{H}_2$ be as in the last proposition 5.1.2. Let $\tilde{U} := h(U)$. Then, for $(a,\delta) \in \mathcal{H}_2$,

(i) *the germ of the inverse image of (a,δ), i.e. $f^{-1}(a,\delta) \cap \tilde{U}$ is equivalent (i.e. via an ambient diffeomorphism $h : U \to \tilde{U}$) to $\phi_b^{-1}(\delta)$, where*

$$\phi_b : U_b \to V_2$$
$$n \mapsto \phi(n,b).$$

$b := T^{-1}(a)$ (uniquely defined as an element of V_1), and $U_b = \theta(U \cap \pi_{V_1}^{-1}(b))$ is the b-slice of U. Thus, with the hypotheses on f of the last proposition, a local model for $f^{-1}(a,\delta)$ near 0 is given by the fibre of the finite dimensional smooth map ϕ_b. In particular $f^{-1}(0)$ is locally homeomorphic to $\phi_0^{-1}(0)$.

(ii) *For a fixed a in $\operatorname{Im} T \subset \mathcal{H}_2$, (a,δ) is a regular value for $f_{|\tilde{U}}$ if and only if δ is a regular value for the smooth map $\phi_{b|U_b}$, whose domain and range are finite dimensional. By the usual (finite-dimensional) Morse-Sard theorem applied to $\phi_{b|U_b}$, it follows that for a fixed a, there is a Baire subset of $\delta \in V_2$ such that (a,δ) is a regular value for $f_{|\tilde{U}}$. In case δ happens to be in this Baire subset, the inverse image $f^{-1}(a,\delta) \cap \tilde{U}$ is a smooth manifold of dimension* $= \dim U_b - \dim V_2 = \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) = \operatorname{index} T$. *(Note: A Baire set is a countable intersection of open dense sets.)*

(iii) *If $f : \mathcal{M} \to \mathcal{N}$ is a smooth Fredholm map between Hilbert manifolds, the regular values of f constitute a Baire subset of $\mathcal{N}$.*

Proof. $f_{|\tilde{U}}$ has exactly the same properties as $f \circ h_{|U}$ as far as regular values, inverse images of regular values etc. are concerned. Thus 5.1.2 implies (i) and (ii). (iii) follows from second countability and the fact that countable intersections of Baire sets are Baire. □

COROLLARY 5.1.4

Let everything be as in Proposition 5.1.2. If f is a G-equivariant map, (with G acting orthogonally and linearly on both domain and range as before), then a local (homeomorphic) model for $(\tilde{U} \cap f^{-1}(0))/G$ is $(U_0 \cap \phi_0^{-1}(0))/G$ where $\phi_0 = \phi(-,0)$.

Proof. By (i) of the corollary 5.1.3 above, $\tilde{U} \cap f^{-1}(0)$ is homeomorphic to $U_0 \cap \phi_0^{-1}(0)$ via a G-equivariant local diffeomorphism $h : U \to \tilde{U}$ in the ambient space. Since (iv) of 5.1.2 implies that ϕ is G-equivariant, so is $\phi_0 = \phi(-,0)$, and the corollary follows. □

Thus we have proved the following:

PROPOSITION 5.1.5

(Local finite-dimensional model for a neighbourhood of a singular point in the orbit space): If $f : \mathcal{H}_1 \to \mathcal{H}_2$ is a G-equivariant map between Hilbert spaces on which G acts linearly and orthogonally, such that $T = Df(0)$ is a Fredholm operator, then a local model for the germ of $f^{-1}(0)/G$ at 0 is the germ at 0 of the finite dimensional object $\phi_0^{-1}(0)/G$, where $\phi_0 : \operatorname{Ker} T \to \operatorname{Coker} T$ is a smooth G-equivariant map between finite dimensional spaces (with the restricted action of G).

Another lemma which is useful is the following.

Lemma 5.1.6. *Let $f : \mathcal{M} \to \mathcal{N}_1 \times \mathcal{N}_2$ be a smooth map between Hilbert manifolds, and let $f_i := \pi_i \circ f$ where π_i are the projection maps to $\mathcal{N}_i$ for $i = 1,2$. Then*

(i) *$(a,b) \in \mathcal{N}_1 \times \mathcal{N}_2$ is a regular value of f if a is a regular value of f_1 and b is a regular value of $f_{2|f_1^{-1}(a)}$.*
(ii) *If f is a Fredholm map, then $f_{2|f_1^{-1}(a)}$ is a Fredholm map for all regular values a of f_1. For a regular value a of f_1, there is a Baire subset $U \subset \mathcal{N}_2$ such that (a,b) is a regular value of f for $b \in U$.*

Proof. Let $M_a := f_1^{-1}(a)$. We have the commuting diagram

$$\begin{array}{ccccc} M_a & \hookrightarrow & \mathcal{M} & \xrightarrow{f_1} & \mathcal{N}_1 \\ \downarrow_{f_{|M_a}} & & \downarrow_{f} & & \| \\ \{a\} \times \mathcal{N}_2 & \hookrightarrow & \mathcal{N}_1 \times \mathcal{N}_2 & \longrightarrow & \mathcal{N}_1 \end{array}$$

which leads to the diagram of derivatives:

$$\begin{array}{ccccccccc} 0 & \to & T_x(M_a) & \hookrightarrow & T_x\mathcal{M} & \xrightarrow{Df_1(x)} & T_a\mathcal{N}_1 & \to & 0 \\ & & \downarrow_{Df_{2|M_a}} & & \downarrow_{Df(x)} & & \| & & \\ 0 & \to & T_{f_2(x)}\mathcal{N}_2 & \longrightarrow & T_a\mathcal{N}_1 \oplus T_{f_2(x)}\mathcal{N}_2 & \longrightarrow & T_a\mathcal{N}_1 & \to & 0 \end{array}$$

for all $x \in \mathcal{M}_a$. Clearly $Df(x)$ is surjective iff $Df_2(x)$ is surjective, for all $x \in \mathcal{M}_a$. Thus $(a,f_2(x))$ is a regular value of f iff $f_2(x)$ is a regular value for $f_{2|\mathcal{M}_a}$. Similarly, the Fredholm statement follows because of the snake lemma, for all $x \in \mathcal{M}_a$. □

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Infinitely divisible probabilities on linear p-adic groups

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Abstract. In this paper we extend the work of Shah, on the structure of infinitely divisible probabilities on p-adic linear groups, to give a classification for all such probabilities.

Keywords. p-adic linear group; probability; infinitely divisible; continuously embeddable; the embedding problem.

1. Introduction

Let G be a locally compact group. The set $P(G)$ of all probability measures on G is a topological semigroup, where the multiplication is convolution of measures and the topology is the weak topology. A measure $\mu \in P(G)$ is called infinitely divisible on G if μ has an nth root in $P(G)$ for each $n \in \mathbb{N}$, and μ is said to be continuously embedded on G if there is a continuous one-parameter semigroup $(\mu_t)_{t>0}$ in $P(G)$ such that $\mu = \mu_1$.

The embedding problem is the problem of understanding the relationship between the class of infinitely divisible measures and the class of continuously embedded measures on G, for any locally compact group G ([3], Chapter III).

In [5], Shah has given a complete solution to the embedding problem when G is a p-adic algebraic group and a partial solution when G is a p-adic linear group. For the linear case, her result is as follows.

Theorem ([5], Theorem 4). *Let G be a p-adic linear group, and suppose $\nu \in P(G)$ is infinitely divisible on G. Then there exists a unipotent element x in the centre of $G(\nu)$ (the closed subgroup of G generated by* $\mathrm{supp}(\nu)$*) such that $x\nu$ is continuously embedded on G.*

In [5] Shah also points out that the converse of the above theorem is not true in general.

Our purpose in this note is to show how the arguments of Shah in [5] can be easily extended to give a complete solution to the embedding problem for any p-adic linear group G. We write $Z((\mu_t)_{t>0}, G)$ for the subgroup of elements of G which commute pointwise with $\bigcup_{t>0} \mathrm{supp}(\mu_t)$, where $(\mu_t)_{t>0}$ is any continuous convolution semigroup in $P(G)$. Our result is now stated as follows.

Classification theorem. *Let G be a p-adic linear group. Then $\nu \in P(G)$ is infinitely divisible on G if and only if there exists a continuous one-parameter semigroup $(\mu_t)_{t>0}$ in $P(G)$, and some $x \in Z((\mu_t)_{t>0}, G)$, such that*

(i) *x is infinitely divisible in $Z((\mu_t)_{t>0}, G)$, and*
(ii) $\nu = x\mu_1$.

We remark that the element x appearing here is necessarily unimodular (by Lemm of [5] and the remark before Proposition 5 below) and belongs to the centre of $G(\nu)$ Corollary 6, below). We also note that our result solves one of the open problems (gi as Problem 3) which are stated in the recently published article of McCrudden [4] known results and open problems on the embedding problem.

2. Notation and preliminary results

We begin by recalling some terminology from Shah [5]. By a linear p-adic group we m a (topologically) closed subgroup of some $GL(d, \mathbb{Q}_p)$, where $d \in \mathbb{N}$, p is a prime and is the field of p-adic numbers.

For $n \in \mathbb{N}$, we define

$$N_n = \{m \in \mathbb{N} : p^n \nmid m\}.$$

For any locally compact group G, any $\mu \in P(G)$, and any $N \subseteq \mathbb{N}$, we write

$$R(N, \mu, G) = \{\lambda^m \in P(G) : \lambda^n = \mu \text{ for some } n \in N,\ 1 \leq m \leq n\}.$$

We also write $R_n(\mu, G)$ for the closed set of all nth roots of μ in $P(G)$. Clearly $R_n(\mu, G$ $R(N_n, \mu, G)$, for each $n \in \mathbb{N}$.

For each $\mu \in P(G)$ we use $G(\mu)$ to denote the smallest closed subgroup of G cont ing supp(μ), the support of μ. We write $Z(\mu, G)$ for the centraliser of $G(\mu)$ in G, $N(\mu, G)$ for the normaliser of $G(\mu)$ in G. When G is a linear p-adic group, so $G \subseteq$ $(d, \mathbb{Q}_p)$ for some $d \in \mathbb{N}$, we write $\widetilde{G}(\mu)$ for the algebraic closure of $G(\mu)$.

PROPOSITION 1

Let G be a linear p-adic group and let $\mu \in P(G)$. For each $n \in \mathbb{N}$, the set $R(N$ $G \cap \widetilde{G}(\mu))$ is a relatively compact subset of $P(G \cap \widetilde{G}(\mu))$.

Proof. Clearly, $R(N_n, \mu, G \cap \widetilde{G}(\mu)) \subseteq R(N_n, \mu, \widetilde{G}(\mu))$, which is itself relatively com by Proposition 3 of [5].

COROLLARY 2

For any linear p-adic group G, $n \in \mathbb{N}$ and $\mu \in P(G)$, $R_n(\mu, G \cap \widetilde{G}(\mu))$ is a compact su of $P(G \cap \widetilde{G}(\mu))$.

Proof. For $n \in \mathbb{N}$ we have $n \in N_n$ and so $R_n(\mu, G \cap \widetilde{G}(\mu)) \subseteq R(N_n, \mu, G \cap \widetilde{G}($ Proposition 1 now yields the result, since $R_n(\mu, G \cap \widetilde{G}(\mu))$ is closed in $P(G \cap \widetilde{G}(\mu)$

PROPOSITION 3

Let G be a linear p-adic group and let $\mu \in P(G)$ be infinitely divisible on G, then infinitely divisible on $G \cap \widetilde{G}(\mu)$.

Proof. By the proof of Proposition 4 of [5], there is an $n_0 \in \mathbb{N}$ such that for any root μ in $P(G)$, λ^{n_0} is supported on $\widetilde{G}(\mu)$. But as μ is infinitely divisible on G, for any n we can find a $\lambda \in P(G)$ such that $\lambda^{nn_0} = \mu$ and so λ^{n_0} is an nth root of μ supporte $\widetilde{G}(\mu) \cap G$.

COROLLARY 4

Let G be a linear p-adic group and let $\mu \in P(G)$. If μ is infinitely divisible on G, then μ is infinitely divisible and root compact on $G \cap \widetilde{G}(\mu)$, and so μ is rationally embedded on $G \cap \widetilde{G}(\mu)$.

Proof. Follows from Propositions 1 and 3, and Theorem 4.2(i) of [4].

Remark. Corollary 4 implies that for any linear p-adic group G, any $x \in G$ which is infinitely divisible on G is in fact rationally embeddable in G. This is in stark contrast to the real Lie case, where there is a discrete subgroup D of $SL(2,\mathbb{R})$ containing -1, such that -1 is infinitely divisible on D but is not rationally embedded on D, see Remark 6.1 of [2].

PROPOSITION 5

If G be a totally disconnected locally compact group and $(\mu_t)_{t>0}$ is a continuous, one-parameter semigroup in $P(G)$, then for all $s, t \in \mathbb{R}_+^*$, $G(\mu_t) = G(\mu_s)$, and so $Z(\mu_t, G) = Z(\mu_s, G)$.

Proof. Suppose $s \in \mathbb{R}_+^*$, and let

$$\eta : N(\mu_s, G) \to N(\mu_s, G)/G(\mu_s)$$

be the canonical homomorphism. Then by Proposition 1.1 of [1], $(\eta(\mu_t))_{t>0}$ is a continuous one-parameter semigroup in the quotient group

$$N(\mu_s, G)/G(\mu_s),$$

and as this group is totally disconnected, we have $\eta(\mu_t) = 1$ for all $t > 0$.

This implies that $G(\mu_t) \subseteq G(\mu_s)$, for all $t \in \mathbb{R}_+^*$. The same argument with t replacing s then gives $G(\mu_t) = G(\mu_s)$, for all $t, s \in \mathbb{R}_+^*$.

From now on, when dealing with totally disconnected locally compact groups, let $Z((\mu_t)_{t>0}, G)$ denote this common centraliser.

COROLLARY 6

Let G be a totally disconnected locally compact group, let $(\mu_t)_{t>0}$ be a continuous convolution semigroup in $P(G)$, and let $x \in Z(\mu_1, G)$. If $\nu = x\mu_1$, then $x \in Z(G(\nu))$, the centre of $G(\nu)$.

Proof. It is enough to show that $x \in G(\nu)$. Since μ_1 is a two-sided factor of ν, $G(\mu_1)$ normalises $G(\nu)$ by Proposition 1.1 of [1], and so $G(\nu)$ is a closed normal subgroup of the closed subgroup H generated by x and $G(\mu_1)$. By Proposition 5, $\mathrm{supp}(\mu_t) \subseteq H$ for all $t > 0$, and if

$$p : H \to H/G(\nu),$$

is the canonical homomorphism, we have $p(\mu_1) = p(x^{-1}\nu) = p(x^{-1})$, a point mass on $H/G(\nu)$. Since $t \mapsto p(\mu_t)$ is a continuous homomorphism of $\mathbb{R}_+^*$ into the totally disconnected group $H/G(\nu)$, we conclude that $p(\mu_t) = 1$ for all $t > 0$, and so $\mathrm{supp}(\mu_t) \subseteq G(\nu)$, for all $t > 0$. We conclude that $x \in (\mathrm{supp}(\nu))(\mathrm{supp}(\mu_1))^{-1} \subseteq G(\nu)$, giving the result.

3. The characterisation

Let G be any totally disconnected locally compact group , and $(\mu_t)_{t>0}$ be any continuous one-parameter semigroup in $P(G)$. Then it is clear that for any $x \in Z((\mu_t)_{t>0}, G)$ which is infinitely divisible in $Z((\mu_t)_{t>0}, G)$, the measure $\nu = x\mu_1$ is an infinitely divisible measure in $P(G)$.

We now want to show that for G a linear p-adic group, any measure $\nu \in P(G)$ which is infinitely divisible on G is of the above form.

PROPOSITION 7

Let G be a linear p-adic group and suppose $t \mapsto \nu_t$ is a homomorphism of $\mathbb{Q}_+^*$ into $P(G)$. Then there is a continuous homomorphism $t \mapsto \mu_t$ of $\mathbb{R}_+$ into $P(G)$ and a homomorphism $t \mapsto \alpha_t$ of $\mathbb{Q}$ into $Z((\mu_t)_{t>0}, G)$ such that $\nu_t = \mu_t\alpha_t = \alpha_t\mu_t$, for all $0 < t \in \mathbb{Q}$.

Proof. Arguing as in the proof of Theorem 3 of [5], with Proposition 1 above in place of Proposition 3 from [5], we obtain a continuous homomorphism $t \mapsto \mu_t$ of $\mathbb{R}_+$ into $P(G)$ and a homomorphism $t \mapsto \alpha_t$ of $\mathbb{Q}$ into G such that $\nu_t = \mu_t\alpha_t = \alpha_t\mu_t$, for all $t \in \mathbb{Q}_+^*$.

We can now argue as in the last paragraph of the proof of Theorem 3 of [5], to show that $\alpha_t \in Z(\widetilde{G}(\nu_1))$, the centre of $\widetilde{G}(\nu_1)$, for all $t \in \mathbb{Q}$. But as α_t then commutes with every element of $\text{supp}(\nu_1)$ and $(\alpha_1)^{-1}$, it also commutes with $\text{supp}(\mu_1) = (\alpha_1)^{-1}\text{supp}(\nu_1)$ and so lies in $Z(\mu_1, G) = Z((\mu_t)_{t>0}, G)$.

PROPOSITION 8

Let G be a linear p-adic group. Let $\nu \in P(G)$, if ν is infinitely divisible on G then there is a continuous one-parameter semigroup $(\mu_t)_{t>0}$ in $P(G)$ and some $x \in Z((\mu_t)_{t>0}, G)$ which is infinitely divisible in $Z((\mu_t)_{t>0}, G)$ so that $\nu = x\mu_1$.

Proof. By Corollary 4, we have a homomorphism $t \mapsto \nu_t$ from $\mathbb{Q}_+^*$ into $P(G \cap \widetilde{G}(\nu))$ with $\nu_1 = \nu$. Then by Proposition 7 we can find a continuous homomorphism $t \mapsto \mu_t$ of $\mathbb{R}_+$ into $P(G)$ and a homomorphism $t \mapsto \alpha_t$ of $\mathbb{Q}$ into $Z((\mu_t)_{t>0}, G)$ such that $\nu_t = \mu_t\alpha_t = \alpha_t\mu_t$, all $t > 0$. If we choose x to be α_1, it is clear that $\nu = x\mu_1$ and x is contained in and is infinitely divisible in $Z((\mu_t)_{t>0}, G)$.

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A short note on weighted mean matrices

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Abstract. In the present paper we have established a relation between $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices, when considered as bounded operators on l^p, $1 < p < \infty$.

Keywords. Weighted mean matrices.

1. Introduction

Let $\sum a_n$ be an infinite series with partial sum s_n.

If $p_n \geq 0$, $p_0 > 0$, $\sum p_n = \infty$ (so that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$), and

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \to s$$

when $n \to \infty$, then we say that

$$s_n \to s(\overline{N}, p_n).$$

A result concerning the relation between $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices has been given in Hardy [2, Theorem 14].

Theorem A [2]. *If $p_n > 0$, $q_n > 0$, $\sum p_n = \infty$, $\sum q_n = \infty$ and either*

(a) $$\frac{q_{n+1}}{q_n} \leq \frac{p_{n+1}}{p_n} \quad \text{or} \tag{1}$$

(b) $$\frac{p_{n+1}}{p_n} \leq \frac{q_{n+1}}{q_n} \tag{2}$$

and also

$$\frac{P_n}{p_n} \leq H \frac{Q_n}{q_n}, \tag{3}$$

then $\sum a_n = s(\overline{N}, p_n)$ implies $\sum a_n = s(\overline{N}, q_n)$.

In the present paper we have established a relation between the $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices, when considered as bounded operators on l^p, $1 < p < \infty$.

2. Main theorem

Theorem. *Let $\{p_n\}$ and $\{q_n\}$ be positive sequences satisfying the conditions*

$$P_n \left(\frac{q_{n+1}}{p_{n+1}} - \frac{q_n}{p_n} \right) \approx (n+1)^{\alpha}, \quad \text{for some } \alpha \geq 0, \tag{4}$$

$$\frac{P_n}{p_n} \cdot \frac{q_n}{Q_n} \le H,$$

$\{p_n\}$ *is a non decreasing sequence.*

Then $(\overline{N}, q_n)$ *matrix is a bounded linear operator on* l^p *whenever* $(\overline{N}, p_n)$ *matrix i bounded linear operator on* l^p.

Remark. In view of our theorem, Theorem A [2, Theorem 14] of Hardy is partially t when $q_{n+1}/p_{n+1} \ge q_n/p_n$ for l^p spaces, $1 < p < \infty$.

In 1994, Rhoades [4] has obtained sufficeint conditions for certain weighted m matrices to be equivalent to C, the Cesaro matrix of order 1, considered as boun operators on l^p, $1 < p < \infty$.

Theorem B [4]. *Let* $\{a_n\}$ *be a positive sequence satisfying the condition* $(n +$ $(a_{n+1} - a_n) \approx (n+1)^\alpha$ *for some* $\alpha \ge 0$. *Then* $(\overline{N}, a)$ *and* C *are equivalent over* l^p *for* $1 < p <$

When we take $q_n = a_n$ and $p_n = 1$ in our theorem, then the condition of Theorem satisfied.

Thus one part of theorem B of Rhoades that $(\overline{N}, a)$ matrix is a bounded linear oper on l^p whenever $(C, 1)$ matrix is a bounded linear operator on l^p is implied by our theor

A triangular matrix $A = (a_{nk})$ is said to be factorable if $a_{nk} = c_n d_k$, $0 \le k \le n$, $a_{nk} = 0$, $k > n$.

3. Lemmas

For the proof of our theorem, we need the following lemmas.

Lemma 1 [1]. *Let* $1 < p < \infty$, *q the conjugate index of p. If* c_n, $d_k \ge 0$ *and*

$$c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/p-1}$$

for $n = 0, 1, \ldots$ *and some constant K, then A is a bounded operator on* l^p.

Lemma 2. *If*

(i) $a_1 \le a_2 \le \cdots \le a_n$ *and* $b_1 \le b_2 \le \cdots \le b_n$, *then Chebychev's inequality state*

(ii) $\left(\frac{1}{n}\sum_{v=1}^{n} a_v\right)\left(\frac{1}{n}\sum_{v=1}^{n} b_v\right) \le \frac{1}{n}\sum_{v=1}^{n} a_v b_v$.

(iii) $1^p + 2^p + 3^p + \cdots + n^p > \frac{n^{p+1}}{p+1}$, $\quad p > 0$.

The proof may be found in [3], pages 16 and 97.

Lemma 3. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two positive sequences satisfying the conditions* (4) *and* (6) *of the theorem. Then*

(iv) $Q_n \ge \frac{(n+1)^{\alpha+1}}{(\alpha+1)(\alpha+2)}$, $\quad$ for $\alpha \ge 0$.

Remark. It follows that $Q_n \to \infty$ as $n \to \infty$.

Proof. For $\alpha \ge 0$, by condition (4)

$$P_k\left(\frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k}\right) \approx (k+1)^\alpha \Rightarrow \frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k} \approx \frac{(k+1)^\alpha}{P_k}.$$

Now taking summation

$$\sum_{k=0}^{n-1}\left(\frac{q_{k+1}}{p_{k+1}}-\frac{q_k}{p_k}\right)\approx\sum_{k=0}^{n-1}\frac{(k+1)^\alpha}{P_k}\Rightarrow\frac{q_n}{p_n}\approx\sum_{k=0}^{n-1}\frac{(k+1)^\alpha}{P_k}.$$

This implies

$$\frac{q_k}{p_k}\approx\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\Rightarrow q_k\approx p_k\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\Rightarrow\sum_{k=0}^{n}q_k\approx\sum_{k=0}^{n}p_k\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}.$$

Using Chebychev's inequality,

$$\begin{aligned}
Q_n &\geq n\left(\frac{1}{n}\sum_{k=0}^{n}p_k\right)\left(\frac{1}{n}\sum_{k=0}^{n}\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\right)\\
&=\frac{P_n}{n}\sum_{k=0}^{n}\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\\
&\geq\frac{P_n}{n}\sum_{k=0}^{n}\frac{1}{P_k}\sum_{r=0}^{k}(r+1)^\alpha\\
&\geq\frac{P_n}{n}\sum_{k=0}^{n}\frac{1}{P_k}\frac{(k+1)^{\alpha+1}}{(\alpha+1)},\quad\text{by Lemma 2,}\\
&\geq\frac{P_n}{nP_n}\sum_{k=0}^{n}\frac{(k+1)^{\alpha+1}}{(\alpha+1)}\\
&=\frac{1}{n(\alpha+1)}\sum_{k=0}^{n}(k+1)^{\alpha+1}\\
&\geq\frac{1}{n(\alpha+1)}\frac{(n+1)^{\alpha+2}}{(\alpha+2)},\quad\text{by Lemma 2,}\\
\Rightarrow Q_n &\geq\frac{(n+1)^{\alpha+1}}{(\alpha+1)(\alpha+2)}.
\end{aligned}$$

This proves (iv).

Proof of the theorem. Define

$$t_{n,q}=\frac{1}{Q_n}\sum_{k=0}^{n}q_ks_k, \tag{7}$$

$$t_{n,p}=\frac{1}{P_n}\sum_{k=0}^{n}p_ks_k. \tag{8}$$

Solving (8) for s_n and then substituting into (7), we have

$$t_{n,q}=\frac{1}{Q_n}\sum_{k=0}^{n}\left[P_k\,t_{k,p}-P_{k-1}t_{k-1,p}\right]\frac{q_k}{p_k}.$$

By partial summation formula

$$t_{n,q} = \frac{1}{Q_n}\left[\frac{q_{n+1}}{p_{n+1}} P_n t_{n,p} + \sum_{k=0}^{n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right) P_k t_{k,p}\right].$$

Then $t_{n,q} = A_n(t_{n,p})$, where A is the lower triangular matrix with entries

$$a_{nk} = \begin{pmatrix} \frac{P_k}{Q_n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right), & 0 \le k < n \\ \frac{q_n}{p_n}\frac{P_n}{Q_n}, & k = n \\ 0. & k > n \end{pmatrix}$$

Obviously $A = C + D$, where

$$c_{nk} = \frac{P_k}{Q_n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right), \quad 0 \le k \le n$$

and D is the diagonal matrix with diagonal entries

$$d_{nn} = \frac{q_{n+1}}{p_{n+1}}\frac{P_n}{Q_n}.$$

To show that $A \in B(l^p)$ it is sufficient to show that $C \in B(l^p)$ and that D is bounde view of condition (5), D is bounded. In order to show that C is bounded, we use Lemm Choose

$$c_n := \frac{1}{Q_n}$$

and

$$d_k := P_k\left(\frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k}\right) \ge 0.$$

Let $\alpha \ge 0$. Then the condition (4) of the theorem implies that $d_k \approx (k+1)^\alpha$.

By Lemma 1, if we show that

$$c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/p-1},$$

then C is bounded.

Now

$$\begin{aligned} \frac{c_n}{d_n^{1/p-1}} \sum_{k=0}^{n} d_k^q &= \frac{1}{Q_n[(n+1)^\alpha]^{q-1}} \sum_{k=0}^{n} (k+1)^{\alpha q} \\ &\approx \frac{(n+1)^{\alpha q+1}}{Q_n (n+1)^{\alpha q - \alpha}} = \frac{(n+1)^{\alpha+1}}{Q_n} \\ &\le \frac{(n+1)^{\alpha+1}(\alpha+1)(\alpha+2)}{(n+1)^{\alpha+1}} \quad \text{by condition (iv) of Lemma 3} \\ &\le K, \quad \text{for some constant } K. \end{aligned}$$

Thus $t_{n,q} \in l^p$ whenever $t_{n,p} \in l^p$. This proves the theorem.

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$L^1(\mu, X)$ as a constrained subspace of its bidual

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Abstract. In this note we consider the property of being constrained in the bidual, for the space of Bochner integrable functions. For a Banach space X having the Radon–Nikodym property and constrained in its bidual and for $Y \subset X$, under a natural assumption on Y, we show that $L^1(\mu, X/Y)$ is constrained in its bidual and $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$. As an application of these results, we show that, if $L^1(\mu, X)$ admits generalized centers for finite sets and if $Y \subset X$ is reflexive, then $L^1(\mu, X/Y)$ also admits generalized centers for finite sets.

Keywords. Spaces of Bochner integrable functions; vector measures; proximinal subspaces; generalized centers.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let X be a Banach space that is constrained in its bidual i.e., X is the range of a norm one projection when canonically embedded in its bidual. Any dual space, thanks to the canonical projection of Dixmier (See [H], p. 213) is contrained in its bidual. It is well known that the space of integrable functions $L^1(\mu)$ satisfies this property. It is easy to see that X is constrained in its bidual iff X is isometric to the range of a norm one projection in some dual space (see [Lin]). Thus this property is preserved by ranges of norm one projections. In this note we consider the question, when is the space of Bochner integrable functions, $L^1(\mu, X)$, constrained in its bidual? That X should be constrained in its bidual is clearly a necessary condition. c_0, the space of sequences converging to zero, is not constrained in its bidual (it is not even complemented in ℓ^∞, see [H], p. 232). The author in [R3] noted the isometric version of a result of Emmanuele [E1] that, if X is any Banach space that has an isometric copy of c_0, then c_0 is isometric to the range of a norm one projection in $L^1(\mu, X)$. Thus by taking $X = \ell^\infty$ we see that $L^1(\mu, X)$ is not constrained in its bidual. If X is constrained in its bidual and has the Radon–Nikodym property (RNP), it was proved in [R1, R2], that $L^1(\mu, X)$ is constrained in its bidual. Let cabv(μ, X) denote the space of X-valued countably additive measures of bounded variation on $\mathcal{A}$ that are absolutely continuous with respect to μ. Emmanuele and Rao in 1994 (see [E2, R4]) have obtained an internal characterization by showing that when X is constrained in its bidual, $L^1(\mu, X)$ is constrained in its bidual iff it is constrained in cabv(μ, X). Using this, in this short note we exhibit more examples of Banach spaces for which $L^1(\mu, X)$ is constrained in its bidual. Our work is motivated by the recent exposition of the Lindenstrauss' lifting principle (LLP) by Kalton and Pełczyński in [KP]. A closed subspace $Y \subset X$ is said to be proximinal if given $x \in X$ there exists a $y \in Y$ such

that $d(x,Y) = \|x - y\|$. A natural question that can be asked in the context of spaces of Bochner integrable functions is that, if $Y \subset X$ is a proximinal subspace, then is $L^1(\mu, Y)$ a proximinal subspace of $L^1(\mu, X)$? Only recently a counterexample was obtained by Mendoza in [M]. As an interesting consequence of our approach we exhibit new classes of proximinal subspaces Y for which $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$. In this process we obtain new proofs of some well known result on proximinality in $L^1(\mu, X)$.

In the concluding part of the paper we apply these results to study a weaker geometric notion called GC (defined later in the paper) for quotient spaces of Bochner integrable functions. Here our result states that, if $L^1(\mu, X)$ has GC and $Y \subset X$ is reflexive then $L^1(\mu, X/Y)$ has GC.

Our notation and terminology is fairly standard and can be found in [DU, H, L, HWW].

2. Main results

The new examples we exhibit come from quotient spaces and an isometric version of LLP is used to achieve this. These results also give a unified approach to Corollary 1 and Proposition in [R4]. We first recall the isomorphic version of LLP from [KP].

Lindenstrauss lifting principle: Let $Y \subset X$ and Y be complemented in its bidual. Let F be any $\mathcal{L}_1$ space. Every bounded linear operator $T : F \to X/Y$ admits a lifting, i.e., there exists a bounded linear operator $T^\wedge : F \to X$ such that $\pi T^\wedge = T$.

In this paper we assume that X, Y and X/Y are constrained in their bidual and will consider the space $L^1(\mu, X/Y)$. We assume that $L^1(\mu, X)$ is constrained in its bidual and most often we assume that Y has the RNP, so that $L^1(\mu, Y)$ is also constrained in its bidual. Here it may be worth recalling from [DU] that $L^1(\mu, X/Y)$ can be identified with the quotient space $L^1(\mu, X)/L^1(\mu, Y)$. We consider the question, when is $L^1(\mu, X/Y)$ constrained in its bidual?

In what follows we will need norm preserving liftings from $L^1(\mu)$. Even under the assumptions of the above paragraph we do not know if this can always be achieved. Thus we need an extra assumption on the projection which is satisfied in several naturally occurring situations.

Assumption. Let $P : X^{**} \to X$ be a norm one projection. Let $Y \subset X$ be a closed subspace such that $P(Y^{\perp\perp}) = Y$.

Remark 1. This clearly is the case when X is constrained in its bidual and Y is reflexive. Also if $M \subset X^*$ then the Dixmier projection, $Q : X^{***} \to X^*$ defined by $Q(\Lambda) = \Lambda/X$, satisfies the condition in the '*assumption*' precisely when M is weak* closed (Lemma IV.1.1 of [HWW]). We recall from Chapter IV of [HWW] that a Banach space X is a L-summand in its bidual if when X is canonically embedded in its bidual, there exists a projection $P : X^{**} \to X$ such that $\|P(\Lambda)\| + \|\Lambda - P(\Lambda)\| = \|\Lambda\|$ for all $\Lambda \in X^{**}$. Clearly such a X is constrained in its bidual and is also a proximinal subspace of its bidual. It was shown in [Li] that if both X, Y are L-summands in their biduals then the '*assumption*' is satisfied. In Proposition 3 below we exhibit some more situations where the '*assumption*' is satisfied.

The following proposition which identifies Q as the only projection with the above property is probably known but as we are not aware of a reference to it in the literature, we give below its easy proof.

PROPOSITION 1

Suppose $P : X^{***} \to X^*$ is a bounded linear projection such that for every weak closed subspace $M \subset X^*$, $P(M^{\perp\perp}) = M$. Then $P = Q$.*

Proof. Since P and Q are projections with the same range, it is clear enough to show that $\mathrm{Ker}P \subset X^{\perp}$. Let $\Lambda \in \mathrm{Ker}P$ and let $x \in X$. Write $\Lambda = f + \gamma$ where $f \in X^*$ and $\gamma \in X^{\perp}$. Let $M = \mathrm{Ker}(x)$. Then M is a weak* closed subspace of X^* and $M^{\perp\perp} = \{\lambda \in X^{***} : \lambda(x) = 0\}$. Thus $\gamma \in M^{\perp\perp}$. By hypothesis, $P(M^{\perp\perp}) = M$. Therefore $P(\gamma) \in M$. Now $P(\Lambda) = 0 = f(x)$. Hence $\Lambda(x) = 0$. Therefore $\Lambda \in X^{\perp}$.

It follows from Lemma 1.1 in Chapter IV of [HWW] that the '*assumption*' implies that Y is a proximinal subspace of X. This is one of the motivations for considering this approach.

We need the following elementary lemma.

Lemma 1. *Under the 'assumption'*, X/Y is constrained in its bidual.

Proof. Define $Q : X^{**}/Y^{\perp\perp} \to X/Y$ by $Q(\pi^{**}(x^{**})) = \pi(P(x^{**}))$. That Q is well defined is guaranteed by the '*assumption*'. Clearly Q is a projection. Also

$$\begin{aligned} \|Q(\pi^{**}(x^{**}))\| &= \|\pi(P(x^{**}))\| \\ &= d(P(x^{**}), Y) \\ &\leq \|P(x^{**}) - P(\lambda)\| \quad \text{for any} \quad \lambda \in Y^{\perp\perp} \\ &\leq \|x^{**} - \lambda\|. \end{aligned}$$

Therefore $\|Q(\pi^{**}(x^{**}))\| \leq \|\pi^{**}(x^{**})\|$. Hence X/Y is constrained in its bidual.

Before proceeding further we prove a proposition of independent interest that shows the limitations of the '*assumption*'.

PROPOSITION 2

Let Y be a Banach space that is constrained in its bidual. Suppose whenever Y is isometrically embedded in a Banach space X which is constrained in its bidual, the 'assumption' is satisfied. Then Y is reflexive.

Proof. Let X be any Banach space containing (isometrically) Y. We shall show that Y is proximinal in X. It then follows from a result of Pollul [CW] that Y is reflexive. Note that since $Y \subset X \subset X^{**}$, the hypothesis on Y implies that it is proximinal in X^{**}. Therefore Y is proximinal in X and hence reflexive.

We do not know an answer to the corresponding subspace formulation.

Question. Characterize spaces X with the property that the '*assumption*' is satisfied for all $Y \subset X$ that are constrained in their bidual.

The following lemma is the isometric version of the lifting theorem we need and we give below its simple proof for the sake of completeness.

Lemma 2. *Let $Y \subset X$ be such that the 'assumption' is satisfied. For any bounded linear operator $T : L^1(\mu) \to X/Y$, there exists a $T^\wedge : L^1(\mu) \to X$ such that $\pi T^\wedge = T$ and $\|T^\wedge\| = \|T\|$.*

Proof. It follows from Theorem 8 on p. 178 of [L] that there is a $S : L^1(\mu) \to X^{**}$ such tha $\|S\| = \|T\|$ and $\pi^{**}S = T$. Put $T^\wedge = PS$. Clearly $\|T^\wedge\| = \|T\|$. Let $f \in L^1(\mu)$. If $T(f) = \pi(x$ for some $x \in X$, then $x - S(f) \in Y^{\perp\perp}$. By hypothesis we get $P(x - S(f)) \in Y$. Thu $\pi T^\wedge = T$.

Theorem 1. *Let $Y \subset X$ be such that the 'assumption' is satisfied. Suppose Y has the RN and $L^1(\mu, X)$ is constrained in its bidual, then $L^1(\mu, X/Y)$ is constrained in its bidual.*

Proof. We follow the arguments given during the proof of Proposition in [R4]. It is thu enough to show that $L^1(\mu, X/Y)$ is the range of a norm one projection in cabv$(\mu, X/Y$ Since Y has the RNP, and since $L^1(\mu, X)$ is constrained in cabv(μ, X) by Theorem 5 [E2] the required conclusion follows once we show that elements of cabv$(\mu, X/Y)$ can lifted to elements of cabv(μ, X) in a norm preserving way. As in the proof of Propositic in [R4] this can be achieved by using the correspondence between vector measures ar operators on L^1-spaces and the above Lifting result for operators.

Remark 2. Even under the '*assumption*' of the above theorem it not clear if the inclusi $L^1(\mu, Y) \subset L^1(\mu, X)$ satisfies the '*assumption*' (see Proposition 3 below).

In the following Proposition we consider two situations where the explicit knowled of the projection shows that the '*assumption*' is satisfied in the space of Bochn integrable functions and hence the quotient space result of our Theorem can be obtain by simply using Lemma 1 instead of the Lifting and other result of Emmanuele [E2] quotient space valued measures, see also Remark 6 below.

PROPOSITION 3

Suppose X has the RNP and is constrained in its bidual.

1. *Consider the embedding of X as constant functions in $L^1(\mu, X)$.*
2. *Let $Y \subset X$ and the 'assumption' is satisfied. Consider the inclusion $L^1(\mu, Y) \subset L^1(\mu,$*

The 'assumption' is satisfied in both these cases. Thus the subspaces are proximinal a the quotient spaces are constrained in the bidual.

Proof. Let $P : X^{**} \to X$ be a norm one projection such that $P(Y^{\perp\perp}) = Y$. We assu w.l.o.g that μ is a category measure on the Borel σ-field of a compact hyperstone space K. Consider the canonical embedding $X^{**} \subset L^1(\mu, X)^{**}$. We recall from [R1, that a norm one projection $P^\wedge : L^1(\mu, X)^{**} \to L^1(\mu, X)$ was defined by $P^\wedge(\lambda) = (\lambda$ $(K, X^*) \circ P)_a / \mathrm{d}\mu$. Here we used Singer's theorem that identified $C(K, X^*)^*$ as cabv$(X$ and the suffix a indicates the absolutely continuous (w.r.t. μ) part of the measure.

Now if $\lambda \in X^{**}$ then the vector measure under consideration is the Dirac measure $P(\lambda)$ and thus $P^\wedge$ extends P.

Let $Y \subset X$ satisfy the '*assumption*' and let $\lambda \in L^1(\mu, Y)^{\perp\perp}$. We claim that λ can naturally restricted to $C(K, Y^*)$. To see this, let $f \in C(K, Y^*)$. We first observe that it be in a norm preserving way extended to a $f^\wedge \in C(K, X^*)$.

Since K is hyperstonean one can do this easily using properties of extremally disc nected spaces and Stone–Čech compactification (see § 11 of [L]). Equivalently one treat a $f \in C(K, Y^*)$ as a compact operator $T : Y \to C(K)$ and use Theorem 1 on p. 20 [L] to get a norm preserving extension $T^\wedge : X \to C(K)$.

Now one defines the restriction by

$$\lambda(f) = \lambda(f^\wedge).$$

This is well defined since $\lambda \in L^1(\mu, Y)^{\perp\perp}$. This is what is meant by the natural restriction.

By the uniqueness part of Singer's representation theorem, we see that the corresponding vector measure actually takes values in $Y^{\perp\perp}$ and thus $P^\wedge(L^1(\mu, Y)^{**}) = L^1(\mu, Y)$. Hence the conclusion follows.

As a corollary to the proof of the above proposition we have the following result that extends several classical proximinality situations in the space of Bochner integrable functions.

COROLLARY 1

Suppose $Y \subset X$ satisfies the 'assumption' and Y has the RNP. Then $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$.

Proof. We note that the projection $P^\wedge$ defined as above, now has $\mathrm{cabv}(\mu, X)$ as its range. Since Y has the RNP, $P^\wedge(L^1(\mu, Y)^{**}) = L^1(\mu, Y)$. Therefore $L^1(\mu, Y) \subset \mathrm{cabv}(\mu, X)$ is proximinal and in particular it is proximinal in $L^1(\mu, X)$.

Remark 3. Let $Y \subset X^*$ be a weak* closed subspace having the RNP. Since the '*assumption*' is satisfied via the canonical projection, we get that $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X^*)$. If $Y \subset X$ is a reflexive subspace then again since Y is a weak* closed subspace of X^{**} we get a new proof of the classical result that $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$ (see [LC], Theorem 2.13), see also [R5].

If X is a Banach space that is a L-summand in its bidual it is still not known whether $L^1(\mu, X)$ will always be a L-summand in its bidual. It follows from the above corollary that if $Y \subset X$ are both L-summands in their bidual (as mentioned in Remark 1, the '*assumption*' is satisfied in this case) and Y has the RNP, then $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$.

Remark 4. We take this opportunity to point out that Corollary 3.5 of [M] does not lead to a new class of proximinal subspaces, since the author's '*assumption*' "each separable subspace Y is proximinal in X" already implies that X is reflexive. To see this, note that it is enough to show that every separable $Y \subset X$ is reflexive. Now for such a Y, for each closed subspace $Z \subset Y$, the hypothesis implies that Z is proximinal in X and hence in Y. It now follows from the proof of the Theorem on p. 161 of [H] that Y and hence X is reflexive.

Remark 5. Suppose X, Y satisfy the '*assumption*' and X/Y is isometric to some $L^1(\nu)$. Since the identity map on X/Y can now be lifted in a norm preserving way, we get that Y is also the kernel of a norm one projection. Thus $L^1(\mu, Y)$ being the kernel of a norm one projection, is again a proximinal subspace of $L^1(\mu, X)$.

Remark 6. Suppose X, Y and X/Y are all constrained in their biduals. If X has the RNP then $L^1(\mu, X/Y)$ is constrained in its bidual. To see this we only have to observe that the hypothesis implies that X/Y has the RNP, then the conclusion follows from [R2]. In view of Lewis and Stegall characterization of the RNP in terms of factorization of operators on

L^1 spaces (Theorem III.1.8 of [DU]) and the LLP we see that when X has the RNP so does X/Y.

The following corollary which covers the isomorphic case, is an immediate consequence of the LLP and the arguments given during the proof of the above theorem. We note that since we will be applying the LLP when the domain is a $L^1(\nu)$ space, $\sigma = 1$ in the proof of the LLP in [KP].

COROLLARY 2

Suppose X, Y and X/Y are complemented in their bidual. Suppose $L^1(\mu, X)$ is complemented in its bidual and Y has the RNP. $L^1(\mu, X/Y)$ is complemented in its bidual.

It follows from Proposition 2.3 of [KP] that if X is a $L^1(\nu)$ space and $Y \subset X$ is such that X/Y has the RNP and constrained in its bidual (thus $L^1(\mu, X)$ and $L^1(\mu, X/Y)$ are both constrained in their biduals) then there is a projection $Q: X^{**} \to X$ such that $Q(Y^{\perp\perp}) = Y$. However such a projection in general need not be of norm one, as can be seen by taking $Y = \mathrm{Ker}(x^*)$ where $x^* \in X^*$ does not attain its norm.

In the concluding part of the paper, as an application of the proximinality results proved here, we consider a weaker geometric notion for quotient spaces of Bochner integrable functions. We first recall the notion of generalized center (GC) due to Veselý that is related to the existence of weighted Chebyshev centers, from [BR].

DEFINITION

A Banach space X is said to have GC, if every finite collection of closed balls in X^{**} with centers from X (as before, X is canonically embedded in X^{**}) and having non-empty intersection, has an element of X in the intersection.

It is easy to see that if X is constrained in its bidual then it has GC. c_0 has GC and more generally any Banach space whose dual is isometric to a $L^1(\mu)$ has GC (see [BR]).

To facilitate the study of GC the authors of [BR] have introduced the notion of a central subspace. $Y \subset X$ is said to be a central subspace if for every finite collection of elements $\{y_1, \ldots, y_n\}$ in Y and $x \in X$, there exists a $y_0 \in Y$ such that $\|y_i - y_0\| \leq \|y_i - x\|$. It is easy to see that a Banach space X has GC iff it is a central subspace of X^{**} (see [BR]). We also have from [BR] that if $Y \subset Z \subset X$ and Y is proximinal in X and Z is a central subspace of X, then Z/Y is a central subspace of X/Y.

Using the arguments given in [R4] the following proposition is easy to prove.

PROPOSITION 4

$L^1(\mu, X)$ has GC iff X has GC and $L^1(\mu, X)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**})$.

We use the above proposition and give an application of the proximinality results.

Theorem 2. *Let X be such that $L^1(\mu, X)$ has GC. Let $Y \subset X$ be reflexive. Then $L^1(\mu, X/Y)$ has GC.*

Proof. We note that since Y is reflexive, $\mathrm{cabv}(\mu, X^{**}/Y)$ can be identified with the quotient space, $\mathrm{cabv}(\mu, X^{**})/L^1(\mu, Y)$. From the above proposition and the hypothesis, we have that $L^1(\mu, X)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**})$. From the proof of Proposition 3 and the subsequent remark, we have that $L^1(\mu, Y)$ is a proximinal subspace of $\mathrm{cabv}(\mu, X^{**})$. Therefore $L^1(\mu, X/Y)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**}/Y)$ and hence has GC.

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A result on the composition of distributions

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Abstract. Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distribution $(x^r)^{-s}$ is evaluated for $r, s = 1, 2, \ldots$.

Keywords. Distribution; delta-function; composition of distributions; neutrix; neutrix limit.

In the following we let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n,\ \ln^r n: \quad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\,\mathrm{d}x = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [3]..

DEFINITION 1

Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\mathrm{d}x = \langle h(x), \varphi(x)\rangle$$

for all test functions φ with compact support contained in (a, b).

The following theorems were proved in [3] and [4] respectively:

Theorem 1. *The distributions* $(x_-^\mu)_-^\lambda$ *and* $(x_+^\mu)_-^\lambda$ *exists and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ *and* $\lambda\mu \neq -1, -2, \ldots$ *and*

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi\operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ *and* $\lambda\mu = -1, -2, \ldots$.

Theorem 2. *The distribution* $(x_+^r)_-^{-s}$ *exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots,$ *where*

$$c(\rho) = \int_0^1 \ln t\, \rho(t)\, \mathrm{d}t.$$

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$ and not as in Gel'fand and Shilov [5].

We need the following lemmas which can be easily proved by induction:

Lemma 1. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\langle x^{-r}, \varphi(x)\rangle = \int_{-1}^{1} x^{-r}\left[\varphi(x) - \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]\mathrm{d}x + \sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}-1}{(r-k)k!}\varphi^{(k)}(0), \tag{1}$$

for $r = 1, 2, \ldots$.

Lemma 2.

$$\int_{-1}^{1} v^i\rho^{(r)}(v)\mathrm{d}v = \begin{cases} 0, & 0 \leq i < r, \\ (-1)^r r!, & i = r \end{cases} \tag{2}$$

for $r = 0, 1, 2, \ldots$.

We now prove the following theorem.

Theorem 3. *The distribution* $(x^r)^{-s}$ *exists and*

$$(x^r)^{-s} = x^{-rs}, \tag{3}$$

for $r, s = 1, 2, \ldots$.

Proof. We first put

$$[(x^r)^{-s}]_n = (x^r)^{-s} * \delta_n(x) = \frac{(-1)^{s-1}}{(s-1)!}\int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)\mathrm{d}t,$$

and note that

$$\int_{-1}^{1} x^k[(x^r)^{-s}]_n \,\mathrm{d}x = \begin{cases} 0, & rs-k \text{ odd}, \\ 2\int_0^1 x^k[(x^r)^{-s}]_n \mathrm{d}x, & rs-k \text{ even}. \end{cases} \tag{4}$$

Then

$$\begin{aligned}
(-1)^{s-1}(s-1)!\int_0^1 x^k[(x^r)^{-s}]_n \,\mathrm{d}x &= \int_0^1 x^k \int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)\mathrm{d}t\,\mathrm{d}x \\
&= \int_{-1/n}^{1/n} \delta_n^{(s)}(t)\int_0^{n^{-1/r}} x^k \ln|x^r - t|\,\mathrm{d}x\,\mathrm{d}t + \int_{-1/n}^{1/n} \delta_n^{(s)}(t)\int_{n^{-1/r}}^{1} x^k \ln|x^r - t|\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{r}n^{(rs-k-1)/r}\int_{-1}^{1} \rho^{(s)}(v)\int_0^1 u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \frac{1}{r}n^{(rs-k-1)/r}\int_{-1}^{1} \rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v \\
&= I_1 + I_2,
\end{aligned} \tag{5}$$

on using the substitutions $u = nx^r$ and $v = nt$.

It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = 0, \tag{6}$$

for $k = 0, 1, \ldots, rs-2$.

Now,

$$\begin{aligned}
\int_{-1}^{1} \rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v
&= \int_{-1}^{1} \rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}[\ln|1-v/u| + \ln u - \ln n]\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-1}^{1} \rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|1-v/u|\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}$$

since $\int_{-1}^{1}\rho^{(s)}(v)\,\mathrm{d}v = 0$ for $s = 1, 2, \ldots$, by Lemma 2. Further

$$\int_{-1}^{1} \rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|1-v/u|\,\mathrm{d}u\,\mathrm{d}v$$

$$= -\sum_{i=1}^{\infty} \frac{1}{i} \int_{-1}^{1} v^i \rho^{(s)}(v) \int_{1}^{n} u^{(k+1)/r-i-1}\, du\, dv$$

$$= -\sum_{i=1}^{\infty} \frac{r[n^{(k+1)/r-i}-1]}{i(k-ri+1)} \int_{-1}^{1} v^i \rho^{(s)}(v)\, dv,$$

and it follows that

$$\begin{aligned} N\text{-}\lim_{n\to\infty} I_2 &= \frac{1}{s(rs-k-1)} \int_{-1}^{1} v^s \rho^{(s)}(v)\, dv \\ &= \frac{(-1)^s (s-1)!}{rs-k-1}, \end{aligned} \tag{7}$$

on using Lemma 2, for $k = 0, 1, \ldots, rs-2$.

Hence

$$N\text{-}\lim_{n\to\infty} \int_{0}^{1} x^k [(x^r)^{-s}]_n\, dx = -\frac{1}{rs-k-1} \tag{8}$$

for $k = 0, 1, \ldots, rs-2$, on using (5), (6) and (7). Then using (4) and (8), we see that

$$N\text{-}\lim_{n\to\infty} \int_{-1}^{1} x^k [(x^r)^{-s}]_n\, dx = \frac{(-1)^{rs-k-1}-1}{rs-k-1}, \tag{9}$$

for $k = 0, 1, \ldots, rs-1$.

When $k = rs$ (5) still holds but now we have

$$I_1 = \frac{n^{-1/r}}{r} \int_{-1}^{1} \rho^{(s)}(v) \int_{0}^{1} u^{-(r-k-1)/r} \ln|(u-v)/n)|\, du\, dv,$$

and it follows that for any continuous function ψ

$$\lim_{n\to\infty} \int_{0}^{n^{-1/r}} x^{rs} [(x^r)^{-s}] \psi(x) dx = 0. \tag{10}$$

Similarly

$$\lim_{n\to\infty} \int_{-n^{-1/r}}^{0} x^{rs} [(x^r)^{-s}] \psi(x) dx = 0. \tag{11}$$

Next, when $x^r \geq 1/n$, we have

$$\begin{aligned} (-1)^{s-1}(s-1)! [(x^r)^{-s}]_n &= \int_{-1/n}^{1/n} \ln|x^r - t| \delta_n^{(s)}(t) dt \\ &= n^s \int_{-1}^{1} \ln|x^r - v/n| \rho^{(s)}(v) dv \\ &= n^s \int_{-1}^{1} \left[\ln x^r - \sum_{i=1}^{\infty} \frac{v^i}{i n^i x^{ri}} \right] \rho^{(s)}(v) dv \\ &= -\sum_{i=s}^{\infty} \int_{-1}^{1} \frac{v^i}{i n^{i-s} x^{ri}} \rho^{(s)}(v) dv. \end{aligned}$$

follows that

$$|(s-1)![(x^r)^{-s}]_n| \le \sum_{i=s}^{\infty} \int_{-1}^{1} \frac{|v|^i}{in^{i-s}x^{ri}} |\rho^{(s)}(v)|\,\mathrm{d}v \le \sum_{i=s}^{\infty} \frac{K_s}{in^{i-s}x^{ri}},$$

where

$$K_s = \int_{-1}^{1} |\rho^{(s)}(v)|\,\mathrm{d}v,$$

for $s = 1, 2, \ldots$.

If now $n^{-1/r} < \eta < 1$, then

$$(s-1)! \int_{n^{-1/r}}^{\eta} |x^{rs}[(x^r)^{-s}]_n|\,\mathrm{d}x \le K_s \sum_{i=s}^{\infty} \frac{n^{s-i}}{i} \int_{n^{-1/r}}^{\eta} x^{r(s-i)}\mathrm{d}x$$

$$= K_s \sum_{i=s}^{\infty} \frac{n^{-1/r}}{ri} \int_{1}^{n\eta^r} u^{s-i+1/r-1}\,\mathrm{d}u$$

$$= \begin{cases} K_s \displaystyle\sum_{i=s}^{\infty} \frac{n^{-1/r}}{ri(s-i+1/r)}[(n\eta^r)^{s-i+1/r} - 1], & r \ne 1, \\ K_s \displaystyle\sum_{i=s,\, i\ne s+1}^{\infty} \frac{n^{-1}}{i(s-i+1)}[(n\eta)^{s-i+1} - 1] + K_s \frac{n^{-1}\ln(n\eta)}{s+1}, & r = 1. \end{cases}$$

follows that

$$\lim_{n\to\infty} \int_{n^{-1/r}} |[(x^r)^{-s}]_n|\,\mathrm{d}x = O(\eta),$$

for $r, s = 1, 2, \ldots$.

Thus, if ψ is a continuous function

$$\lim_{n\to\infty} \left| \int_{n^{-1/r}}^{\eta} x^{rs}[(x^r)^{-s}]_n \psi(x)\mathrm{d}x \right| = O(\eta), \tag{12}$$

for $r, s = 1, 2, \ldots$.

Similarly,

$$\lim_{n\to\infty} \left| \int_{-\eta}^{-n^{-1/r}} x^{rs}[(x^r)^{-s}]_n \psi(x)\mathrm{d}x \right| = O(\eta), \tag{13}$$

for $r, s = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's theorem we have

$$\varphi(x) = \sum_{k=0}^{rs-1} \frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^{rs}}{(rs)!}\varphi^{(rs)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\langle [(x^r)^{-s}]_n, \varphi(x) \rangle = \int_{-1}^{1} [(x^r)^{-s}]_n \varphi(x)\mathrm{d}x$$

$$= \sum_{k=0}^{rs-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k[(x^r)^{-s}]_n\,\mathrm{d}x + \int_{-n^{-1/r}}^{n^{-1/r}} \frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n \varphi^{(rs)}(\xi x)\mathrm{d}x$$

$$+\int_{n^{-1/r}}^{\eta}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x+\int_{\eta}^{1}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x$$
$$+\int_{-\eta}^{-n^{-1/r}}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x+\int_{-1}^{-\eta}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x.$$

Using (11) to (15) and noting that the sequence $[(x^r)^{-s}]_n$ converges uniformly to x^{-rs} on the intervals $[-1,-\eta]$ and $[\eta,1]$, it follows that

$$\begin{aligned}N\text{-}\lim_{n\to\infty}\langle[(x^r)^{-s}]_n,\varphi(x)\rangle &= \sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)+O(\eta)\\ &\quad+\int_{\eta}^{1}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x+\int_{-1}^{-\eta}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x.\end{aligned}$$

Since η can be made arbitrarily small, it follows that

$$\begin{aligned}N\text{-}\lim_{n\to\infty}\langle[(x^r)^{-s}]_n,\varphi(x)\rangle &= \sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)+\int_{-1}^{1}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x\\ &= \int_{-1}^{1}x^{-rs}\left[\varphi(x)-\sum_{k=0}^{rs-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]\mathrm{d}x\\ &\quad+\sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)\\ &= \langle x^{-rs},\varphi(x)\rangle,\end{aligned}$$

on using (1). This proves (3) on the interval $[-1,1]$. However, (3) clearly holds on any interval not containing the origin, and the proof is complete.

In the corollary, the distribution $(x+i0)^{-s}$ is defined by

$$(x+i0)^{-s}=x^{-s}+\frac{i\pi(-1)^s}{(s-1)!}\delta^{(s-1)}(x), \tag{14}$$

for $s=1,2,\ldots$, see Gel'fand and Shilov [5].

COROLLARY 3.1

The distribution $(x^r+i0)^{-s}$ *exists and*

$$(x^r+i0)^{-s}=(x+i0)^{-rs} \tag{15}$$

for $r=1,3,5,\ldots$ *and* $s=1,2,\ldots$ *and*

$$(x^r+i0)^{-s}=x^{-rs}, \tag{16}$$

for $r=2,4,6,\ldots$ *and* $s=1,2,\ldots$.

Proof. It was proved in [2] that

$$\delta^{(s)}(x^r)=\frac{s!}{r(rs+r-1)!}\delta^{(rs+r-1)}(x), \tag{17}$$

for $r=1,3,5,\ldots$ and $s=0,1,2,$ and

$$\delta^{(s)}(x^r)=0, \tag{18}$$

for $r = 2, 4, 6, \ldots$ and $s = 0, 1, 2, \ldots$. Equation (15) follows immediately from (3), (14) and (17) and (16) follows immediately from (3), (14) and (18).

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Existence of solutions of neutral functional integrodifferential equation in Banach spaces

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Abstract. In this paper we prove the existence of mild solutions for neutral functional integrodifferential equation in a Banach space. The results are obtained by using the Schaefer fixed-point theorem.

Keywords. Neutral integrodifferential equation; Schaefer fixed point theorem.

1. Introduction

The theory of neutral delay differential equations has been extensively studied in the literature [1–3, 6, 8]. Recently Hernandez and Henriquez [4] obtained some existence results for neutral functional differential equations in Banach spaces. In [5] they have established the existence of periodic solutions for the same kind of equations. In both papers they have used the semigroup theory and the Sadovski fixed point principle.

The purpose of this paper is to prove the existence of mild solutions for neutral functional integrodifferential equation of the form

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-g(t,x_t)] = Ax(t)+\int_0^t f(s,x_s)\mathrm{d}s, \quad t\in J=[0,b],\\ x_0=\phi, \quad \text{on } [-r,0],\end{gathered} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in $X, f: J\times C\to X$ and $g: J\times C\to X$ are continuous functions. Here $C=C([-r,0],X)$ is the Banach space of all continuous functions $\phi:[-r,0]\to X$ endowed with the norm $\|\phi\|=\sup\{|\phi(\theta)|: -r\leq\theta\leq 0\}$. Also for $x\in C([-r,b],X)$ we have $x_t\in C$ for $t\in[0,b]$, $x_t(\theta)=x(t+\theta)$ for $\theta\in[-r,0]$.

2. Preliminaries

In order to define the concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$x'(t)=Ax(t)+f(t),$$

whose properties are well-known [9], we associate problem (1) to the integral equation

$$\begin{aligned}x(t) = {} & \lambda T(t)[\phi(0)-g(0,\phi)]+\lambda g(t,x_t)+\lambda\int_0^t AT(t-s)g(s,x_s)\mathrm{d}s\\ & +\lambda\int_0^t T(t-s)\int_0^s f(\tau,x_\tau)\mathrm{d}\tau\,\mathrm{d}s, \quad t\in[0,b].\end{aligned} \tag{2}$$

DEFINITION

A function $x : (-r, b) \to X$, $b > 0$, is called a mild solution of the Cauchy problem (1) if $x_0 = \phi$; the restriction of $x(.)$ to the interval $[0, b)$ is continuous and for each $0 \leq t < b$ the function $AT(t-s)F(s, x_s)$, $s \in [0, t)$, is integrable and the integral equation (2) is satisfied.

We need the following fixed point theorem due to Schaefer [10].

Schaefer Theorem. *Let S be a convex subset of a normed linear space E and $0 \in S$. Let $F : S \to S$ be a completely continuous operator and let*

$$\zeta(F) = \{x \in S;\ x = \lambda Fx \quad \text{for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that:

(i) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in X such that

$$|T(t)| \leq M_1, \quad \text{for some } M_1 \geq 1 \quad \text{and } |AT(t)| \leq M_2, M_2 \geq 0.$$

(ii) For each $t \in J$ the function $f(t, .) : C \to X$ is continuous and for each $x \in C$ the function $f(., x) : J \to X$ is strongly measurable.

(iii) For every positive integer k there exists $\alpha_k \in L^1(0, b)$ such that

$$\sup_{\|x\| \leq k} |f(t, x)| \leq \alpha_k(t), \quad \text{for } t \in J \text{ a.e.}$$

(iv) The function g is completely continuous and such that the operator

$$G : C([-r, 0], X) \to C([0, b], X)$$

defined by $(G\phi)(t) = g(t, \phi)$ is compact.

(v) There exists constants $c_1 < 1$ and $c_2 > 0$ such that

$$|g(t, \phi)| \leq c_1 \|\phi\| + c_2, \quad t \in J, \quad \phi \in C.$$

(vi) There exists an integrable function $m : [0, b] \to [0, \infty)$ such that

$$|f(t, \phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in C,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function.

(vii)

$$\int_0^b \hat{m}(s) \mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s + \Omega(s)},$$

where

$$c = \frac{1}{1 - c_1}[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + M_2 c_2 b],$$

and

$$\hat{m}(t) = \max\{M_2 c_1/(1 - c_1),\ M_1 m(t)/M_2 c_1\}.$$

3. Main result

Theorem. *If the assumptions (i) to (vii) are satisfied, then the problem (1) has a mild solution on $[-r, b]$.*

Proof. To prove the existence of a mild solution of (1) we apply Schaefer theorem. First we obtain *a priori* bounds for the solutions of the problem (3) as in [7]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\lambda g(t,x_t)] = \lambda Ax(t) + \lambda \int_0^t f(s,x_s)\mathrm{d}s, \quad \lambda \in (0,1)\ t \in J = [0,b],$$
$$x_0 = \phi. \tag{3}$$

Let x be a mild solution of the problem [2]. From

$$x(t) = \lambda T(t)[\phi(0) - g(0,\phi)] + \lambda g(t,x_t) + \lambda \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s$$
$$+\lambda \int_0^t T(t-s)\int_0^s f(\tau,x_\tau)\mathrm{d}\tau\,\mathrm{d}s,$$

we have

$$|x(t)| \le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|x_t\| + c_2 + M_2\int_0^t (c_1\|x_s\| + c_2)\mathrm{d}s$$
$$+ M_1 \int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\|)\mathrm{d}\tau\,\mathrm{d}s.$$

We consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,b]$ by the previous inequality we have

$$\mu(t) \le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2c_1\int_0^{t^*}\mu(s)\mathrm{d}s$$
$$+ M_2c_2b + M_1\int_0^{t^*}\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$\le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2c_1\int_0^t \mu(s)\mathrm{d}s$$
$$+ M_2c_2b + M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s$$

or

$$\mu(t) \le \frac{1}{1-c_1}\Big\{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + M_2c_2b$$
$$+ M_2c_1\int_0^t \mu(s)\mathrm{d}s + M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s\Big\}.$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \ge 1$.

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$c = v(0) = \frac{1}{1-c_1}\{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + M_2c_2b\},$$

$\mu(t) \le v(t), 0 \le t \le b$ and

$$v'(t) = \frac{1}{1-c_1}M_2c_1\mu(t) + \frac{1}{1-c_1}M_1\int_0^t m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau$$

$$\leq \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau$$
$$\leq \frac{1}{1-c_1} M_2 c_1 \left\{ v(t) + \frac{M_1}{M_2 c_1} \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau \right\}.$$

Let

$$w(t) = v(t) + \frac{M_1}{M_2 c_1} \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau.$$

Then $w(0) = v(0)$, $v(t) \leq w(t)$, and

$$w'(t) = v'(t) + \frac{M_1}{M_2 c_1} m(t)\Omega(v(t))$$
$$\leq \frac{1}{1-c_1} M_2 c_1 w(t) + \frac{M_1}{M_2 c_1} m(t)\Omega(w(t))$$
$$\leq \hat{m}(t)\{w(t) + \Omega(w(t))\}.$$

This implies

$$\int_{w(0)}^{w(t)} \frac{\mathrm{d}s}{s+\Omega(s)} \leq \int_0^b \hat{m}(s)\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s+\Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K$, $t \in [0,b]$ and hence $\mu(t) \leq K$, $t \in [0,b]$, $\|x_t\| \leq \mu(t)$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions m and Ω.

In the second step we rewrite the problem (1) as follows. For $\phi \in C$ define $\hat{\phi} \in C_b$, $C_b = C([-r,b],X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ T(t)\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r,b]$, it is easy to see that y satisfies

$$y_0 = 0$$
$$y(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s$$
$$+ \int_0^t T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

if and only if x satisfies

$$x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s$$
$$+ \int_0^t T(t-s) \int_0^s f(\tau, x_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

and $x_0 = \phi$.

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and $F : C_b^0 \to C_b^0$, by

$$(Fy)(t) = 0, \quad -r \le t \le 0$$

$$(Fy)(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s$$
$$+ \int_0^t T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s, \quad 0 \le t \le b.$$

It will now be shown that F is a completely continuous operator.

Let $B_k = \{y \in C_b^0 : \|y\|_1 \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$. Then if $0 < t_1 < t_2 < b$,

$$\begin{aligned}
|(Fy)(t_1) - (Fy)(t_2)| &\le |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + \left|\int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]g(s, y_s + \hat{\phi}_s)\mathrm{d}s\right| \\
&\quad + \left|\int_{t_1}^{t_2} AT(t_2 - s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s\right| \\
&\quad + \left|\int_0^{t_1} [T(t_1 - s) - T(t_2 - s)]\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s\right| \\
&\quad + \left|\int_{t_1}^{t_2} T(t_2 - s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s\right| \\
&\le |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + \int_0^{t_1} |A[T(t_1 - s) - T(t_2 - s)]|(c_1\|y_s + \hat{\phi}_s\| + c_2)\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |AT(t_2 - s)|(c_1\|y_s + \hat{\phi}_s\| + c_2)\mathrm{d}s| \\
&\quad + \int_0^{t_1} |T(t_1 - s) - T(t_2 - s)|\int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |T(t_2 - s)|\int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

where $k' = k + \|\hat{\phi}\|$. The right hand side is independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since g is completely continuous and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_k is uniformly bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela–Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$ we define

$$(F_\epsilon y)(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^{t-\epsilon} AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s$$
$$+ \int_0^{t-\epsilon} T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

$$= -T(t)g(0,\phi)+g(t,y_t+\hat{\phi}_t)+T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)g(s,y_s+\hat{\phi}_s)\mathrm{d}s$$

$$+T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)\int_0^s f(\tau,y_\tau+\hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t)=\{(F_\epsilon y)(t) : y\in B_k\}$ is precompact in X for every ϵ, $0<\epsilon<t$. Moreover for every $y\in B_k$ we have

$$\begin{aligned}|(Fy)(t)-(F_\epsilon y)(t)| &\le \int_{t-\epsilon}^t |AT(t-s)g(s,y_s+\hat{\phi}_s)|\mathrm{d}s\\ &\quad+\int_{t-\epsilon}^t |T(t-s)\int_0^s f(\tau,y_\tau+\hat{\phi}_\tau)\mathrm{d}\tau|\mathrm{d}s\\ &\le \int_{t-\epsilon}^t |AT(t-s)g(s,y_s+\hat{\phi}_s)|\mathrm{d}s\\ &\quad+\int_{t-\epsilon}^t |T(t-s)|\int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fy)(t) : x\in B_k\}$. Hence the set $\{(Fy)(t) : y\in B_k\}$ is precompact in X.

It remains to show that $F : C_b^0\to C_b^0$ is continuous. Let $\{y_n\}_0^\infty\subseteq C_b^0$ with $y_n\to y$ in C_b^0. Then there is an integer r such that $\|y_n(t)\|\le r$ for all n and $t\in J$, so $y_n\in B_r$ and $y\in B_r$. By (iii), $f(t,y_{n_t}+\hat{\phi}_t)\to f(t,y_t+\hat{\phi}_t)$ for each $t\in J$ and since $|f(t,y_{n_t}+\hat{\phi}_t)-f(t,y_t+\hat{\phi}_t|$ $\le 2\alpha_{r'}(t)$, $r'=r+\|\hat{\phi}\|$ and also g is completely continuous, we have by dominated convergence theorem

$$\begin{aligned}\|Fy_n-Fy\| &= \sup_{t\in J}|[g(t,y_{n_t}+\hat{\phi}_t)-g(t,y_t+\hat{\phi}_t)]\\ &\quad+\int_0^t AT(t-s)[g(s,y_{n_s}+\hat{\phi}_s)-g(s,y_s+\hat{\phi}_s)]\mathrm{d}s\\ &\quad+\int_0^t T(t-s)\int_0^s [f(\tau,y_{n_\tau}+\hat{\phi}_\tau)-f(\tau,y_\tau+\hat{\phi}_\tau)]\mathrm{d}\tau\,\mathrm{d}s|\\ &\le |g(t,y_{n_t}+\hat{\phi}_t)-g(t,y_t+\hat{\phi}_t)|\\ &\quad+\int_0^b |AT(t-s)||g(s,y_{n_s}+\hat{\phi}_s)-g(s,y_s+\hat{\phi}_s)|\mathrm{d}s\\ &\quad+\int_0^b |T(t-s)|\int_0^s |f(\tau,y_{n_\tau}+\hat{\phi}_\tau)-f(\tau,y_\tau+\hat{\phi}_\tau)|\mathrm{d}\tau\,\mathrm{d}s\to 0.\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F)=\{y\in C_b^0 : y=\lambda Fy, \lambda\in(0,1)\}$ is bounded, since for every solution y in $\zeta(F)$ the function $x=y+\hat{\phi}$ is a mild solution of (3), for which we have proved that $\|x\|_1\le K$ and hence

$$\|y\|_1\le K+\|\hat{\phi}\|.$$

Consequently by Schaefer's theorem the operator F has a fixed point in C_b^0. This means that the problem (1) has a mild solution.

4. Example

Consider the following partial integrodifferential equation of the form

$$\frac{\partial}{\partial t}[z(y,t) - p(s,z(y,t-r))] = \frac{\partial^2}{\partial y^2} z(y,t) + \int_0^t q(s,z(y,s-r))\mathrm{d}s,$$
$$0 \le y \le \pi, \quad t \in J \tag{4}$$
$$z(0,t) = z(\pi,t) = 0, \quad t \ge 0$$
$$z(t,y) = \phi(y,t), \quad -r \le t \le 0$$

where ϕ is continuous. Let

$$f(t,w_t)(y) = q(t,w(t-y)), \quad 0 \le y \le \pi$$

and

$$g(t,w_t)(y) = p(t,w(t-y)).$$

Take $X = L^2[0,\pi]$ and define $A : X \to X$ by $Aw = w''$ with domain $D(A) = \{w \in X, w, w'$ are absolutely continuous, $w'' \in X$, $w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w,w_n)w_n, \quad w \in D(A)$$

where $w_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1,2,3,\ldots$ is the orthogonal set of eigenvectors of A. It is well-known that A is the infinitesimal generator of an analytic semigroup $T(t), t \ge 0$ in X and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w,w_n)w_n, \quad w \in X.$$

Since the analytic semigroup $T(t)$ is compact there exist constants $N \ge 1$ and $N_1 > 0$ such that $|T(t)| \le N$ and $|AT(t)| \le N_1$ for each $t \ge 0$.

Further the function $p : J \times [0,\pi] \to [0,\pi]$ is completely continuous and there exist constants $n_1 < 1$ and $n_2 > 0$ such that

$$\|p(t,w(t-y))\| \le n_1(\|w\|) + n_2,$$

and also there exists an integrable function $l : J \to [0,\infty)$ such that

$$\|q(t,w(t-y))\| \le l(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing and

$$\int_0^b \hat{n}(s)\mathrm{d}s < \int_c^{\infty} \frac{\mathrm{d}s}{s + \Omega_1(s)},$$

where

$$c = \frac{1}{1-n_1}[N(\|\phi\| + n_1\|\phi\| + n_2) + n_2 + N_1 n_2 b]$$

and

$$\hat{n}(t) = \max\{N_1 n_1/(1-n_1), Nl(t)/N_1 n_1\}.$$

Since all the conditions of the above theorem are satisfied, the system (4) has a mild solution on $[-r, b]$.

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On hypothesis K for biquadrates

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Abstract. It is proved that the number of ways of expressing a large positive integer n as the sum of four biquadrates is

$$O\left[\frac{n^{1/2}(\log\log n)^8}{\log n}\right].$$

Keywords. Biquadrates; Waring's problem.

In order to advance further their researches on Waring's problem, Hardy and Littlewood in the sixth memoir [5] of their series on Partitio Numerorum introduced their hypothesis K to the effect that, if $r_{l,k}(n)$ denote the number of representations of n as the sum of l non-negative kth powers, then

$$r_k(n) = r_{k,k}(n) = O(n^{\epsilon}).$$

Since, however, it was applied through the consequential but weaker relation

$$\sum_{m\leq x} r_k^2(n) = O[x^{1+\epsilon}], \tag{0}$$

it has been the latter hypothesis rather than the former that has engaged the attention of most scholars studying the usual problems of Waring's type. Inevitable in the case $k=3$ for which hypothesis K was disproved by Mahler [13], this tendency has anyway been strengthened by the perception that at present the best way of bounding the left-side of (0) is not to substitute individual estimates for $r_k(n)$ into it. Indeed, in the only known case $k=3$ where hypothesis K is known to fail, the relation (0) has been shown to be true by Heath–Brown [6] and the author [12] independently on the assumption of the Riemann hypothesis for Hasse–Weil L-functions defined over certain cubic three-folds. Also, for completeness, we should mention that an exhaustive discussion of lower bounds for the sum in (0) is given for all $k>2$ in our paper [10].

Notwithstanding its current failure to meet the problems it was introduced to settle, hypothesis K is in itself the subject of an interesting aspect of Waring's problem. The ultimate object being still to determine whether the conjecture be true for any $k>3$, it is desirable in the meantime to seek out as much information as possible about $r_k(n)$. So far as lower bounds are concerned, we already have Erdös's relation [1]

$$r_k(n) = \Omega[n^{A/\log\log n}]$$

that shows that hypothesis K would be almost best possible when true. On the other side, the problem of approaching $r_k(n)$ from above seems much harder, so much so that any upper bound better than the relatively trivial would be a useful starting point in our studies on the question.

In the present paper we treat the case $k = 4$, where the bound

$$r_4(n) = O[n^{(1/2)+\epsilon}]$$

is an obvious consequence of the representation of $r_4(n)$ as

$$\sum_{x^4+Y^4\le n} r_{2,4}(n - x^4 - Y^4)$$

and of the relation $r_{2,4}(m) = O\{(m+1)^\epsilon\}$ that is restated in (2) below. Here the ϵ in the exponent can be removed by a slight advance that involves expressing $r_4(n)$ as

$$\sum_{0\le m\le n} r_{2,4}(m)r_{2,4}(n-m) \le \sum_{0\le m\le n} r_{2,4}^2(m)$$

and an appeal to Greaves's asymptotic formula ([3], see also our paper [11] that is cited later). But this estimate is almost devoid of interest, not so much because of its weakness but because it has been attained without taking into account any special features relating to the expression of a number as the sum of four biquadrates. Accordingly, we now look more carefully at the estimation of $r_4(n)$ and prove that

$$r_4(n) = O\left[\frac{n^{1/2}(\log\log n)^8}{\log n}\right].$$

Doubtless this is far from what is best possible but represents the first step into territory beyond that delineated by the solitary properties of the arithmetical function $r_{2,4}(n)$.

Since the structure of the method will be abundantly clear from the sequel, it suffices for now to state that it depends on two differences between the arithmetical properties of the polynomials $u^4 + v^4$ and $n - u^4 - v^4$.

In what follows, the letters l, m, X, Y, Z, W denote non-negative integers; p is a positive prime number; n is an integer that is to be regarded as tending to infinity, all stated inequalities being valid when it is sufficiently large; $A_1, A_2, \ldots$ are positive absolute constants; ϵ is a (small) positive number which is not necessarily the same at each occurrence and on which the constants implied by the O-notation at most depend. The notations $p|l$ and $p\|l$ have their customary meanings, namely, respectively, that p divides l and that p is the highest power of p that divides l; $p \not\| l$ is the negation of $p\|l$; $d\|l$ denotes the conjunction of $p\|l$ for all prime divisors of p of a square-free number d, a condition that is sometimes re-expressed as a quasi-congruence $l \doteqdot 0, \bmod d$, whose solutions for given d arrange themselves in residue classes, $\bmod d^2$. The functions $d(m)$ and $\omega(m)$ are, respectively, the number of divisors of m and the number of distinct prime factors of m.

The main property of the arithmetical function $r(n) = r_{2,4}(n)$ required below is established in our paper [11], although an equally suitable but weaker version was derived in our anterior paper [9] without recourse to any application of Deligne's theories to the estimation of exponential sums. This is that, if $\nu(x)$ be the number of integers not exceeding x that are expressible as the sum of two biquadrates in essentially more than

one way, then

$$\nu(x) = O[x^{(5/12)+\epsilon}], \tag{1}$$

to which statement we append the obvious relation

$$r(m) \le r_2(m) \le 4d(m) = O(m^\epsilon) \quad (m > 0) \tag{2}$$

that will be needed on several occasions.

The mechanism of the treatment depends on our introducing a number

$$N = n^{\frac{A}{\log\log n}} \tag{3}$$

defined in terms of a suitably chosen small positive constant[1] A, by means of which we bring in the special function $\omega_n(m)$ that is the number of distinct prime divisors p of m that satisfy the conditions

$$p \le N; \quad p \nmid n; \quad p \equiv 1, \quad \bmod 8. \tag{4}$$

To direct this instrument to the problem clearly, we shall consider instead of $r_4(n)$ the number $R(n)$ of solutions of

$$X^4 + Y^4 + Z^4 + W^4 = n \tag{5}$$

for which neither $X^4 + Y^4$ nor $Z^4 + W^4$ is zero, where

$$r_4(n) = R(n) + O(n^\epsilon) \tag{6}$$

because the donation of the excluded solutions to $r_4(n)$ is $2r(n) = O(n^\epsilon)$ by (2). Then first $R(n)$ itself obviously does not exceed twice the number $R^*(n)$ of solutions of (5) for which both

$$X^4 + Y^4, \quad Z^4 + W^4 > 0 \quad \text{and} \quad \omega_n(X^4 + Y^4) \ge \omega_n(Z^4 + W^4). \tag{7}$$

Next, for an appropriate value of A_1,

$$R^*(n) = R^\dagger(n) + R^\$(n), \tag{8}$$

in which $R^\dagger(n)$ is the part of $R^*(n)$ answering to the condition

$$\omega_n(X^4 + Y^4) \le A_1 \log\log n = N_1, \text{ say}, \tag{9}$$

and $R^\$(n)$ is the residual part that is less than or equal to

$$\begin{aligned}\sum_{\substack{0<m<n\\ \omega_n(n-m)>N_1}} r(n-m)r(m) &\le 2 \sum_{\substack{0<m<n\\ \omega_n(n-m)>N_1}} r(n-m) + \sum_{\substack{0<m<n\\ r(m)>2}} r(n-m)r(m)\\ &= 2 \sum_{\substack{0<m<n\\ \omega_n(n-m)>N_1}} r(n-m) + O[n^\epsilon \nu(n)]\\ &= 2 \sum_{\substack{0<m<n\\ \omega_n(n-m)>N_1}} r(n-m) + O[n^{(5/12)+\epsilon}] \end{aligned} \tag{10}$$

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[1] It may be helpful to indicate that the later constants A_1 and A_4 must be taken large enough for certain key inequalities to be valid. The choice of a sufficiently small A then ensures the truth of other vital relations.

by (2) and (1). But the last sum above is

$$\sum_{\substack{0<X^4+Y^4<n\\ \omega_n(X^4+Y^4)>N_1}} 1 < 2^{-N_1} \sum_{0<X^4+Y^4<n} d(X^4+Y^4) = 0[n^{1/2}\log^{(1-A_1\log 2)} n]$$

$$= O\left[\frac{n^{1/2}}{\log n}\right] \tag{11}$$

by the asymptotic formula

$$\sum_{0<X^4+Y^4\le n} \mathrm{d}(X^4+Y^4) \sim A_2 n^{1/2}\log n$$

that is implicit in the penultimate paragraph of our paper [8]; alternatively, more elementarily but perhaps less accessibly, we may equally well use the bounds $O[n^{1/2}\log^{A_3} n]$ or $O[n^{1/2}\log n]$ for the divisor sum that are obtainable, respectively, by methods due to van der Corput [14] or Erdös [2]. Thus, in summation of what has so far been achieved, we complete the first stage of the estimation by deducing the inequality

$$r_4(n) \le 2R^{\dagger}(n) + O\left[\frac{n^{1/2}}{\log n}\right] \tag{12}$$

from (6), (8), (10) and (11).

Going on to the primary element $R^{\dagger}(n)$ in $r_4(n)$ and letting the function $\rho(n)$ be 1 or 0 according as n is or is not a sum of two biquadrates, we first have from (5), (7) and (9) that

$$\begin{aligned} R^{\dagger}(n) &= \sum_{\substack{0<m<n\\ \omega_n(m)\le\omega_n(n-m)\le N_1}} r(n-m)r(m) \\ &\le 2 \sum_{\substack{0<m<n\\ \omega_n(m)\le\omega_n(n-m)\le N_1}} \rho(n-m)r(m) + \sum_{\substack{0<m<n\\ r(n-m)>2}} r(n-m)r(m) \\ &= 2R^{\ddagger}(n) + O[n^{(5/12)+\epsilon}], \quad \text{say,} \end{aligned} \tag{13}$$

in which the remainder term has been derived as in (10). Secondly, for integers s, t such that

$$0 \le s \le t \le N_1, \tag{14}$$

let

$$R^{\ddagger}_{s,t}(n) = \sum_{\substack{0<m<n\\ \omega_n(m)=s;\,\omega_n(n-m)=t}} \rho(n-m)r(m) \tag{15}$$

and write

$$R^{\ddagger}(n) = \sum_{0\le s\le t\le N_1} R^{\ddagger}_{s,t}(n). \tag{16}$$

Each sum $R^{\ddagger}_{s,t}(n)$ is then itself split up into sums $S(P_s, Q_t) = S(P_s, Q_t, n)$ that are the contributions due to values of m for which the square-free numbers $P_s = p_1 \cdots p_s$ and $Q_t = q_1 \cdots q_t$ are the respective products (possibly empty) of all the primes of type (4) that divide m and $n-m$, this being the final step in our decomposition of $r_4(n)$ into other sums.

To examine $S(P_s, Q_t)$ we partially revert to an earlier notation by replacing m by $Z^4 + W^4$ with the consequent disappearance of the function $r(m)$, deducing that the conditions of summation in $S(P_s, Q_t)$ imply that

$$Z^4 + W^4 = P_s l_1, \quad n - Z^4 - W^4 = Q_t l_2, \tag{17}$$

where l_1 must be indivisible by any prime p of type (4) not dividing P_s and where l_2 likewise must be indivisible by any such prime not dividing Q_t. But, owing to the middle condition in (4), we see that

$$(P_s, Q_t) = 1 \tag{18}$$

and that automatically $(l_1, Q_t) = (l_2, P_s) = 1$. Hence, without loss, we may merely impose the stipulation that $l_1 l_2$ be indivisible by primes p satisfying the conditions

$$(4) \quad \text{and} \quad p \nmid P_s Q_t, \tag{19}$$

any square-free product of which (including 1) will be later denoted by d^*. Furthermore, we note for future convenience that (3), (4), (9) and (14) imply the inequalities

$$P_s, Q_t \leq N^{N_1} = n^{AA_1} = n^{1/100}. \tag{20}$$

The limitations on l_1 and l_2 identified above will be exploited by means of a sieve method. Somewhat similarly, the restrictive effect of the latent presence of the function $\rho(n - Z^4 - W^4)$ in the definition of $R^{\ddagger}_{s,t}(n)$ will be measured by another sieve method that takes advantage of a simple criterion for the non-vanishing of $\rho(l)$. Suppose that $l = X^4 + Y^4$ and that $p \| l$. Then the congruence $X^4 + Y^4 \equiv 0, \bmod p$, has a solution for which $p \nmid Y$ so that the congruence $\Omega^4 + 1 \equiv 0, \bmod p$, is soluble and thus $p \equiv 1, \bmod 8$ when p is odd. Hence the numbers l representable as the sum of two biquadrates are contained in the set of *all* integers m (including zero) with the feature that $p \not\| m$ for any odd prime p in a suitable set of $\mathcal{S}$ of primes that are incongruent to 1, mod 8. Having then indicated that we shall find it convenient here to choose $\mathcal{S}$ to consist of the primes p answering to the conditions

$$p \leq N; \quad p \nmid 2n; \quad p \not\equiv 1, \bmod 8, \tag{21}$$

and then to denote any square-free product of such primes by δ, we are ready to activate some simple sieving machinery through two lemmata of a familiar type. First we have

Lemma 1. *Let $s_1, \ldots, s_r$ be the elementary symmetric functions of degrees $1, \ldots, r$, respectively, in numbers $a_1, \ldots, a_r$ that lie between 0 and 1 (in the non-strict sense), it being understood that $s_j = 0$ for $j > r$. Then*

$$1 + \sum_{0<i\leq u} (-1)^i s_i - \prod_{0<i\leq r} (1 - a_i)$$

is non-positive or non-negative according as the positive integer u is odd or even. Hence

$$0 \leq 1 + \sum_{0<i\leq u} (-1)^i s_i \leq \prod_{0<i\leq r} (1 - a_i) + s_{u+1}$$

when u is even.

Easily proved by induction, this proposition is the basis of the following easy form of Brun's method that has been named *Brun's Pure Sieve* by Halberstam and Richert in their monograph [4].

Lemma 2. Let $\mathcal{T}$ denote a finite set of prime numbers to each of which there is attached a property that a specified type of mathematical entity E may or may not have, a typical square-free product of such primes being denoted by d. Also, let the notation $d \cdot | \cdot E$ mean that E possesses all the properties appertaining to the prime factors of d. Then, for any even positive integer u,

$$\sum_{\substack{d\cdot|\cdot E\\ \omega(d)\le u}} \mu(d)$$

is a non-negative function of E that equals 1 when E does not enjoy any of the properties related to $\mathcal{T}$.

More or less described in the language of our tract [7], this version of a well-known result follows from the previous lemma by taking $a_1 = \cdots = a_r = 1$ in the case where the properties possessed by E correspond to precisely $r > 0$ primes in $\mathcal{T}$.

Returning to the argument before the lemmata, we have shown that $S(P_s, Q_t)$ is not more than the number of solutions of (17) for which $0 < Z^4 + W^4 < n$ and for which Z, W have been contained by two sieving processes that permit us to relax the former condition to

$$Z, W \le n^{1/4} \tag{22}$$

when determining an upper bound. The first sifting of the pairs Z, W appertains to the divisibility properties of l_1 and l_2 and is brought into the analysis by means of the majorant weight[2]

$$\sum_{\substack{d^*|l_1 l_2\\ \omega(d^*)\le N_2}} \mu(d^*) \tag{23}$$

of Lemma 2, where

$$N_2 = 2[A_4 \log\log n] \tag{24}$$

is an even number determined by a suitably large value of A_4. Similarly, the needs of the second sifting process are catered for by the majorant

$$\sum_{\substack{\delta\|(n-Z^4-W^4)\\ \omega(\delta)\le N_2}} \mu(\delta), \tag{25}$$

the interpretation of which depends on our earlier convention concerning the meaning of $\delta\|m$ for composite (square-free) numbers δ. In conclusion, therefore, $S(P_s, Q_t)$ is bounded above by the sum $T(P_s, Q_t)$ of the product of (23) and (25) taken over all solutions of (17) that conform to (22).

Some further preparations are needed before the detailed estimation of $T(P_s, Q_t)$ begins. First, (20) is complemented by the inequalities

$$d^* \le n^{1/100}, \quad \delta \le n^{1/100} \tag{26}$$

that stem from (3), (24), (19), (21), and the conditions of summation in (23) and (25) when A is sufficiently small; secondly, since (17) and (19), respectively, imply that

[2] Note that $\omega(d^*) = \omega_n(d^*)$.

$(l_1, l_2)|n$ and $(d^*, n) = 1$, the condition $d^*|l_1 l_2$ in (23) is tantamount to the unique expression of d^* as $d_1^* d_2^*$, where $d_1^*|l_1$ and $d_2^*|l_2$; lastly, we should note that the solutions of the quasi-congruence

$$n - Z^4 - W^4 \doteqdot 0, \quad \bmod \delta,$$

fall into a number $\tau(\delta)$, say, of complete residue class pairs, mod δ^2, where plainly $\tau(\delta)$ is multiplicative in the square-free argument δ. Then, recognizing $T(P_s, Q_t)$ as a triple sum and arranging that the outer summations are over d^* and δ, we have

$$T(P_s, Q_t) = \sum_{\omega(d_1^* d_2^*), \omega(\delta) \le N_2} \mu(d_1^* d_2^*)\mu(\delta) Z(P_s, Q_t, d_1^*, d_2^*, \delta), \tag{27}$$

where $Z(P_s, Q_t, d_1^*, d_2^*, \delta)$ is the number of solutions in Z, W of the conditions

$$Z^4 + W^4 \equiv 0, \bmod P_s d_1^*; \quad n - Z^4 - W^4 \equiv 0, \bmod Q_t d_2^*;$$
$$n - Z^4 - W^4 \doteqdot 0, \bmod \delta, \tag{28}$$

that are constrained by (22). Next, since all the relevant moduli are coprime owing to (19) and (21), the solutions of (28) alone are congruent, $\bmod P_s Q_t d_1^* d_2^* \delta^2$, to $\psi_1(P_s d_1^*)$, $\psi_2(Q_t d_2^*)\tau(\delta)$ incongruent pairs B, C, $\bmod P_s Q_t d_1^* d_2^* \delta^2$, where $\psi_1(k_1)$ and $\psi_2(k_2)$ are, respectively, the numbers of incongruent solutions of $Z^4 + W^4 \equiv 0$, $\bmod k_1$, and $n - Z^4 - W^4 \equiv 0$, $\bmod k_2$. Therefore

$$\begin{aligned} Z(P_s, Q_t, d_1^*, d_2^*, \delta) &= \sum_{B,C} \left[\sum_{\substack{Z \le n^{1/4} \\ Z \equiv B, \bmod P_s Q_t d_1^* d_2^* \delta^2}} 1 \right] \left[\sum_{\substack{W \le n^{1/4} \\ W \equiv C, \bmod P_s Q_t d_1^* d_2^* \delta^2}} 1 \right] \\ &= \sum_{B,C} \left[\frac{n^{1/4}}{P_s Q_t d_1^* d_2^* \delta^2} + O(1) \right] \left[\frac{n^{1/4}}{P_s Q_t d_1^* d_2^* \delta^2} + O(1) \right] \\ &= \frac{n^{1/2} \psi_1(P_s d_1^*) \psi_2(Q_t d_2^*) \tau(\delta)}{P_s^2 Q_t^2 d_1^{*2} d_2^{*2} \delta^4} + O\left[\frac{n^{1/4} \psi_1(P_s d_1^*) \psi_2(Q_t d_2^*) \tau(\delta)}{P_s Q_t d_1^* d_2^* \delta^2} \right] \end{aligned}$$

because

$$P_s Q_t d_1^* d_2^* \delta^2 \le n^{6/100} < n^{1/4}$$

by (20) and (26). Consequently, by the multiplicativity of $\psi_1(k_1)$ and $\psi_2(k_2)$ and the coprimality of $P_s Q_t$ and $d_1^* d_2^*$,

$$\begin{aligned} Z(P_s, Q_t, d_1^*, d_2^*, \delta) &= \frac{n^{1/2} \psi_1(P_s) \psi_2(Q_t)}{P_s^2 Q_t^2} \cdot \frac{\psi_1(d_1^*) \psi_2(d_2^*)}{d_1^{*2} d_2^{*2}} \cdot \frac{\tau(\delta)}{\delta^4} \\ &\quad + O\left[\frac{n^{1/4} \psi_1(P_s) \psi_2(Q_t)}{P_s Q_t} \cdot \frac{\psi_1(d_1^*) \psi_2(d_2^*)}{d_1^* d_2^*} \cdot \frac{\tau(\delta)}{\delta^2} \right], \end{aligned} \tag{29}$$

to advance from which we need some more information about $\psi_1(k_1), \psi_2(k_2)$, and $\tau(\delta)$.

For a prime p congruent to 1, mod 8, there is one solution, mod p, of $X^4 + Y^4 \equiv 0$, mod p, for which $X \equiv 0$, mod p, and $4(p-1)$ other solutions so that

$$\psi_1(p) = 4p - 3, \tag{30}$$

whereas, for $p \nmid n$,

$$\psi_2(p) = p + O[p^{1/2}] \tag{31}$$

by a familiar estimate due to Weil. Also

$$\tau(p) = p^2\psi(p) - \psi(p^2) = p^2\psi(p) - p\psi(p) = p(p-1)\psi(p) = p^3 + O[p^{5/2}] \tag{32}$$

for $p \nmid 2n$ with the implication that

$$\tau(\delta) = \delta^3 \prod_{p|\delta}\left\{1 + O\left[\frac{1}{p^{1/2}}\right]\right\} = O[\delta^{3+\epsilon}]. \tag{33}$$

Furthermore, we easily confirm that

$$\Phi(d^*) = \frac{1}{d^{*2}} \sum_{d_1^* d_2^* = d^*} \psi_1(d_1^*)\psi_2(d_2^*) \tag{34}$$

is a multiplicative function in the square-free argument d^*; this by its genesis does not exceed[3] 1 because, being the number of incongruent solutions of

$$(Z^4 + W^4)(n - Z^4 - W^4) \equiv 0, \ \bmod d^*,$$
$$\sum_{d_1^* d_2^* = d^*} \psi_1(d_1^*)\psi_2(d_2^*)$$

does not exceed d^{*2}. From (30) and (31) we then deduce that

$$\Phi(p) = \frac{1}{p^2}[\psi_1(p) + \psi_2(p)] = \frac{1}{p^2}[5p + O[p^{1/2}]] = \frac{5}{p} + O\left[\frac{1}{p^{3/2}}\right] \tag{35}$$

for p satisfying (19) and also that

$$d^*\Phi(d^*) = O[6^{\omega(d^*)}] = O[d^{*\epsilon}]. \tag{36}$$

We are ready to revert to (27) and (29), which with (34) imply that

$$T(P_s, Q_t) = \frac{n^{1/2}\psi_1(P_s)\psi_2(Q_t)}{P_s^2 Q_t^2}\left[\sum_{\omega(d^*)\le N_2} \mu(d^*)\Phi(d^*)\right]\left[\sum_{\omega(\delta)\le N_2} \frac{\mu(\delta)\tau(\delta)}{\delta^4}\right]$$
$$+ \left\{\frac{n^{1/4}\psi_1(P_s)\psi_2(Q_t)}{P_s Q_t}\left[\sum_{\omega(d^*)\le N_2} d^*\Phi(d^*)\right]\left[\sum_{\omega(\delta)\le N_2} \frac{\tau(\delta)}{\delta^2}\right]\right\}, \tag{37}$$

[3] As in most sieving situations, this inequality is actually implicit in the underlying data; however, it seems helpful to supply a direct proof in the current circumstances.

where the remainder term is

$$O\left[n^{(1/4)+(1/50)}\,\frac{\psi_1(P_s)\psi_2(Q_t)}{P_s^2Q_t^2}\sum_{d^*\leq n^{1/100}} d^{*\epsilon}\sum_{\delta\leq n^{1/100}}\delta^{1+\epsilon}\right]$$

$$\neq O\left[n^{(1/4)+(1/50)+(1/100)+(1/50)+\epsilon}\,\frac{\psi_1(P_s)\psi_2(Q_t)}{P_s^2Q_t^2}\right]=O\left[\frac{n^{1/3}\psi_1(P_s)\psi_2(Q_t)}{P_s^2Q_t^2}\right] \tag{38}$$

by (20), (26), (36) and (33). As for the explicit term in this, Lemma 1 with (19) first shews that

$$0\leq\sum_{\omega(d^*)\leq N_2}\mu(d^*)\Phi(d^*)\leq\prod_{\substack{p\equiv1,\bmod 8\\ p\leq N;\,p+nP_sQ_t}}[1-\Phi(p)]+\sum_{\omega(d^*)=N_2+1}\Phi(d^*)$$

$$\leq\prod_{\substack{p\equiv1,\bmod 8\\ p\leq N;\,p+nP_sQ_t}}[1-\Phi(p)]+\frac{1}{(N_2+1)!}\left[\sum_{\substack{p\equiv1,\bmod 8\\ p\leq N}}\Phi(p)\right]^{N_2+1}.$$

In this, the first term on the second line is

$$\prod_{\substack{p\equiv1,\bmod 8\\ p\leq N;\,p+nP_sQ_t}}\left\{1-\frac{5}{p}+O\left[\frac{1}{p^{3/2}}\right]\right\}=O\left\{\prod_{\substack{p\equiv1,\bmod 8\\ p\leq N;\,p+nP_sQ_t}}\left[1-\frac{5}{p}\right]\right\}$$

$$=O\left\{\left[\frac{nP_sQ_t}{\phi(nP_sQ_t)}\right]^5\prod_{\substack{p\leq N\\ p\equiv1,\bmod 8}}\left[1-\frac{1}{p}\right]^5\right\}=O\left[\frac{(\log\log n)^{25/4}}{\log^{5/4}n}\right]$$

by (35) and then (3), while the second term does not exceed

$$\left\{\frac{e}{N_2+1}\sum_{\substack{p\leq n\\ p\equiv1,\bmod 8}}\left[\frac{5}{p}+O\left[\frac{5}{p^{3/2}}\right]\right]\right\}^{N_2+1}\leq\left[\frac{5e}{4(N_2+1)}(\log\log n+A_5)\right]^{N_2+1}$$

$$\leq\left[\frac{e}{A_4}\right]^{2[A_4\log\log n]+1}=O\left[\frac{1}{\log^{5/4}n}\right]$$

provided that A_4 in (24) be large enough, the inference being that the first sum appearing in the main term of (37) is non-negative and

$$O\left[\frac{(\log\log n)^{25/4}}{\log^{5/4}n}\right].$$

Since the second sum in this term can be shown through a parallel method to be non-negative and not more than

$$\prod_{\substack{p\not\equiv1,\bmod 8\\ p+2n;\,p\leq N}}\left[1-\frac{\tau(p)}{p^4}\right]+\sum_{\omega(\delta)=N_2+1}\frac{\tau(\delta)}{\delta^4}$$

$$\leq \prod_{\substack{p\not\equiv 1, \bmod 8\\ p+2n; p\leq N}} \left[1-\frac{1}{p}+O\left[\frac{1}{p^{3/2}}\right]\right]+\frac{1}{(N_2+1)!}\left\{\sum_{\substack{p\leq n\\ p\not\equiv 1, \bmod 8}}\left[\frac{1}{p}+O\left[\frac{1}{p^{3/2}}\right]\right]\right\}^{N_2+1}$$
$$= O\left[\frac{(\log\log n)^{7/4}}{\log^{3/4} n}\right],$$

by (32), we conclude finally from (37) and (38) that

$$S(P_s, Q_t) \leq T(P_s, Q_t) = O\left[\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\cdot\frac{\psi_1(P_s)\psi_2(Q_t)}{P_s^2 Q_t^2}\right].$$

With the estimation of $S(P_s, Q_t)$, the major part of the investigation has been completed. Considering all possible values of P_s and Q_t for assigned indices s, t compatible with (14), we first deduce that $R^{\ddagger}_{s,t}(n)$ in (15) is

$$O\left[\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\sum_{(P_s,Q_t)=1}\frac{\psi_1(P_s)\psi_2(Q_t)}{P_s^2Q_t^2}\right]$$
$$= O\left\{\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\cdot\frac{1}{s!t!}\left[\sum_{\substack{p+n; p\leq N\\ p\equiv 1, \bmod 8}}\frac{\psi_1(p)}{p^2}\right]^s\left[\sum_{\substack{p+n; p\leq N\\ p\equiv 1, \bmod 8}}\frac{\psi_2(p)}{p^2}\right]^t\right\}$$
$$= O\left\{\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\cdot\frac{1}{s!t!}\left[\sum_{\substack{p\leq n\\ p\equiv 1, \bmod 8}}\left[\frac{4}{p}+O\left[\frac{1}{p^2}\right]\right]\right]^s\left[\sum_{\substack{p\leq n\\ p\equiv 1, \bmod 8}}\left[\frac{1}{p}+O\left[\frac{1}{p^{3/2}}\right]\right]\right]^t\right\}$$
$$= O\left[\frac{n^{1/2}(\log\log n)^8\left[\frac{1}{4}\log\log n + A_6\right]^{s+t}}{\log^2 n}\cdot\frac{4^s}{s!t!}\right]$$

because of (30) and (31). Next, substituting this in (16) and then waiving the condition $t \leq N_1$, we obtain an estimate

$$R^{\ddagger}(n) = O\left\{\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\sum_{q=0}^{\infty}\frac{\left[\frac{1}{4}\log\log n + A_6\right]^q}{q!}\sum_{\substack{s\leq t\\ s+t=q}}\binom{q}{s}4^s\right\}$$

containing an inner sum that does not exceed

$$4^{q/2}\sum_{s\leq(1/2)q}\binom{q}{s} \leq 4^{q/2}2^q = 4^q.$$

Therefore

$$R^{\ddagger}(n) = O\left[\frac{n^{1/2}(\log\log n)^8}{\log^2 n}\sum_{q=0}^{\infty}\frac{[\log\log n + 4A_6]^q}{q!}\right] = O\left[\frac{n^{1/2}(\log\log n)^8}{\log n}\right],$$

which with (12) and (13) yields the following theorem.

heorem. *Let $r_4(n)$ be the number of representations of n as the sum of four biquadrates. hen*

$$r_4(n) = O\left[\frac{n^{1/2}(\log\log n)^8}{\log n}\right]$$

s $n \to \infty$.

At the expense of a little extra calculation the exponent of $\log\log n$ in the theorem ould be reduced to 6. Also, it is not impossible that this exponent could be yet further ecreased by a more intensive variation in our techniques. Yet such an effort would be tterly incommensurate with the improvement obtained. Other methods must be sought in ie quest for results that more closely reflect the likely true size of $r_4(n)$.

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Rings with all modules residually finite

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Abstract. Define a ring A to be RRF (resp. LRF) if every right (resp. left) A-module is residually finite. Refer to A as an RF ring if it is simultaneously RRF and LRF. The present paper is devoted to the study of the structure of RRF (resp. LRF) rings. We show that all finite rings are RF. If A is semiprimary, we show that A is RRF $\Longleftrightarrow A$ is finite $\Longleftrightarrow$ A is LRF. We prove that being RRF (resp. LRF) is a Morita invariant property. All boolean rings are RF. There are other infinite strongly regular rings which are RF. If $A/J(A)$ is of bounded index and A does not contain any infinite family of orthogonal idempotents we prove:

(i) A an RRF ring $\Longrightarrow A$ right perfect and $A/J(A)$ finite (hence $A/J(A)$ finite semi-simple artinian).
(ii) A an LRF ring $\Longrightarrow A$ left perfect and $A/J(A)$ finite.

If A is one sided quasi-duo (left or right immaterial) not containing any infinite family of orthogonal idempotents then (i) and (ii) are valid with the further strengthening that $A/J(A)$ is a finite product of finite fields.

Keywords. Residual finiteness; semiprimary rings; perfect rings; primitive factors; PF rings.

1. Introduction

All the rings we consider will be associative rings with an identity element $1 \neq 0$ and all the modules considered will be unital modules. In what follows A denotes a ring. As usual mod $- A$ (resp. $A -$ mod) will denote the category of right (resp. left) A-modules. Unless otherwise mentioned we will be working in mod $- A$. Recall (Definition 2.1 in [10]) that an A-module M is *said* to be residually finite if given any $x \neq 0$ in M, we *can* find a submodule N of M (depending on x) with $x \notin N$ and M/N finite. Note that we want M/N to be actually finite. It is straightforward to see that any submodule of a residually finite module is residually finite and that the direct product $\prod_{\alpha\in\Delta} M_\alpha$ (hence the direct sum $\bigoplus_{\alpha\in\Delta} M_\alpha$) of any family $\{M_\alpha\}_{\alpha\in\Delta}$ of residually finite modules, is itself residually finite. A is said to be residually finite as a ring if given any $a \neq 0$ in A, there exists a two sided ideal I of A with $a \notin I$ and A/I finite. From Proposition 2.14 in [10] we see that A is residually finite as a ring $\Longleftrightarrow$ A is residually finite in mod $- A \Longleftrightarrow A$ is residually finite in $A -$ mod.

DEFINITION 1.1

A is called an RRF (resp. LRF) ring if every right (resp. left) A-module is residually finite. A will be called an RF ring if it is both RRF and LRF.

As remarked earlier we will be dealing mainly with right sided notions. All our results have left sided analogues which will not be stated explicitly. Sometimes one sided assumptions yield two sided results. We will indicate these very clearly. For any module M, we denote the injective hull (resp. the Jacobson radical) of M by $E(M)$ (resp. $J(M)$).

Our first main result is a characterization of RRF rings as the rings satisfying the condition that $E(S)$ is finite for every simple right module S (Proposition 2.1). Using this characterization we show that any finite ring is RF (Theorem 2.1) and that a semiprimary ring A is RRF $\Longleftrightarrow A$ is finite $\Longleftrightarrow A$ is LRF (Theorem 2.2). If A is a one sided PF ring we prove that A is RRF $\Longleftrightarrow A$ is finite $\Longleftrightarrow A$ is LRF (Theorem 2.3).

In § 3 we first construct infinite commutative von Neumann regular rings which are RF. In fact any boolean ring is RF. Next we show that being an RRF (resp. LRF) ring is a Morita invariant property. The class $\mathcal{V}$ of right V-rings A with the property that every simple right A-module S is finite is included in the class of RRF rings. Our proof of the Morita invariance of RRF rings will also show that $\mathcal{V}$ is a Morita invariant class.

It is straightforward to see that any RRF ring A is a right max ring, in the sense that every right A-module admits a maximal submodule. Then appealing to Hirano's recent results on right max rings [4], we show that if A is an RRF ring not admitting any infinite family of orthogonal idempotents and further satisfying the condition that $A/J(A)$ is of finite index of nilpotence, then A is right perfect with $A/J(A)$ finite (Theorem 4.1). Combining this with the result of Hua-Ping Yu (Corollary 2.4 in [12]) asserting that for any one sided quasi-duo ring A, the factor ring $A/J(A)$ is reduced we get our final structure theorem (Theorem 4.2)

Finally in § 5 we mention some open problems which to us look like very challenging questions.

2. Characterization and examples of RRF rings

Any finite module M is clearly residually finite. If S is a simple A-module since every non zero submodule of $E(S)$ contains S, for any non zero $x \in S$ the only submodule of $E(S)$ not containing x is the zero submodule of $E(S)$. Hence $E(S)$ residually finite $\Longrightarrow E(S)$ finite.

PROPOSITION 2.1

The following conditions are equivalent for a ring A.

1. A *is* RRF.
2. *For every simple right A-module S, its injective hull $E(S)$ is finite.*
3. *There exists a cogenerator M for* mod $- A$ *with M residually finite.*

Proof. (1) $\Longrightarrow$ (2) is an immediate consequence of the fact that $E(S)$ is residually finite in mod $- A$, for every simple right A-module S whenever A is RRF and the observation $E(S)$ is residually finite if and only if it is actually finite.

(2) $\Longrightarrow$ (3): Let $\mathcal{S}$ be a set of representatives for isomorphism classes of simple right A-modules. Then $M = \prod_{S \in \mathcal{S}} E(S)$ is a cogenerator for mod $- A$ which is residually finite because each $E(S)$ is finite, hence residually finite.

(3) $\Longrightarrow$ (1): Any N in mod $- A$ is isomorphic to a submodule of M^{Δ} for some Δ, where M^{Δ} is the direct product of copies of M indexed by Δ. The residual finiteness of M implies that of N. Hence A is RRF. ■

In our next two results we will be using a result due to Rosenberg and Zelinsky [8]. For the sake of the readers we will state this result. This is also proved in [11]. Refer to Theorem 11.2(1) in [11].

PROPOSITION 2.2

Let A be a semiprimary ring and $J = J(A)$. Let $J^n = 0$ (with n an integer ≥ 1). Let E be an injective right A-module. Then E is a module of finite length if and only if $\mathrm{Hom}((J^i/J^{i+1})_A, \mathrm{Soc}(E_A))$ *is of finite length as a right A-module (where the right A-module structure on* $\mathrm{Hom}((J^i/J^{i+1})_A, \mathrm{Soc}(E_A))$ *arises from the left A-module structure on J^i/J^{i+1}) for* $0 \leq i \leq n-1$.

Here J^0 should be interpreted as A.

Theorem 2.1. *Any finite ring is an* RF *ring.*

Proof. Let A be a finite ring. We will show that A is RRF, the proof that A is LRF is similar. Let S be any simple right A-module and $E = E(S)$. Since S is a quotient of A_A and A is finite, we see that S is finite. Clearly $\mathrm{Soc}(E_A) = S$. From $J^i \leq A_A$ for $0 \leq i \leq n-1$ and the finiteness of A we see that J^i/J^{i+1} are all finite implying that $\mathrm{Hom}((J^i/J^{i+1})_A, \mathrm{Soc}(E_A)) = \mathrm{Hom}((J^i/J^{i+1})_A, S_A)$ are finite for $0 \leq i \leq n-1$. From Proposition 2.2 we conclude that $E(S)$ is of finite length in mod-A. Since A is finite we see that $E(S)$ itself is finite. ■

Theorem 2.2. *Let A be a semiprimary ring. Then A is* RRF $\Longleftrightarrow$ *A is finite* $\Longleftrightarrow$ *A is* LRF.

Proof. Because of Theorem 2.1, to prove the equivalence A is RRF $\Longleftrightarrow$ A is finite we have only got to prove the implication A an RRF ring $\Longrightarrow$ A is finite, under the assumption that A is semiprimary. From Proposition 2.1 we see that $E = E(S)$ is finite for every simple right A-module S. In particular each simple right A-module is finite. By assumption, A/J is semisimple artinian and $J^n = 0$ for some integer $n \geq 1$, where $J = J(A)$. Every simple module over A is annihilated by J and the simple A/J-modules are the same as the simple modules of A. The semisimple artinian nature of A/J implies that A/J is finite. Since $J^i \cdot J = J^{i+1} = J \cdot J^i$ see that J^i/J^{i+1} is actually an A/J-bimodule, and $\mathrm{Hom}((J^i/J^{i+1})_A, \mathrm{Soc}(E_A)) = \mathrm{Hom}((J^i/J^{i+1})_A, S_A) = \mathrm{Hom}((J^i/J^{i+1})_{A/J}, S_{A/J})$. The finiteness of $E = E(S)$ implies that $E(S)$ is of finite length over A. From Proposition 2.2 we conclude that $S_{A/J}$ occurs with a finite multiplicity in the semisimple module $(J^i/J^{i+1})_{A/J}$. Since A/J is finite there are only finitely many isomorphism classes of simple right A/J-modules. It follows that J^i/J^{i+1} is finite for $0 \leq i \leq n-1$ and hence $J^0 = A$ itself is finite. When A is semiprimary, the proof of the equivalence A is finite $\Longleftrightarrow$ A is LRF is similar. ■

Recall that A is said to be a right PF (pseudo-Frobenius) ring if A is an injective cogenerator in mod $-A$. It is well-known that this is equivalent to requiring A_A to be injective with $\mathrm{Soc}(A_A)$ finitely generated and essential in A_A (Proposition 12.5.2 in [6]). We have a similar definition and characterization of a left PF ring.

Theorem 2.3. *Let A be a one sided* PF *ring (left or right immaterial). Then the following are equivalent:*

1. *A is* RF,
2. *A is* RRF,

3. *A is* LRF,
4. *The right module A_A is residually finite,*
5. *A is residually finite as a ring,*
6. *The left module $_AA$ is residually finite,*
7. *A is finite.*

Proof. For any ring A we have proved the equivalence of (4), (5) and (6) (Proposition 2.14 in [10]). Now assume that A is right PF. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) are trivial. Since A_A is a cogenerator, Proposition 2.1 yields the implication (4) $\Longrightarrow$ (2). Again from Proposition 2.1 we see that $E(S)$ is finite for every simple right A-module S. Since $\mathrm{Soc}(A_A)$ is finitely generated and $E(\mathrm{Soc}\, A_A) = A$ we immediately see that A is finite. This proves (2) $\Longrightarrow$ (7). Also (7) $\Longrightarrow$ (2) and (7) $\Longrightarrow$ (3) are consequences of Theorem 2.1 and (3) $\Longrightarrow$ (6) is trivial. This completes the proof of Theorem 2.3. ■

When A is left PF the proof is similar. We omit the proof.

Remark 2.4. When the equivalent conditions of Theorem 2.5 are satisfied A turns out to be a finite QF (quasi-Frobenius) ring.

Remark 2.5. To make sure that Theorems 2.2 and 2.3 deal with different classes of rings even though the conclusion appears to be the same (finiteness of A) we give an example of a semiprimary ring which is not right or left PF and a PF ring which is not semiprimary.

Example 2.6. Let K be a finite field and V_K a vector space over K of finite dimension $n \geq 2$. Let $A = K \times V$ additively. Let multiplication in A be given by $(\lambda, v)(\mu, w) = (\lambda\mu, w\lambda + v\mu)$ for any λ, μ in K and v, w in V. Then A is a commutative local ring which is also finite. $0 \times V$ is the unique maximal ideal of A. Since A is finite it is semiprimary. If W is a nonzero proper subspace of V_K , then $0 \times W$ is an ideal of A with $r_A l_A(0 \times W) \neq 0 \times W$.

From Proposition 12.4.1 of [6] we see that A_A is not a cogenerator.

Also if W is a proper subspace of V, then $0 \times W$ is a finitely generated ideal of A with $l_A r_A(0 \times W) \neq 0 \times W$. From Proposition 12.4.2 of [6] we see that A_A is not injective. Thus A fails to be PF for two reasons. In this example, as usual l_A, r_A stand for the appropriate annihilators. In [3] Carl Faith has considered similar examples.

Osofsky has already constructed a commutative, local PF ring which is not semiprimary (Example 1 in [7]). The same example is also discussed in Carl Faith's book (Example 23.34.1 in [2]).

3. Morita invariance of RRF rings

All the examples of RRF (resp. LRF) rings considered in § 2 turned out to be finite rings. Any boolean ring A is a commutative von Neumann regular ring, hence a V-ring. For any maximal ideal I of A it is known that $A/I \simeq \mathbb{Z}/2\mathbb{Z}$ as a ring. It follows that every simple module over A is finite (having exactly two elements). Since A is a V-ring, simple modules are injective. From Proposition 2.1 we see immediately that such an A is an RF ring. Since there is an abundant supply of infinite boolean rings we have an abundant supply of infinite RF rings. Towards the end of this section we will construct other families of infinite RF rings.

PROPOSITION 3.1

Let p be a given prime number, K_α a finite field of characteristic p with order $K_\alpha = p^{n_\alpha}$, where α ranges over a certain indexing set J. Suppose there exists an integer $n \geq 1$ with $n_\alpha \leq n$ for all $\alpha \in J$. Then $A = \prod_{\alpha \in J} K_\alpha$ is a V-ring with the property that all its simple modules are finite.

Proof. Clearly A is a commutative von Neumann regular ring, hence a V-ring. Let I be any maximal ideal of A and $K = A/I$. Then K is a field. Since for any $\mathbf{a} = (x_\alpha)_{\alpha \in J}$ in A with $x_\alpha \in K_\alpha$ we have $p\mathbf{a} = 0$ we see that $p\lambda = 0$ for every $\lambda \in K$. Hence K is of characteristic p. If $l =$ the least common multiple of $\{n_\alpha | \alpha \in J\}$ (which exists because $1 \leq n_\alpha \leq n$ for all $\alpha \in J$) we see that $\mathbf{a}^{p^l} = \mathbf{a}$ for any $\mathbf{a} \in A$. Hence $\lambda^{p^l} = \lambda$ for every $\lambda \in K$. Thus every $\lambda \in K$ is a root of the polynomial $X^{p^l} - X$ over the prime field of K. Since $X^{p^l} - X$ can have at most a finite number of roots in K it follows that K is a finite field. It follows that every simple A-module is finite. ■

Theorem 3.1. *The class of* RRF (*resp.* LRF) *rings is a Morita invariant class, consequently so is the class of* RF *rings.*

Proof. We will give the proof in the case of RRF rings. Let A and B be Morita equivalent rings with A an RRF ring. Then there exists a progenerator ${}_AP$ with $B \simeq \mathrm{End}({}_AP)$ such that $-\otimes_A P : \mathrm{mod} - A \longrightarrow \mathrm{mod} - B$ and $-\otimes_B P^* : \mathrm{mod} - B \longrightarrow \mathrm{mod} - A$ give inverse equivalences of categories. Since $B \simeq \mathrm{End}({}_AP)$, we have a natural right action of B on P, which in turn gives a left B-action on $P^* = \mathrm{Hom}({}_AP, {}_AA)$. The right action of A on the second factor yields a right A-action on P^*. It is well-known that any Morita equivalence $F : \mathrm{mod} - A \longrightarrow \mathrm{mod} - B$ establishes a bijective correspondence between simple modules in the two categories and that for any $M \in \mathrm{mod} - A$, $F(E(M))$ is an injective hull of $F(M)$ in $\mathrm{mod} - B$ (Proposition 21.6, 21.8 in [1]).

Thus the injective hulls of simple modules in $\mathrm{mod} - B$ are of the form $E(S) \otimes_A P$ up to isomorphism. Since P is a direct summand of A^k for some integer $k \geq 1$ and $E(S)$ is finite we immediately see that $E(S) \otimes_A P$ is finite. This proves that B is an RRF ring. ■

Remark 3.2. If A is a right V-ring with every simple right A−module S finite, since $S = E(S)$ the above proof shows that every right simple module over B is finite and injective. Hence if $\mathcal{V}$ denotes the class of right V-rings A with all simple right A-modules finite, then $\mathcal{V}$ is a Morita invariant class.

The ring A in Proposition 3.1 is a right as well as a left V-ring with the property that all its right as well as left simple modules are finite. It follows that for any integer $n \geq 1$, the matrix ring $M_n(A)$ is a right and left V-ring with all its right as well as left simple modules finite. Similarly, for any boolean ring A the matrix rings $M_n(A)$ are two-sided V-rings with simple modules on either side finite. All these constitute examples of *RF* rings.

PROPOSITION 3.2

Let A be an RRF (*resp.* LRF) *ring. Then any factor ring A/I by an ideal I which is right (resp. left) primitive is a finite simple artinian ring.*

Proof. Let A be RRF and I an ideal of A which is right primitive. Then there exists a simple right A-module A with $I = r_A(S)$. Since A is RRF we see that S is finite. In

particular $\mathrm{End}(S_A)$ is a finite division ring, hence a field. Writing K for $\mathrm{End}(S_A)$, since S is finite the left K-module ${}_KS$ is finite dimensional over K. By Jacobson's density theorem A/I is canonically isomorphic to a dense subring of $\mathrm{End}({}_KS)$. Since $\dim_K S < \infty$, we see that $A/I \simeq \mathrm{End}({}_KS)$ which is a finite simple artinian ring. □

Let A be an RRF ring and $0 \neq M \in \mathrm{mod} - A$. For any $0 \neq x \in M$ there exists a submodule N of M with $x \notin N$ and M/N finite. Since M/N has a maximal submodule it follows that M has a maximal submodule.

Let us call an element $a \in A$ right regular if $r_A(a) = 0$ (some refer to such an a as a left non-zero divisor in A).

Lemma 3.1. *Let A be an* RRF *ring. Then any right regular element of A is a unit in A.*

Proof. Immediate consequence of Proposition 3.2 above and Lemma 2 in [4]. ■

Remark 3.3. Let A be an RRF ring. Then any ring B which is Morita equivalent to A is also an RRF ring by Theorem 3.1. Hence the analogues of Proposition 3.2 and Lemma 3.1 are also valid for B. In particular for any integer $n \geq 1$ any factor ring of $M_n(A)$ by a right primitive ideal of $M_n(A)$ is finite simple artinian. Ever right regular element in $M_n(A)$ is a unit in $M_n(A)$.

Recall that $M \in \mathrm{mod} - A$ is said to be co-hopfian if every injective endomorphism $f: M \to M$ is an automorphism. From Theorem 1.3 in [9] and the comments above we conclude that if A is RRF and B is Morita equivalent to A then B is co-hopfian in $\mathrm{mod} - B$.

4. Structure of RRF rings satisfying some additional restrictions

PROPOSITION 4.1

Any factor ring of an RRF *ring is* RRF.

Proof. Let A be an RRF ring and I an ideal of A. Any $M \in \mathrm{mod} - A/I$ can be regarded as an A-module via the canonical quotient map $\eta : A \to A/I$. Proposition 4.1 is an immediate consequence of the observation that the A-submodules of M are the same as the A/I-submoduels of M. ■

Recall that a ring A is said to have a bounded index of nilpotence if there exists an integer $n_0 \geq 1$ such that whenever a is a nilpotent element of A we have $a^{n_0} = 0$.

Theorem 4.1. *Let A be an* RRF *ring satisfying*

(a) *A does not contain any infinite family of orthogonal idempotents and*
(b) *$A/J(A)$ has a bounded index of nilpotency.*

Then A is right perfect and $A/J(A)$ is finite.

Proof. As already observed any RRF ring is a right max ring. From Proposition 3.2 we see that any factor ring of A by an ideal which is right primitive is a finite simple artinian ring. It follows from a recent result of Hirano (Theorem 2 in [4]) that A is right perfect. From Proposition 4.1 we see that $A/J(A)$ is RRF. From Theorem 2.2 we see that $A/J(A)$ is finite. ■

We now recall the definition of a right (resp. left) quasi-duo ring. A ring A is called right (resp. left) quasi-duo if every maximal right (resp. left) ideal in A is two sided. We write 'A is one sided quasi-duo' if it is either right or left quasi-duo.

Theorem 4.2. *Let A be a one sided quasi-duo ring (left or right immaterial). Suppose A is an* RRF *ring not containing any infinite family of orthogonal idempotents. Then A is right perfect with $A/J(A)$ a product of finitely many finite fields.*

Proof. Any factor ring of a right (resp. left) quasi-duo ring is right (resp. left) quasi-duo. If A is one sided quasi-duo it follows from Corollary 2.4 in [12] that $A/J(A)$ is a reduced ring, thus $A/J(A)$ has index of nilpotency 1. From Theorem 4.1 we see that A right perfect with $A/J(A)$ a finite semi-simple artinian ring. In particular $A/J(A) \simeq M_{n_1}(K_1) \times \cdots \times M_{n_r}(K_r)$ where each K_j is a finite field and n_j are integers ≥ 1. Since $A/J(A)$ is reduced it follows that $n_j = 1$ for all j, hence $A/J(A) \simeq K_1 \times \cdots \times K_r$. ∎

We leave the analogues of Theorems 4.1 and 4.2 for LRF rings to the reader.

5. Some open problems

1. Let A be a right perfect ring. Find effective criteria for the injective hull $E(S_A)$ of a simple right A-module S_A to be of finite length (resp. artinian) in mod $- A$. Such effective criteria will help in strengthening Theorems 4.1 and 4.2 considerably. When A is semiprimary the results of Rosenberg and Zelinsky [8] give such effective criteria.
2. Find an example of an RRF ring which is not LRF.
3. Obtain some structure theorems for rings A satisfying the condition that all finitely generated right A-modules are residually finite. For instance the ring of integers $\mathbb{Z}$ has this property. There are many infinitely generated abelian groups which are not residually finite, for instance any non-zero divisible abelian group.

Finally we wish to comment that in [5] Hirano studies rings A satisfying the condition that $E(S)$ has finite length for every simple right A-module S_A. He calls them right $\pi - V$ rings. RRF rings studied by us in the present paper satisfy a very much stronger condition than the right $\pi - V$ rings of Hirano.

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Holomorphic and algebraic vector bundles on 0-convex algebraic surfaces

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Abstract. Let X be a smooth complex algebraic surface such that there is a proper birational morphism $f : X \to Y$ with Y an affine variety. Let X_{hol} be the 2-dimensional complex manifold associated to X. Here we give conditions on X which imply that every holomorphic vector bundle on X is algebraizable and it is an extension of line bundles. We also give an approximation theorem of holomorphic vector bundles on X_{hol} (X normal algebraic surface) by algebraic vector bundles.

Keywords. Holomorphic vector bundle; algebraic vector bundle; complex surface; 0-convex complex surface.

1. Introduction

Let X be a 2-dimensional complex smooth algebraic variety and let X_{hol} be the associated complex manifold. If X_{hol} is not compact (e.g. if it is Stein) usually X_{hol} admits holomorphic vector bundles that are not algebraizable and non-isomorphic algebraic vector bundles which are isomorphic as holomorphic vector bundles. In this paper we want to study some cases in which these phenomena do not occur. Furthermore, we will give interesting examples of manifolds such that every (holomorphic or algebraic) vector bundle on them is an extension of line bundles. This is always true for smooth compact curves, but false on every projective variety of dimension > 1. In all this paper we will fix the following notations. Let X be a complex, smooth, connected quasi-projective 2-dimensional algebraic variety. Let X_{hol} be the associated complex analytic manifold. For any coherent algebraic sheaf U, let U_{hol} be the associated analytic coherent sheaf. We assume that X is a proper modification of an affine irreducible variety Y, i.e. we assume the existence of an affine 2-dimensional normal variety Y and a proper surjective map $f : X \to Y$. Since Y_{hol} is a Stein variety, the map $f_{\text{hol}} : X_{\text{hol}} \to Y_{\text{hol}}$ induced by f shows that X_{hol} is holomorphically 0-convex. We do not assume that Y is smooth. Let T be the union of all complete 1-dimensional algebraic subvarieties of X and set $S := f(T)$. Hence S is finite. We assume $T \neq \emptyset$, i.e. X_{hol} not Stein, because if $T = \emptyset$ our general result (Theorem 2.1) has as assumption the triviality of every holomorphic vector bundle on X_{hol} whose restriction to a small bidisc is trivial, i.e. it has as assumption the triviality of every holomorphic vector bundle on X_{hol}. Our methods do not give any new result for Stein or affine varieties. The aim of this paper is the proof of the following results 1.1 and 1.2.

Theorem 1.1. *Let T be a smooth connected projective complex curve of genus $q \geq 0$. Let $X := \mathbf{V}(L)$ be the total space of a line bundle L on T. Assume $\deg(L) < -q$ and $h^1(T, L^*) = 0$. Let $u : X \to T$ be the projection. Then every holomorphic vector bundle E on X is algebraizable and it is an extension of line bundles of the form $u^*(U)$ with $U \in \mathrm{Pic}_{\mathrm{hol}}(T_{\mathrm{hol}}) \cong \mathrm{Pic}(T)$.*

By Riemann–Roch as a particular case of 1.1 we have the following result.

COROLLARY 1.2

Let T be a smooth connected projective complex curve of genus $q \geq 0$. Let $X := \mathbf{V}(L)$ be the total space of a line bundle L on T. Assume $\deg(L) < \min\{0, 2-2q\}$. Let $u : X \to T$ be the projection. Then every holomorphic vector bundle E on X is algebraizable and it is an extension of line bundles of the form $u^*(U)$ with $U \in \mathrm{Pic}_{\mathrm{hol}}(T_{\mathrm{hol}}) \cong \mathrm{Pic}(T)$.

To prove these results we will prove a more general result (see Theorem 2.1) in which Y is not assumed to be a cone. At the end of the paper we will give 'an approximation theorem' for holomorphic vector bundles on X_{hol} by algebraic vector bundles (see Theorem 3.1).

2. The main results and the proof of 1.1

Set $S := f(T)$. Let $\mathbf{I}_T$ be the ideal sheaf of T in X and $\mathbf{J}$ the ideal sheaf of T_{hol} in X_{hol}. Let $T^{(n)}$ (resp. $T^{(n)}_{\mathrm{hol}}$), $n \geq 0$, be the nth infinitesimal neighborhood of T in X (resp. of T_{hol} in X_{hol}) (hence with $\mathbf{I}_T^{n+1}$ (resp. $\mathbf{J}^{n+1}$) as ideal sheaf). Hence $T^{(0)} := T$. Since X and X_{hol} are smooth, T is a Cartier divisor of T and T_{hol} is a Cartier divisor on X_{hol}. Hence the conormal sheaves $\mathbf{I}_T/\mathbf{I}_T^2$ and $\mathbf{J}/\mathbf{J}^2$ are line bundles on T and T_{hol} and we have $(\mathbf{I}_T/\mathbf{I}_T^2)^{\otimes n} \cong S^n(\mathbf{I}_T/\mathbf{I}_T^2) \cong \mathbf{I}_T^n/\mathbf{I}_T^{n+1}$ and $(\mathbf{J}/\mathbf{J}^2)^{\otimes n} \cong S^n(\mathbf{J}/\mathbf{J}^2) \cong \mathbf{J}^n/\mathbf{J}^{n+1}$ for all integers $n > 0$. Note that by GAGA $\mathrm{Pic}(T^{(n)}) \cong \mathrm{Pic}_{\mathrm{hol}}(T^{(n)}_{\mathrm{hol}})$. Hence from the exact sequence on X:

$$0 \to \mathbf{I}_T^{n+1} \to \mathbf{O}_{T^{(n)}} \to \mathbf{O}_{T^{(n-1)}} \to 0 \tag{1}$$

and the corresponding exact sequence $(1)_{\mathrm{hol}}$ on X_{hol} we obtain the following exact sequence in the Zariski topology:

$$0 \to S^n(\mathbf{I}_T/\mathbf{I}_T^2) \to \mathbf{O}_{T^{(n)}} \to \mathbf{O}_{T^{(n-1)}} \to 0 \tag{2}$$

and the corresponding exact sequence $(2)_{\mathrm{hol}}$ on X_{hol} in the Euclidean topology. Let T^{∞} (resp. $T^{\infty}_{\mathrm{hol}}$) be the formal scheme (resp. formal analytic space) which is the limit of the schemes (resp. complex spaces) $T^{(n)}$ (resp. $T^{(n)}_{\mathrm{hol}}$). Hence $T^{\infty}_{\mathrm{hol}}$ is the completion of X_{hol} along T_{hol}.

The aim of this section is the proof of the following result.

Theorem 2.1. *Let X be a smooth (non affine) quasi-projective complex surface such that there is a proper birational morphism $f : X \to Y$ with Y affine. Let $T \neq \emptyset$ be the union of all the complete algebraic curves contained in X. Assume that the following conditions are satisfied:*

(i) *every holomorphic line bundle on X_{hol} is algebraic;*
(ii) *the irreducible components of T are smooth;*
(iii) *for every irreducible component D of T and for a general $P \in D$ there is a closed algebraic subvariety V of X, $V \cong \mathbf{A}^1$ such that V intersects transversally D and exactly at P;*

(iv) $H^1(T, S^n(\mathbf{I}_T/\mathbf{I}_T^2)) = 0$ *for every* $n > 0$;
(v) *there is a fundamental system of neighborhoods* $\{U_n\}_{n\in N}$ *of* T_{hol} *in* X_{hol} *(in the Euclidean topology) such that every holomorphic rank* 2 *vector bundle* M *on* X_{hol} *with* $M|U_n$ *trivial for some* n *is trivial.*

Then every holomorphic vector bundle on X_{hol} *is algebraizable and is an extension of line bundles.*

We will see that all these conditions are satisfied in the case needed to prove Theorem 1.1 and hence Corollary 1.2.

Lemma 2.2. *Let* F *be an algebraic vector bundle on* X *and* F_{hol} *the associated holomorphic vector bundle on* X_{hol}. *Then* $H^1(X,F)$, $H^1(T^\infty, F|T^\infty)$, $H^1(T^\infty_{\text{hol}}, F_{\text{hol}}|T^\infty_{\text{hol}})$ *and* $H^1(X_{\text{hol}}, F_{\text{hol}})$ *are finite dimensional vector spaces with the same dimension.*

Proof. For all integers $n > 0$ we will use the tensorisazion with F (resp. F_{hol} in the complex topology) of the exact sequence (2) (resp. $(2)_{\text{hol}}$). Since F is locally free these sequences are exact. Since T is contracted by f, there is an integer m such that for all $n > m$ we have $H^1(T^{(n)}, F\otimes S^n(\mathbf{I}_T/\mathbf{I}_T^2)) = H^1(T^{(n)}_{\text{hol}}, F_{\text{hol}}\otimes S^n(\mathbf{I}_T/\mathbf{I}_T^2)) = 0$. Since $\dim(T) = 1$, we have $h^2(X,G) = h^2(T_{\text{hol}}, G_{\text{hol}}|T_{\text{hol}}) = 0$ for every algebraic coherent sheaf G on X. Hence by GAGA applied to every $T^{(s)}$ the cohomology groups $H^1(T^\infty, F|T)$ and $H^1(T^\infty_{\text{hol}}, F_{\text{hol}}|T_{\text{hol}})$ are finite dimensional vector spaces with the same dimension. Since Y_{hol} is Stein, Y is affine and f and f_{hol} are proper, the Leray spectral sequence of f and the formal function Theorem for proper maps (see [6], Th. III.11.1 for the algebraic case and ([1], Ch. III), for the analytic case) gives $H^1(T^\infty_{\text{hol}}, F_{\text{hol}}|T^\infty_{\text{hol}}) \cong H^1(X_{\text{hol}}, F_{\text{hol}})$.

Lemma 2.3. *Let* A, B *be algebraic vector bundles on* X *and* A_{hol} *and* B_{hol} *the corresponding holomorphic vector bundles on* X_{hol}. *Then* $\mathrm{Ext}^1(X;A,B)$, $\mathrm{Ext}^1(T^\infty;A|T^\infty,B|T^\infty)$, $\mathrm{Ext}^1(T^\infty_{\text{hol}};A|T^\infty_{\text{hol}},B|T^\infty_{\text{hol}})$ *and* $\mathrm{Ext}^1(X_{\text{hol}};A_{\text{hol}},B_{\text{hol}})$ *are finite dimensional vector spaces with the same dimension.*

Proof. Since A and B are locally free, we have $\mathrm{Ext}^1(X;A,B) \cong H^1(X,\mathrm{Hom}(A,B))$ and $\mathrm{Ext}^1(X_{\text{hol}};A_{\text{hol}},B_{\text{hol}}) \cong H^1(X_{\text{hol}},\mathrm{Hom}(A_{\text{hol}},B_{\text{hol}}))$ and $F := \mathrm{Hom}(A,B)$ is an algebraic vector bundle. Hence the result follows from Lemma 2.2.

Lemma 2.4. *Assume* $H^1(T, S^n(\mathbf{I}_T/\mathbf{I}_T^2)) = 0$ *for every* $n > 0$. *Let* A *(resp.* B*) be a formal vector bundle on* T^∞ *(resp.* T^∞_{hol}*). If* $A|T$ *(resp.* $B|T_{\text{hol}}$*) is trivial, then* A *(resp.* B*) is trivial.*

Proof. We will prove the triviality of A, since the same proof works for B. Tensor the exact sequence (2) for $n = 1$ with A. Since $H^1(T, \mathbf{I}_T/\mathbf{I}_T^2) = 0$ and $A|T$ is trivial we obtain that a base of $H^0(T, A|T)$ lifts to sections of $A|T^{(1)}$ whose restriction to T are linearly independent. Hence these sections induce a trivialization of $A|T^{(1)}$. And so on using (2) for all n.

Lemma 2.5. *Assume* $H^1(T, S^n(\mathbf{I}_T/\mathbf{I}_T^2)) = 0$ *for every* $n > 0$. *Let* A *(resp.* B*) be an algebraic (resp. analytic) vector bundle on* X *(resp.* X_{hol}*). If* $A|T$ *(resp.* $B|T_{\text{hol}}$*) is trivial, then there is a Zariski (resp. Euclidean) neighborhood* U *(resp.* V*) of* T *such that* $A|U$ *(resp.* $B|V$*) is trivial.*

Proof. By Lemma 2.4 $A|T^\infty$ and $B|T^\infty_{\text{hol}}$ are trivial. It is sufficient to extend a base of $H^0(T^\infty, A|T)$ (resp. $H^0(T^\infty_{\text{hol}}, B|T^\infty_{\text{hol}})$) to a Zariski neighborhood U (resp. Euclidean neighborhood V) of T. Since $S := f(T)$ is finite, S has a fundamental system of neighborhoods in Y (resp. Y_{hol}) formed by affine (resp. Stein) neighborhoods. By Theorem A for affine varieties (resp. Stein spaces) the extension of the sections follows from the formal function Theorem for proper maps (see [6], Th. III.11.1) for the algebraic case and ([1], Ch. III) for the analytic case.

Lemma 2.6. *Assume that every holomorphic vector bundle F on X_{hol} with* $\text{rank}(F) \le 2$ *is algebraizable* (*resp. it is an extension of holomorphic* (*resp. algebraic*) *line bundles*). *Then every holomorphic vector bundle on X_{hol} is algebraizable* (*resp. it is an extension of holomorphic* (*resp. algebraic*) *line bundles*).

Proof. Let H be a very ample line bundle on X. Let E be a vector bundle on X_{hol} with $r := \text{rank}(E) \ge 3$. We will use induction on r. Since $f_{\text{hol}^*}(E)$ is a coherent sheaf and Y_{hol} is Stein, $f_{\text{hol}^*}(E)$ is generated by global sections. Note that $f_{\text{hol}^*}(E)$ is locally free outside S. Hence we may find a sufficiently general global section of $f_{\text{hol}^*}(E)$ (i.e. of E) generating a subbundle of $f_{\text{hol}^*}(E)$ outside a small neighborhood of S. This general section generates a rank 1 subsheaf $R, R \cong \mathbf{O}_{X_{\text{hol}}}$, of E which is a subbundle of E (i.e. with E/R locally free) outside a small neighborhood of T_{hol}. Since $\dim(T) = 1$ and T is projective, we may twist E by a high power $H^{\otimes m}_{\text{hol}}, m > 0$, of H_{hol} so that $E \otimes H^{\otimes m}_{\text{hol}}$ is generated by global sections in a Euclidean neighborhood of T (Theorems A and B for projective morphisms between complex spaces (see [1], Ch. III, for a more general case)). Taking $E \otimes H^{\otimes m}_{\text{hol}}$ instead of E in the first part of the proof, we obtain that $(H^{\otimes m}_{\text{hol}})^*$ is a subbundle of E. Let E' be the rank $r-1$ quotient $E/(H^{\otimes m}_{\text{hol}})^*$. We conclude by the inductive assumption on r and Lemma 2.3.

Proof of Theorem 2.1. According to Lemma 2.6, in order to check that every holomorphic vector bundle on X_{hol} is algebraizable it is sufficient to prove that every holomorphic vector bundle E on X_{hol} with $\text{rank}(E) = 2$ is algebraizable. In addition, Lemma 2.3 says that to prove all the assertions of 2.1 it is sufficient to show that E is an extension (as holomorphic bundle) of two holomorphic line bundles. Fix an irreducible component D of T (hence smooth). If D is rational we fix a curve $V \cong \mathbf{A}^1$ such that V intersects D exactly at one point, P. If D has genus $q > 0$ we fix finitely many such curves, $V(1), \ldots, V(s)$ with $P(i) := D \cap V(i)$ sufficiently general and such that there is an effective divisor $\Sigma_{1 \le i \le s} n_i P(i)$, $n_i \in \mathbf{N}$ defining a large multiple of a very ample divisor H on D and a divisor $\Sigma_{1 \le i \le s} m_i P(i)$, $m_i \in \mathbf{N}$, defining $\det(E|D)^{\otimes m}$ for some $m > 0$. This is possible with, say, $s = 3q$. Set $V := V(1) \cup \cdots \cup V(s)$. Since every holomorphic vector bundle on a one-dimensional stein manifold is trivial ([3], Th. 30.1), $E|V$ is trivial and there are several surjective maps (as analytic coherent sheaves) $E|V \to \mathbf{O}_{V_{\text{hol}}}$. Fix one of them, $\mathbf{t}$, and set $E' := \text{Ker}(\mathbf{t})$. E' is a vector bundle on X_{hol} because V is a Cartier divisor of X_{hol} (see e.g. [7]). Note that $E'|D$ is obtained from $E|D$ making the elementary transformation (in the sense of [7]) corresponding to $\mathbf{t}|D$. Hence we see that there is a large integer m and a vector bundle E'' on X_{hol} such that we have an exact sequence

$$0 \to E'' \to E(mH_{\text{hol}}) \to M \to 0 \tag{3}$$

with M trivial rank 1 sheaf of $\mathbf{O}_{V_{\text{hol}}}$-modules and with $E''|D$ trivial. We may do the same construction for all components of T with the same integer m and obtain an analytic vector bundle, E'' on X_{hol}, E'' fitting in (3) with M structural sheaf of the union of finitely

many divisors isomorphic to $\mathbf{A}^1$ and with $E''|D$ trivial. By Lemmas 2.3 and 2.4, $E''|U_n$ is trivial for some n. Hence E'' is trivial by our assumption on U_n. The existence of such E'' and the fact that M is supported on an algebraic divisor, U', and it is a trivial line bundle on U gives the existence of $R \in \mathrm{Pic}_{\mathrm{hol}}(X_{\mathrm{hol}})$ such that R is a subbundle of E, i.e. by $E/R \in \mathrm{Pic}_{\mathrm{hol}}(X_{\mathrm{hol}})$. Hence E is an extension of two line bundles, as wanted.

Proof of Theorem 1.1. We will check that X satisfies all the assumptions of 2.1. Note that $\mathrm{Pic}_{\mathrm{hol}}(X_{\mathrm{hol}}) \cong u^*_{\mathrm{hol}}(\mathrm{Pic}_{\mathrm{hol}}(T_{\mathrm{hol}})) \cong u^*(\mathrm{Pic}(T))$ (e.g. use the exponential sequence and the fact that X is homotopic to T). Let $T \subset X$ be the 0-section. Note that T is contractible in the algebraic category (say by the morphism $f: X \to Y$) and the corresponding contraction Y is an affine variety. Let U be any neighborhood of T in X_{hol} in the Euclidean topology and let G be a holomorphic vector bundle on X_{hol} with $G|U$ trivial. Since Y is normal by definition of contraction, we have $f^*_{\mathrm{hol}}(\mathbf{O}_{X_{\mathrm{hol}}}) = \mathbf{O}_{Y_{\mathrm{hol}}}$. This equality and the triviality of $G|U$ imply that $f^*_{\mathrm{hol}}(G)$ is a vector bundle G' on Y_{hol} and that $f^*_{\mathrm{hol}}(G') \cong G$. Since Y is contractible, G' is trivial as topological vector bundle. Since Y is Stein, by a theorem of Grauert (Oka's principle [4]) G' is trivial. Hence G is trivial and we have checked the last assumption of 2.1 needed to apply 2.1.

3. An approximation result

In this short section we will give the following easy approximation theorem for holomorphic vector bundles on X_{hol} by a sequence of algebraic vector bundles.

Theorem 3.1. *Let X be a normal 2-dimensional complex variety (X not affine) and such that there is a proper birational morphism $f: X \to Y$ with Y affine. Assume that every holomorphic line bundle on X_{hol} is algebraic. Fix an increasing sequence $\{U_n\}_{n\in N}$ of open subsets of X_{hol} (in the Euclidean topology) which are relatively compact in X_{hol} and with $\cup_{n\in N} U_n = X_{\mathrm{hol}}$. Fix a holomorphic vector bundle F on X. Then there is a sequence of algebraic vector bundles $\{E_n\}_{n\in N}$ such that $(E_n)_{\mathrm{hol}}|U_n \cong F|U_n$ (as holomorphic vector bundles) for every $n \in N$.*

Proof. We may assume X connected and hence we may assume that F has constant rank, r. Since the case $r = 1$ is assumed to be true, we may assume $r > 1$. Let T be the exceptional set of f (hence T union of the 1-dimensional compact subvarieties of X_{hol}, if any, and $S := f(T)$ is finite). First we assume $r = 2$. Using Theorem A for the coherent sheaf $f_{\mathrm{hol}^*}(F)$ and the fact that f is an isomorphism outside a set with finite image in Y, we obtain the existence of a global section of F which generates a subbundle of F outside a Euclidean neighborhood V of T and outside a discrete set Z of X_{hol} with $Z \cap \mathrm{Sing}(X) = \emptyset$; the last condition is easily satisfied because $\mathrm{Sing}(X)$ is finite and we just need to take a section of F not vanishing at any point of $\mathrm{Sing}(X)$. We claim that, as in the proof of Lemma 2.6, we may find $M \in \mathrm{Pic}_{\mathrm{hol}}(X_{\mathrm{hol}})$ and $L \in \mathrm{Pic}_{\mathrm{hol}}(X_{\mathrm{hol}})$ such that F fits in the exact sequence on X_{hol}

$$0 \to M \to F \to L \otimes \mathbf{I}_Z \to 0. \tag{4}$$

The only difference with respect to the proof of 2.6 is that in (4) we take Z with its reduced structure. This is a kind of Bertini's theorem, but it is not essential. If Z is just a 0-dimensional unreduced subspace of X_{hol} the proof below will work with just notational modifications (see below). Set $Z[n] := Z \cap U_n$. Hence each $Z[n]$ is a finite 0-dimensional

subscheme of X because Z is discrete and U_n relatively compact in X_{hol}. For each $n > 0$ we consider the set of all extensions (with M, L and $Z[n]$) fixed

$$0 \to M \to A[n] \to L \otimes \mathbf{I}_{Z[n]} \to 0 \tag{5}$$

both in the analytic and in the algebraic category. By Lemma 2.3 if $Z[n] = \emptyset$ this set of extensions is a finite dimensional vector space and gives the same objects in both categories. Assume $Z[n] \neq \emptyset$. We use that Z is locally a complete intersection (this being true even if Z is not reduced) and contained in the smooth part of X. We use the local to global spectral sequence of the Ext-functors and the fact that $H^2(X, F) = H^2(X_{\mathrm{hol}}, G) = 0$ for all coherent sheaves F and G. As in [2] (which corrects the corresponding calculations and statements given in ([4], pp. 726–731)) we obtain that the vector space $\mathrm{Ext}^1(X; L \otimes \mathbf{I}_{Z[n]}, M)$ is the direct sum of $H^1(X, M \otimes L^*)$ and a finite dimensional vector space $W(n)$ with $\dim(W(n)) = \mathrm{card}(Z[n]_{\mathrm{red}})$ and that the same is true for the extensions as analytic sheaves. Furthermore, the proof of Lemma 2.2 shows that if $T \subset U_n$ (which is the case for large n) we have $\mathrm{Ext}^1(U_n; L \otimes \mathbf{I}_{Z[n]}, M) \cong W(n) \oplus \mathrm{Ext}^1(X_{\mathrm{hol}}; L, M) \cong W(n) \oplus \mathrm{Ext}^1(X; L, M) \cong \mathrm{Ext}^1(X, L \otimes \mathbf{I}_{Z[n]}, M)$ (the algebraic extensions). Hence for every such large $nF|U_n$ induces an extension $e[n]$ as in (5). This extension is algebraizable, i.e. it gives a coherent algebraic sheaf E_n which is locally free outside $Z[n]$. Furthermore, in [2] (correcting [5]) it is stated and proved the necessary and sufficient conditions for the local freeness of $A[n]$ (Cayley–Bacharach condition). This condition is satisfied for the extension $e[n]$ because F is locally free. Hence E_n is locally free and by construction $(E_n)_{\mathrm{hol}}|U_n \cong F|U_n$ for large n. Taking iterated extensions we obtain the case $r > 2$ as in the proof of Lemma 2.6.

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Means, homomorphisms, and compactifications of weighted semitopological semigroups

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Abstract. We consider some almost periodic type function algebras on a weighted semitopological semigroup, and using the set of multiplicative means on each of these algebras, we define their corresponding weighted semigroup compactifications. This will constitute an effective tool for investigating the properties of the function algebras concerned. We also show that these compactifications do not retain all the nice properties of the ordinary semigroup compactifications unless we impose some restrictions on the weight functions.

Keywords. Weighted semigroup; multiplicative mean; (weakly) almost periodic function; semigroup compactification.

1. Preliminaries

During the past decade harmonic analysis on weighted semigroups has enjoyed considerable attention, and a good deal of results have been proved in this connection (see for example [1, 4, 5, 6]).

In this article we continue our previous investigation [7], concerning function algebras on weighted semigroups, and try to use semigroup compactification techniques for characterizing these algebras. We shall also show that these techniques, although very effective for some weighted semigroups, do not carry over all the nice properties of the (non-weighted) semigroup case.

Throughout this paper, S denotes a locally compact Hausdorff semitopological semigroup, unless otherwise stated. A mapping $w : S \to (0, \infty)$ is called a weight function on S if $w(st) \leq w(s)w(t)$ for all $s, t \in S$, with w and w^{-1} locally bounded (i.e. bounded on compact subsets of S). w is said to be multiplicative if $w(st) = w(s)w(t)$ for all $s, t \in S$. We denote by Ω the map $(s, t) \to w(st)/(w(s)w(t))$ on $S \times S$ into $(0, 1]$. We assume Ω to be a separately continuous function. We also denote by ${}_s\Omega$ the map $t \to w(st)/(w(s)w(t))$. We denote by $B(S)$ the set of all bounded complex-valued functions on S, while the set of all continuous functions in $B(S)$ is denoted by $C(S)$, and the set of all elements in $C(S)$ which vanish at infinity by $C_0(S)$.

DEFINITIONS 1.1

(i) Let w be a weight function on S, then we shall denote by $B(S, w)$ (resp. $C(S, w)$, $C_0(S, w)$), the set of all functions $f : S \to C$ such that $f/w \in B(S)$ (resp. $f/w \in C(S)$, $f/w \in C_0(S)$) and $\|f\|_w := \sup\{|f(x)|/w(x) : x \in S\}$.

Furthermore, with the multiplication $\odot$ given by $f \odot g := (f \cdot g)/w$, usual addition, scalar multiplication, and complex conjugation $B(S,w)$ (resp. $C(S,w)$, $C_0(S,w)$) is a C^*-algebra.

(ii) By $\mathcal{LUC}(S,w)$ resp. $\mathcal{RUC}(S,w)$ we mean the set of all $f \in C(S,w)$ such that the map $s \to L_s f/(w(s)w)$ (resp. $s \to R_s f/(w(s)w)$) from S into $C(S)$ is norm continuous. Note that for each $f \in C(S,w)$ and $s \in S$ we define $L_s f$ by $L_s f = f(st)$, and $R_s f$ by $R_s f(t) = f(ts)$, $(t \in S)$.

(iii) Let $f \in C(S,w)$ then f is called w-weakly almost (resp. w-almost) periodic if the set $\{L_s f/(w(s)w) : s \in S\}$ is weakly (resp. norm) relatively compact in $C(S)$. The set of all w-weakly almost (resp. w-almost) periodic functions on S is denoted by $\mathcal{WAP}(S,w)$ (resp. $\mathcal{AP}(S,w)$). Let $\mathcal{F}$ be a subset of $C(S)$, or $C(S,w)$, we then say that $\mathcal{F}$ is translation invariant, if for each $f \in \mathcal{F}$ and $s \in S$, $L_s f, R_s f \in \mathcal{F}$. Note that, from the separate continuity of Ω, we deduce that $C(S,w)$ is translation invariant.

Remark 1.2. (i) If w is a weight function on S such that $\{x \in S : w(x) < \epsilon\}$ is finite for some $\epsilon > 0$, then $w(x) \geq 1$, for all $x \in S$ (see [5]).

(ii) It is interesting to note that the measurable weight functions on locally compact groups are bounded away from zero and infinity, on compact sets (see Proposition 2.1 of [5]). Thus if S is a locally compact group and w is a measurable weight function on S, then we can drop the local boundedness of w and w^{-1}.

(iii) In [8] it is shown that if S is a topological semigroup with a Borel measurable weight function w such that Ω satisfies the so-called Grothendieck's double limit property, as defined in [2], and is continuous on $S \times S$, then $\mathcal{WAP}(S,w)$ is a translation invariant C^*-subalgebra of $C(S,w)$, and w is the identity element of $\mathcal{WAP}(S,w)$. For $\mathcal{LUC}(S,w)$ and $\mathcal{AP}(S,w)$, we have the following two theorems, in which, we must note that, since a subset of $C(S)$ is norm relatively compact if and only if it is totally bounded, a function $f \in C(S,w)$ is in $\mathcal{AP}(S,w)$ if and only if for each $\epsilon > 0$ there exists a finite subset K of S such that

$$\min\left\{\left\|\frac{L_s f}{w(s)w} - \frac{L_t f}{w(t)w}\right\| : t \in K\right\} < \epsilon \quad \text{for all } s \in S.$$

Or equivalently

$$\min\left\{\left\|\frac{R_s f}{w(s)w} - \frac{R_t f}{w(t)w}\right\| : t \in K\right\} < \epsilon \quad \text{for all } s \in S.$$

Theorem 1.3. *Let w be a weight function on S, such that $_s\Omega^{-1}$ is bounded for all $s \in S$, and the map $_s\Omega^{-1}$ is norm continuous, then $\mathcal{LUC}(S,w)$ is a translation invariant C^*-subalgebra of $C(S,w)$ which contains w as the identity element.*

Proof. We shall first prove that for every $t \in S$ and $f, g \in \mathcal{LUC}(S,w)$ both $f \odot g$ and $L_t f$ (and $R_t f$) belong to $\mathcal{LUC}(S,w)$. To this end, we only need to see that

$$\left\|\frac{L_{s_\alpha}(f \odot g)}{w(s_\alpha)w} - \frac{L_s(f \odot g)}{w(s)w}\right\| \leq \left\|\frac{L_{s_\alpha} f}{w(s_\alpha)w} \cdot \frac{L_{s_\alpha} g}{w(s_\alpha)w} - \frac{L_s f}{w(s)w} \cdot \frac{L_s g}{w(s)w}\right\|$$
$$\times \left\|\frac{w(s_\alpha)w}{L_{s_\alpha} w}\right\| + \left\|\frac{w(s_\alpha)w}{L_{s_\alpha} w} - \frac{w(s)w}{L_s w}\right\| \left\|\frac{L_s f}{w(s)w} \cdot \frac{L_s g}{w(s)w}\right\|.$$

Note that since $s \to (L_s f/(w(s)w)) \cdot (L_s g/(w(s)w))$ and $s \to {}_s\Omega^{-1}$ are norm continuous,

$$\left\| \frac{L_{s_\alpha} f}{w(s_\alpha)w} \cdot \frac{L_{s_\alpha} g}{w(s_\alpha)w} - \frac{L_s f}{w(s)w} \cdot \frac{L_s g}{w(s)w} \right\| \to 0, \quad \left\| \frac{w(s_\alpha)w}{L_{s_\alpha} w} - \frac{w(s)w}{L_s w} \right\| \to 0$$

and

$$\left\| \frac{L_{s_\alpha} L_t f}{w(s_\alpha)w} - \frac{L_s L_t f}{w(s)w} \right\| \leq w(t) \left(\left\| \frac{L_{ts_\alpha} f}{w(ts_\alpha)w} \right\| \cdot |{}_t\Omega(s_\alpha) - {}_t\Omega(s)| \right.$$
$$\left. + \left\| \frac{L_{ts_\alpha} f}{w(ts_\alpha)w} - \frac{L_{ts} f}{w(ts)w} \right\| \right).$$

(Note that since the map $s \to L_{ts} f/(w(ts)w)$ is norm continuous and Ω is separately continuous, $\|L_{ts_\alpha} f/(w(ts_\alpha)) - L_{ts} f/(w(ts)w)\| \to 0$ and $|{}_s\Omega(s_\alpha) - {}_t\Omega(s)| \to 0$.) Similarly, one can show that $R_t f \in \mathcal{LUC}(S, w)$, (cf. [3]).

To prove the completeness of $\mathcal{LUC}(S, w)$, we show that $\mathcal{LUC}(S, w)$ is closed in the Banach space $C(S, w)$. So we assume that $\{f_n\}$ is a sequence in $\mathcal{LUC}(S, w)$ such that $\|f_n - f\|_w \to 0$ (i.e. $\|f_n/w - f/w\| \to 0$) for some $f \in C(S, w)$. To see that $f \in \mathcal{LUC}(S, w)$, note that $\|L_s((f_n/w) - (f/w))\| \to 0$ (i.e. $\|L_s f_n/(L_s w) - L_s f/(L_s w)\| \to 0$) for all s, and

$$\left\| \frac{L_s f_n}{w(s)w} - \frac{L_s f}{w(s)w} \right\| \leq \frac{w(s)w}{L_s w} \left\| \frac{L_s f_n}{w(s)w} - \frac{L_s f}{w(s)w} \right\| = \left\| \frac{L_s f_n}{L_s w} - \frac{L_s f}{L_s w} \right\|,$$

so

$$\left\| \frac{L_s f_n}{w(s)w} - \frac{L_s f}{w(s)w} \right\| \to 0.$$

Now let $s_\alpha \to s$, then since

$$\left\| \frac{L_{s_\alpha} f}{w(s_\alpha)w} - \frac{L_s f}{w(s)w} \right\| \leq 2 \left\| \frac{f}{w} - \frac{f_n}{w} \right\| + \left\| \frac{L_{s_\alpha} f_n}{w(s_\alpha)w} - \frac{L_s f_n}{w(s)w} \right\|,$$

we see that

$$\left\| \frac{L_{s_\alpha} f}{w(s_\alpha)w} - \frac{L_s f}{w(s)w} \right\| \to 0,$$

i.e. $f \in \mathcal{LUC}(S, w)$.

Obviously since the map $s \to {}_s\Omega^{-1}$ from S into $C(S)$ is norm continuous, the map $s \to {}_s\Omega$ from S into $C(S)$ is norm continuous, and so $w \in \mathcal{LUC}(S, w)$. Hence w is the identity element of $\mathcal{LUC}(S, w)$.

COROLLARY 1.4

If S is a group and $w \in \mathcal{LUC}(S, w)$, then $\mathcal{LUC}(S, w)$ is a translation invariant C^*-subalgebra of $C(S, w)$, (cf. [3]).

Proof. Since $w \in \mathcal{LUC}(S, w)$, the map $s \to {}_s\Omega$ from S into $C(S)$ is norm continuous. On the other hand S is a group, thus for all s and t in S, we have

$$\frac{w(s)w(t)}{w(st)} = \frac{w(s)w(s^{-1}st)}{w(st)} \leq \frac{w(s)w(s^{-1})w(st)}{w(st)} = w(s)w(s^{-1}),$$

i.e. ${}_s\Omega^{-1}$ is bounded for all s in S. And from the fact that

$$\left\| \frac{1}{{}_{s_\alpha}\Omega} - \frac{1}{{}_s\Omega} \right\| = \left\| \frac{{}_s\Omega - {}_{s_\alpha}\Omega}{{}_{s_\alpha}\Omega_s\Omega} \right\|$$

and the norm continuity of the map $s \to {}_s\Omega$, we conclude that the map $s \to {}_s\Omega^{-1}$ from S into $C(S)$ is norm continuous. So by the above theorem $\mathcal{LUC}(S,w)$ is a translation invariant C^*-subalgebra of $C(S,w)$. Note that if ${}_s\Omega^{-1}$ is bounded for all s in S, then the map $s \to {}_s\Omega$ is norm continuous if and only if the map $s \to {}_s\Omega^{-1}$ is norm continuous.

Remark 1.5. If $w \in \mathcal{LUC}(S,w)$, i.e. the map $s \to {}_s\Omega$ from S into $C(S)$ is norm continuous, then from the fact that $L_sf/(w(s)w) = L_s(f/w) \cdot {}_s\Omega$ we conclude that $\{f | (f/w) \in \mathcal{LUC}(S)\} \subseteq \mathcal{LUC}(S,w)$, and if ${}_s\Omega^{-1}$ (resp. Ω_s^{-1}) is also bounded for all s in S (for example, if S is a group or w is a multiplicative weight function on S) then we have $\{f : (f/w) \in \mathcal{LUC}(S)\} = \mathcal{LUC}(S,w)$, (cf. [10]).

Theorem 1.6. *Let w be a weight function on S, such that $\{{}_s\Omega^{-1} : s \in S\}$ is relatively compact in $C(S)$, then $\mathcal{AP}(S,w)$ is a translation invariant C^*-subalgebra of $C(S,w)$, with the identity element w.*

Proof. If f and g are in $\mathcal{AP}(S,w)$, since $(L_sf \odot g)/(w(s)w) = (L_sf/(w(s)w)) \cdot (L_sg/(w(s)w)) \cdot {}_s\Omega^{-1}$ and $\{(L_sf/(w(s)w)) \cdot (L_sg/w(s)w)) \cdot {}_s\Omega^{-1} : s \in S\}$ is a norm relatively compact subset of $C(S)$, then $\{(L_sf \odot g)/(w(s)w) : s \in S\}$ is a relatively compact subset of $C(S)$, i.e. $f \odot g \in \mathcal{AP}(S,w)$.

To see that $\mathcal{AP}(S,w)$ is translation invariant, let $f \in \mathcal{AP}(S,w)$ and $x, s \in S$, and $\epsilon > 0$ be arbitrary, then by the remark (1.2) (iii), there exists a finite subset K of S such that

$$\min\left\{ \left\| \frac{L_sf}{w(s)w} - \frac{L_tf}{w(t)w} \right\| : t \in K \right\} < \frac{\epsilon}{w(x)}$$

but

$$\left\| \frac{L_sR_xf}{w(s)w} - \frac{L_tR_xf}{w(t)w} \right\| \le w(x) \left\| \frac{L_sf}{w(s)w} - \frac{L_tf}{w(t)w} \right\|,$$

therefore

$$\min\left\{ \left\| \frac{L_sR_xf}{w(s)w} - \frac{L_tR_xf}{w(t)w} \right\| : t \in K \right\} < \epsilon,$$

i.e. $\{L_sR_xf/(w(s)w) : s \in S\}$ is totally bounded, thus $R_xf \in \mathcal{AP}(S,w)$.

Similarly we can show that $L_xf \in \mathcal{AP}(S,w)$.

To prove the completeness of $\mathcal{AP}(S,w)$, we show that $\mathcal{AP}(S,w)$ is closed in $C(S,w)$. Therefore we assume that (f_n) is a sequence in $\mathcal{AP}(S,w)$ such that

$$\|f_n - f\|_w \to 0 \left(\text{i.e. } \left\| \frac{f_n}{w} - \frac{f}{w} \right\| \to 0 \right) \text{ for some } f \in C(S,w).$$

To see that $f \in \mathcal{AP}(S,w)$, let $s \in S$ and $\epsilon > 0$ be arbitrary, then there exists n_0 in N such that $\|(f_n/w) - (f/w)\| < \epsilon/3$ for all $n \ge n_0$.

Since $f_{n_0} \in \mathcal{AP}(S,w)$, there exists a finite subset K of S such that

$$\min\left\{ \left\| \frac{L_sf_{n_0}}{w(s)w} - \frac{L_tf_{n_0}}{w(s)w} \right\| : t \in K \right\} < \frac{\epsilon}{3}.$$

However

$$\left\|\frac{L_sf}{w(s)w}-\frac{L_tf}{w(s)w}\right\| \leq \|L_s\|\left\|\frac{f}{w}-\frac{f_{n_0}}{w}\right\|+\left\|\frac{L_sf_{n_0}}{w(ts)w}-\frac{L_tf_{n_0}}{w(t)w}\right\|+\|L_t\|\left\|\frac{f_{n_0}}{w}-\frac{f}{w}\right\|$$
$$\leq \frac{2\epsilon}{3}+\left\|\frac{L_sf_{n_0}}{w(s)w}-\frac{L_tf_{n_0}}{w(t)w}\right\|.$$

Thus

$$\min\left\{\left\|\frac{L_sf}{w(s)w}-\frac{L_tf}{w(t)w}\right\| : t\in K\right\}<\epsilon.$$

i.e. $\{L_sf/(w(s)w): s\in S\}$ to totally bounded, thus $f\in \mathcal{AP}(S,w)$.

To see that $w\in \mathcal{AP}(S,w)$, note that $1\leq {}_s\Omega^{-1}$ for each $s\in S$, and since $C=\{{}_s\Omega^{-1}: s\in S\}$ is norm relatively compact, there exists a positive number M such that $1\leq \|f\|\leq M$ for each $f\in \overline{C}$ (where $\overline{C}$ is the norm closure of C in $C(S)$).

So the map $f\to 1/f$ from $\overline{C}$, into $C(S)$, which we denote by F, is norm continuous. Thus $F(\overline{C})$ is compact (and hence is closed), but $D=\{{}_s\Omega : s\in S\}$ is a subset of $F(\overline{C})$, so $\overline{D}$ is a closed subset of the compact set $F(\overline{C})$, thus $\overline{D}$ is compact (in norm topology), i.e. $\{L_sw/(w(s)w): s\in S\}$ is norm relatively compact, thus $w\in \mathcal{AP}(S,w)$.

Remark 1.7. (i) As we see in the proof of the above theorem, if $\{{}_s\Omega^{-1}: s\in S\}$ is relatively norm compact, then $\{{}_s\Omega : s\in S\}$ is relatively norm compact. The converse is however false. To see this let $S=N$, with the discrete topology and the left zero multiplication given by $m\odot n=m$ and $w(n)=n$, then $L_sw/(w(s)w)=1/w$ so $w\in \mathcal{AP}(S,w)$, but ${}_s\Omega^{-1}=w(s)w/(L_sw)=w$, which is not an element of $C(S)$ (since it is not bounded). This example also shows that the norm compactness of the set $\{{}_s\Omega^{-1} : s\in S\}$ is a sufficient condition, but not a necessary one (see (2.21)).
(ii) Another condition equivalent to the norm relative compactness of $\{{}_s\Omega^{-1}: s\in S\}$ is that $w\in \mathcal{AP}(S,w)$ and there exists $M>0$ such that $M\leq {}_s\Omega$ for all $s\in S$ (i.e. the set $\{{}_s\Omega: s\in S\}$ is norm relatively compact, and bounded away from zero).

2. The main results

Let S be a semigroup with weight function w, and suppose that $\tilde{\mathcal{F}}$ is a linear subspace of $B(S,w)$, put $\mathcal{F}=\{f/w: f\in \mathcal{F}\}$, then $\tilde{\mathcal{F}}\subseteq B(S)$. It is easy to see that, $\tilde{\mathcal{F}}$ is a linear subspace (resp. subalgebra) of $B(S,w)$ if and only if $\mathcal{F}$ is a linear subspace (resp. subalgebra) of $B(S)$.

DEFINITION 2.1

Let $\tilde{\mathcal{F}}$ be a linear subspace of $B(S,w)$, and let $\tilde{\mathcal{F}}_r$ denote the set of all real-valued members of $\tilde{\mathcal{F}}$. A mean on $\tilde{\mathcal{F}}$ is a linear functional $\tilde{\mu}$ on $\tilde{\mathcal{F}}$ with the property that

$$\inf_{s\in S}\frac{f}{w}(s)\leq \tilde{\mu}(f)\leq \sup_{s\in S}\frac{f}{w}(s)\quad (f\in \tilde{\mathcal{F}}_r).$$

The set of all means on $\tilde{\mathcal{F}}$ is denoted by $M(\tilde{\mathcal{F}})$. If $\tilde{\mathcal{F}}$ is also an algebra and if $\tilde{\mu}\in M(\tilde{\mathcal{F}})$ satisfies

$$\tilde{\mu}(f\odot g)=\tilde{\mu}(f)\tilde{\mu}(g)\quad (f,g\in \tilde{\mathcal{F}}),$$

then $\tilde{\mu}$ is said to be multiplicative. The set of all multiplicative means on $\mathcal{F}$ will be denoted by $MM(\tilde{\mathcal{F}})$.

Example 2.2. Let $S=[0,1]$, with the ordinary multiplication and usual topology, and let $w(s)=1+s$ for all $s\in S$, then $\tilde{\mu}$ which is defined on $C(S,w)$ by

$$\tilde{\mu}(f)=\int_0^1 \frac{f(x)}{w(x)}\,\mathrm{d}x$$

is a mean on $C(S,w)$.

Remark 2.3. (i) Let $\tilde{\mathcal{F}}$ be a linear subspace (resp. subalgebra) of $B(S,w)$, and for each $\tilde{\mu}\in M(\tilde{\mathcal{F}})$ (resp. $MM(\tilde{\mathcal{F}})$), let μ be defined on $\tilde{\mathcal{F}}$ as follows: $\mu(f/w):=\tilde{\mu}(f)$, then it is evident that $\mu\in M(\mathcal{F}))$ (resp. $MM(\mathcal{F})$) and

$$M(\tilde{\mathcal{F}})=\{\tilde{\mu}:\mu\in M(\mathcal{F})\}\ \text{(resp. } MM(\tilde{\mathcal{F}})=\{\tilde{\mu}:\mu\in MM(\mathcal{F})\}).$$

In fact there is a one-to-one correspondence between $M(\tilde{\mathcal{F}})$ and $M(\mathcal{F})$ (resp. $MM(\tilde{\mathcal{F}})$ and $MM(\mathcal{F})$).
(ii) Let $\tilde{\mathcal{F}}$ be a conjugate closed linear subspace of $B(S,w)$ such that $w\in\tilde{\mathcal{F}}$, and let $\tilde{\mu}$ be a mean on $\tilde{\mathcal{F}}$, then $\tilde{\mu}$ is positive, $\tilde{\mu}(w)=1$ and $\tilde{\mu}$ is a bounded linear functional on $\tilde{\mathcal{F}}$ with $\|\tilde{\mu}\|=1$.

DEFINITION 2.4

Let $\tilde{\mathcal{F}}$ be a conjugate closed, linear subspace of $B(S,w)$ such that $w\in\tilde{\mathcal{F}}$.

(i) For each $s\in S$ define $\tilde{\epsilon}(s)\in M(\tilde{\mathcal{F}})$ by $\tilde{\epsilon}(s)(f)=(f/w)(s)\ (f\in\tilde{\mathcal{F}})$. The mapping $\tilde{\epsilon}:S\to M(\tilde{\mathcal{F}})$ is called the evaluation mapping. If $\tilde{\mathcal{F}}$ is also an algebra, then $\tilde{\epsilon}(S)\subseteq MM(\tilde{\mathcal{F}})$.
(ii) Let $\tilde{X}=M(\tilde{\mathcal{F}})$ (resp. $\tilde{X}=MM(\tilde{\mathcal{F}})$, if $\tilde{\mathcal{F}}$ is a subalgebra) be furnished with the relative weak* topology. For each $f\in\tilde{\mathcal{F}}$ the function $\hat{f}\in C(\tilde{X})$ is defined by

$$\hat{f}(\tilde{\mu}):=\tilde{\mu}(f)\quad(\tilde{\mu}\in\tilde{X}).$$

Furthermore, we define $\hat{\tilde{F}}:=\{\hat{f}:f\in\tilde{\mathcal{F}}\}$.

Remark 2.5. (i) The mapping $f\to\hat{f}:\tilde{\mathcal{F}}\to C(\tilde{X})$ is clearly linear and multiplicative if $\tilde{\mathcal{F}}$ is an algebra and $\tilde{X}=MM(\tilde{\mathcal{F}})$. Also it preserves complex conjugation, and is an isometry, since for any $f\in\tilde{\mathcal{F}}$

$$\begin{aligned}\|\hat{f}\| &= \sup\{|\hat{f}(\tilde{\mu})|:\tilde{\mu}\in\tilde{X}\}=\sup\{|\tilde{\mu}(f)|:\tilde{\mu}\in\tilde{X}\}\\ &= \sup\left\{\left|\mu\left(\frac{f}{w}\right)\right|:\mu\in X\right\}\le\sup\left\{\left|\mu\left(\frac{f}{w}\right)\right|:\mu\in C(X)^*,\ \|\mu\|\le 1\right\}\\ &= \left\|\frac{f}{w}\right\|=\|f\|_w=\sup\left\{\left|\frac{f}{w}(s)\right|:s\in S\right\}=\sup\{|\tilde{\epsilon}(s)(f)|:s\in S\}\\ &= \sup\{|\hat{f}(\tilde{\epsilon}(s))|:s\in S\}\le\|\hat{f}\|,\end{aligned}$$

where $X=M(\mathcal{F})$. Note that $\hat{f}(\tilde{\epsilon}(s))=\tilde{\epsilon}(s)(f)=(f/w)(s)(f\in\tilde{\mathcal{F}},s\in S)$. This identity may be written in terms of the dual map $\tilde{\epsilon}^*:C(\tilde{X})\to C(S,w)$ as $\tilde{\epsilon}^*(\hat{f})=f$ for $f\in\tilde{\mathcal{F}}$.

(ii) Let $\tilde{\mathcal{F}}$ be a conjugate closed linear subspace of $B(S,w)$, containing w. Then $M(\tilde{\mathcal{F}})$ is convex and weak* compact, $co(\tilde{\epsilon}(S))$ is weak* dense in $M(\tilde{\mathcal{F}})$, $\tilde{\mathcal{F}}^*$ is the weak* closed linear span of $\tilde{\epsilon}(S)$, $\tilde{\epsilon}: S \to M(\tilde{\mathcal{F}})$ is weak* continuous, and if $\tilde{\mathcal{F}}$ is also an algebra, then $MM(\tilde{\mathcal{F}})$ is weak* compact and $\tilde{\epsilon}(S)$ is weak* dense in $MM(\tilde{\mathcal{F}})$.
(iii) Let $\tilde{\mathcal{F}}$ be a C^*-subalgebra of $B(S,w)$, containing w. If $\tilde{X}$ denotes the space $MM(\tilde{\mathcal{F}})$ with the relative weak* topology, and if $\tilde{\epsilon}: S \to \tilde{X}$ denotes the evaluation mapping, then the mapping $f \to \hat{f} : \tilde{\mathcal{F}} \to C(\tilde{X})$ is an isometric isomorphism with inverse $\tilde{\epsilon}^* : C(\tilde{X}) \to \tilde{\mathcal{F}}$.
(iv) Let $\tilde{\mathcal{F}}$ be a norm closed, conjugate closed, linear subspace of $B(S,w)$, containing w, and let $\tilde{X}$ denote the weak* compact convex set $M(\tilde{\mathcal{F}})$. Then the mapping $f \to \hat{f}$ is a linear isometry of $\tilde{\mathcal{F}}$ onto $AF(\tilde{X})$, the space of affine functions in $C(\tilde{X})$. The inverse of this mapping is $\tilde{\epsilon}^* : AF(\tilde{X}) \to \tilde{\mathcal{F}}$, where $\tilde{\epsilon}: S \to \tilde{X}$ is the evaluation mapping.

DEFINITION 2.6

Let $\tilde{\mathcal{F}}$ be a linear subspace of $B(S,w)$ such that $w \cdot L_s(f/w)$, $w \cdot R_s(f/w) \in \tilde{\mathcal{F}}$, for each $f \in \tilde{\mathcal{F}}$, and each $s \in S$ (i.e. $\mathcal{F}$ is translation invariant).

(i) For $\tilde{\mu} \in \tilde{\mathcal{F}}^*$, the left (resp. right) introversion operator determined by $\tilde{\mu}$ is the mapping $\tilde{T}_{\tilde{\mu}} : \tilde{\mathcal{F}} \to B(S,w)$ (resp. $\tilde{U}_{\tilde{\mu}} : \tilde{\mathcal{F}} \to B(S,w)$) defined by $\tilde{T}_{\tilde{\mu}} f)(s) = w(s) \cdot \mu(L_s(f/w))$ (resp. $(\tilde{U}_{\tilde{\mu}} f)(s) = w(s) \cdot \mu(R_s(f/w)))$. (In fact $\tilde{T}_{\tilde{\mu}} = w \cdot T_\mu(f/w)$ and $\tilde{U}_{\tilde{\mu}} = w \cdot U_\mu(f/w)$, see [2].)
(ii) Let $w \in \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ be conjugate closed then $\tilde{\mathcal{F}}$ is said to be left introverted (resp. left m-introverted) if $\tilde{T}_{\tilde{\mu}} \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{F}}$ for all $\tilde{\mu} \in M(\tilde{\mathcal{F}})$ (resp. $MM(\tilde{\mathcal{F}})$), and $\tilde{\mathcal{F}}$ is said to be right introverted (resp. m-introverted) if $\tilde{U}_{\tilde{\mu}} \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{F}}$ for all $\tilde{\mu} \in M(\tilde{\mathcal{F}})$ (resp. $MM(\tilde{\mathcal{F}})$). $\tilde{\mathcal{F}}$ is said to be introverted (resp. m-introverted) if $\tilde{\mathcal{F}}$ is both left and right introverted (resp. m-introverted). Note that by [2; 2.1.4] and linearity of the mapping $\tilde{\mu} \to \tilde{T}_{\tilde{\mu}}$, the set $M(\tilde{\mathcal{F}})$ in the above definition may be replaced by $\tilde{\mathcal{F}}^*$.
(iii) Let $w \in \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ be conjugate closed, define

$$\tilde{Z}_{\tilde{T}} := \{\tilde{\nu} \in \tilde{\mathcal{F}}^* : \tilde{T}_{\tilde{\nu}} \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{F}}\},$$

and

$$\tilde{Z}_{\tilde{U}} := \{\tilde{\mu} \in \tilde{\mathcal{F}}^* : \tilde{U}_{\tilde{\mu}} \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{F}}\}.$$

If $\tilde{\mu} \in \tilde{\mathcal{F}}^*$ and $\tilde{\nu} \in \tilde{Z}_{\tilde{T}}$ define $\tilde{\mu}\tilde{\nu} : \tilde{\mathcal{F}} \to C$ by

$$\tilde{\mu}\tilde{\nu}(f) := \tilde{\mu}(\tilde{T}_{\tilde{\nu}} f) \quad (f \in \tilde{\mathcal{F}}).$$

If $\tilde{\mu} \in \tilde{Z}_{\tilde{U}}$ and $\tilde{\nu} \in \tilde{\mathcal{F}}^*$ define $\tilde{\mu} \star \tilde{\nu} : \tilde{\mathcal{F}} \to C$ by

$$\tilde{\mu} \star \tilde{\nu}(f) := \tilde{\nu}(\tilde{U}_{\tilde{\mu}} f) \quad (f \in \tilde{\mathcal{F}}).$$

(iv) An admissible subspace of $B(S,w)$ is a norm closed, left introverted subspace of $B(S,w)$. An m-admissible subalgebra of $B(S,w)$ is a left m-introverted C^*-subalgebra of $B(S,w)$.

The space $B(S,w)$ itself is obviously admissible, as is $\tilde{\mathcal{F}} = \{\alpha w : \alpha \in C\}$. Less trivial examples of admissible spaces are $C(S,w)$, when S is a compact semitopological semigroup, and $\mathcal{LUC}(S,w)$, when S is a topological group and w is multiplicative. (These are direct consequences of the definitions (2.6). Note that when w is a muliplicative weight function, then $\mathcal{LUC}(S,w) = \{f \in C(S,w) : f/w \in \mathcal{LUC}(S)\}$, see also [2; 2.2.10

and 4.4.3].) Note that, for an admissible subspace $\tilde{\mathcal{F}}$, $M(\tilde{\mathcal{F}})$ is a subsemigroup of $\tilde{Z}_{\tilde{U}} = \tilde{\mathcal{F}}$, under the multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$. Analogously, for m-admissible subalgera $\tilde{\mathcal{F}}$ of $B(S,w)$, $MM(\tilde{\mathcal{F}})$ is a subalgebra of $\tilde{Z}_{\tilde{T}}$.

Remark 2.7. (i) If $\tilde{\mathcal{F}}$ is an admissible subspace of $B(S,w)$, then $\tilde{\mathcal{F}}^*$ is a Banach algebra under the dual space norm and multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$. Furthermore with respect to the weak* topology and multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$, $\tilde{\mathcal{F}}^*$ is a right toplogical, affine subsemigroup of $\tilde{\mathcal{F}}^*$, $co(\tilde{\epsilon}(S)) \subseteq \Lambda(M(\tilde{\mathcal{F}}))$, (hence $\Lambda(M(\tilde{\mathcal{F}}))$ is dense in $M(\tilde{\mathcal{F}})$), and $\tilde{\epsilon} : S \to M(\tilde{\mathcal{F}})$ is a homomorphism, where $\Lambda(M(\tilde{\mathcal{F}})) = \{\tilde{\nu} :$ the map $\tilde{\mu} \to \tilde{\nu}\tilde{\mu}$ is continuous$\}$.
(ii) If $\tilde{\mathcal{F}}$ is an m-admissible subalgebra of $B(S,w)$, then with respect to the weak* topology and multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$, $MM(\tilde{\mathcal{F}})$ is a compact, right topological semigroup, $\tilde{\epsilon}(S) \subseteq \Lambda(MM(\tilde{\mathcal{F}}))$ (hence $\Lambda(MM(\tilde{\mathcal{F}}))$ is dense in $MM)\tilde{\mathcal{F}})$, and $\tilde{\epsilon} : S \to MM(\tilde{\mathcal{F}})$ is homomorphism.

Examples 2.8. Let S be a compact semitopological semigroup, then by [**2**; 2.22.12], $C(S,w)^*$ is introverted, hence both products $\tilde{\mu}\tilde{\nu}$ and $\tilde{\mu} \star \tilde{\nu}$ are defined on $C(S,w)^*$ and

(a) $\tilde{\mu}\tilde{\nu} = \tilde{\mu} \star \tilde{\nu}$ for all $\tilde{\mu},\tilde{\nu} \in C(S,w)^*$,
(b) with respect to the weak* topology and multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$, $C(S,w)^*$ is a semitopological semigroup, hence $M := M(C(S,w))$ is a compact semitopological affine semigroup; and
(c) If S is a topological semigroup, then so in M.

DEFINITION 2.9

Let (S_1,w_1) and (S_2,w_2) be weighted semigroups. A continuous homomorphism θ of S_1 into S_2 is called a homomorphism of (S_1,w_1) into (S_2,w_2) if $w_2 \circ \theta = w_1$. (We abbreviate it as $\theta : (S_1,w_1) \to (S_2,w_2)$.)

Examples 2.10. (i) Let (S,w) be a weighted semitopological semigroup, and let T be a subsemigroup of S, then the inclusion mapping $i : T \to S$, is a continuous homomorphism of (T,w) into (S,w).
(ii) Let $S_1 = N$ with the ordinary addition, and discrete topology, and let $w_1(n) = 1 + 2n$, let also $S_2 = Z$ with the ordinary addition, and let $w_2(m) = 1 + |m|$, then $\theta : (S_1,w_1) \to (S_2,w_2)$ defined by $\theta(s_1) = 2s_1$, is a continuous homomorphism from (S_1,w_1) into (S_2,w_2).
(iii) Let $S_1 = (0,1)$, with the usual topology and multiplication $s_1t_1 = \min\{s_1,t_1\}$, and $w_1(s_1) = 3 - s_1$, and let $S_2 = [2,3]$ with the usual topology and multiplication $s_2t_2 = \max\{s_2,t_2\}$, and $w_2(s_2) = s_2$, then $\theta : (S_1,w_1) \to (S_2,w_2)$, defined by $\theta(s_1) = 3 - s_1$ is a continuous homomorphism from (S_1,w_1) into (S_2,w_2).

Remark 2.11. If $\theta : (S_1,w_1) \to (S_2,w_2)$ is a continuous homomorphism, then $\theta^* C(S_2,w_2)$ is a translation invariant C^*-subalgebra of $C(S_1,w_1)$.

Theorem 2.12. *Let $\theta : (S_1,w_1) \to (S_2,w_2)$ be a continuous homomorphism, and let $\theta^* : C(S_2,w_2) \to C(S_1,w_1)$ denote its dual map, then*

(i) $\theta^* \mathcal{WAP}(S_2,w_2) \subseteq \mathcal{WAP}(S_1,w_1)$,
(ii) $\theta^* \mathcal{AP}(S_2,w_2) \subseteq \mathcal{AP}(S_1,w_1)$,
(iii) $\theta^* \mathcal{LUC}(S_2,w_2) \subseteq \mathcal{LUC}(S_1,w_1)$.

In particular if S_1 is a subsemigroup of S_2, and w is a weight function on S_2, then

(i)' $\mathcal{WAP}(S_2,w)|_{S_1} \subseteq \mathcal{WAP}(S_1,w)$,
(ii)' $\mathcal{AP}(S_2,w)|_{S_1} \subseteq \mathcal{AP}(S_1,w)$,
(iii)' $\mathcal{LUC}(S_2,w)|_{S_1} \subseteq \mathcal{LUC}(S_1,w)$.

Proof. (i) Let $f \in \mathcal{WAP}(S_2,w_2)$, and $\{x_n\},\{y_m\}$ be sequences in S_1, since

$$\frac{\theta^*(f)(x_n y_m)}{w_1(x_n)w_1(y_m)} = \frac{f(\theta(x_n)\theta(y_m))}{w_2(\theta(x_n))w_2(\theta(y_m))}$$

and

$$\lim_n \lim_m \frac{f(\theta(x_n)\theta(y_m))}{w_2(\theta(x_n))w_2(\theta(y_m))} = \lim_m \lim_n \frac{f(\theta(x_n)\theta(y_m))}{w_2(\theta(x_n))w_2(\theta(y_m))},$$

so

$$\lim_n \lim_m \frac{\theta^*(f)(x_n y_m)}{w_1(x_n)w_1(y_m)} = \lim_m \lim_n \frac{\theta^*(f)(x_n y_m)}{w_1(x_n)w_2(y_m)},$$

i.e. $\theta^*(f) \in \mathcal{WAP}(s_1,w_1)$.
(ii) Let $f \in \mathcal{AP}(S_2,w_2)$ and s_1, $x \in S_1$. Since

$$\frac{L_{S_1}\theta^*(f)}{w_1(s)w_1} = \theta^*\left(\frac{L_{\theta(s_1)}f}{w_2(\theta(s_1))w_2}\right)$$

and

$$\left\{\frac{L_{s_2}f}{w_2(s_2)w_2} : s_2 \in S_2\right\}$$

is a relatively compact subset of $C(S_2)$, and since

$$A := \left\{\frac{L_{\theta(s_1)}f}{w_2(\theta(s_1))w_2} : s_1 \in S_1\right\} \subseteq \left\{\frac{L_{s_2}f}{w_2(s_2)w_2} : s_2 \in S_2\right\},$$

thus $\overline{A}$ is a compact subset of $C(S_2)$, and so $\theta^*(\overline{A})$ is compact, and since $\overline{\theta^*(A)} \subseteq \overline{\theta^*(\overline{A})} = \theta^*(\overline{A})$, so $\theta^*(A)$ is a relatively compact subset of $C(S_1)$, i.e. $\theta^*(f) \in \mathcal{AP}(S_1,w_1)$.
(iii) Let $f \in \mathcal{LUC}(S_2,w_2)$, and $s_1 \in S_1$. Since θ is continuous, the map $s_1 \to L_{\theta(s_1)}f/(w_2(\theta(s_1))w_2)$ from S_1 into $C(S_2)$ is (norm) continuous, but $\theta^* : C(S_2) \to C(S_1)$ is also (norm) continuous, so the map $s_1 \to \theta^*(L_{\theta(s_1)}f/(w_2(\theta(s_1))w_2))$ from S_1 into $C(S_1)$ is (norm) continuous, and since $L_{s_1}\theta^*(f)/(w_1(s_1)w_1) = \theta^*(L_{\theta(s_1)}f/(w_2(\theta(s_1))w_2))$, the map $s_1 \to L_{s_1}\theta^* f/(w_1(s_1)w_1)$ from S_1 into $C(S_1)$ is (norm) continuous, i.e. $\theta^*(f) \in \mathcal{LUC}(S_1,w_1)$.
For the second assertion ((i)', (ii)', (iii)'), take θ to be the inclusion mapping.

DEFINITION 2.13

(i) A weighted (semigroup) compactification of a weighted semitopological semigroup (S,w) is a triple $(\psi, X, \overline{w})$, where $(X,\overline{w})$ is a compact, Hausdorff, right topological weighted semigroup, and $\psi : (S,w) \to (X,\overline{w})$ is a continuous weighted homomorphism such that $\overline{\psi(S)} = X$, and $\psi(S) \subseteq \Lambda(X) = \{t \in X$: the function $s \to ts : X \to X$ is continuous$\}$.

(ii) Let (S,w) be a weighted semitopological semigroup and let P be a property of compactifications $(\psi,X,\overline{w})$ of (S,w). A weighted P-compactification of (S,w) is a compactification of (S,w) which has the given property P. Affine weighted semigroup compactifications and affine weighted semigroup P-compactifications are defined analogously.

(iii) Let $(\psi,X,\overline{w})$ and (φ,Y,w_1) be two compactifications of a weighted semitopological semigroup (S,w). A continuous homomorphism π from $(X,\overline{w})$ onto (Y,w_1) such that $\pi\circ\psi=\varphi$ is called a homomorphism from $(\psi,X,\overline{w})$ onto (φ,Y,w_1), and is denoted by $\pi\colon(\psi,X,\overline{w})\rightarrow(\varphi,Y,w_1)$. If such a homomorphism exists, then (φ,Y,w_1) is said to be a factor of $(\psi,X,\overline{w})$, and $(\psi,X,\overline{w})$ is said to be an extension of (φ,Y,w_1), and we write $(\psi,X,\overline{w})\geq(\varphi,Y,w_1)$. Note that if π is a homomorphism of $(\psi,X,\overline{w})$ onto (φ,Y,w_1) and π_1 is a homomorphism of (φ,Y,w_1) onto $(\psi,X,\overline{w})$, then $\pi_1\circ\pi:(X,\overline{w})\rightarrow(Y,w_1)$ is a continuous function whose restriction to the dense subset $\psi(S)$ is the identity function. It follows readily that $\pi_1=\pi^{-1}$. In this setting we say that π is an isomorphism of $(\psi,X,\overline{w})$ onto (φ,Y,w_1), or that $(\psi,X,\overline{w})$ is isomorphic to (φ,Y,w_1), and we write $(\psi,X,\overline{w})\cong(\varphi,Y,w_1)$.

Theorem 2.14. *If $(\psi,X,\overline{w})$ is a weighted compactification of a weighted semitopological semigroup (S,w), then $\psi^*C(X,\overline{w})$ is an m-admissible subalgebra of $C(S,w)$.*

Proof. Since $w=\psi^*(\overline{w})$, thus $w\in\psi^*C(X,\overline{w})$, so by Remark (2.13) it is enough to show that for each $f\in C(X,\overline{w})$, each $s\in S$, and each $\tilde{\mu}\in MM(\psi^*C(X,\overline{w}))$, $w\cdot L_s(\psi^*(f)/w)$, $w\cdot R_s(\psi^*(f)/w)$, $\tilde{T}_{\tilde{\mu}}(\psi^*(f))\in\psi^*C(X,\overline{w})$, (which we denote by $\tilde{\mathcal{F}}$, i.e. $\tilde{\mathcal{F}}=\psi^*C(X,\overline{w})$). First note that ${}_x\overline{\Omega}\in C(X)$ for all $x\in X$, (where ${}_x\overline{\Omega}:X\rightarrow C$ is defined by ${}_x\overline{\Omega}(y)=\overline{w}(xy)/(\overline{w}(x)\overline{w}(y))$, but X is compact, thus $\overline{w}$ is bounded away from zero and also bounded, thus $1/{}_x\overline{\Omega}\in C(X)$ for all $x\in X$, so $\overline{w}\cdot(1/{}_x\overline{\Omega})\in C(X,\overline{w})$, and therefore $\psi^*(\overline{w}\cdot(1/{}_x\overline{\Omega}))\in\tilde{\mathcal{F}}$. Since $(1/{}_s\Omega)\cdot w=\psi^*((1/{}_{\psi(s)}\Omega)\cdot\overline{w})$, therefore $(1/{}_s\Omega)\cdot w\in\tilde{\mathcal{F}}$, (so $(1/{}_s\Omega)\in\mathcal{F}$).

By Remark (2.11) (i) $L_s(\psi^*(f))\in\tilde{\mathcal{F}}$, thus $L_s(\psi^*(f))/w\in\mathcal{F}$, and since $\mathcal{F}$ is an algebra, $L_s(\psi^*(f))/w\cdot(1/{}_s\Omega)\in\mathcal{F}$, and therefore $L_s(\psi^*(f))/w\cdot(1/{}_s\Omega)\cdot w\in\tilde{\mathcal{F}}$.

On the other hand $w\cdot L_s(\psi^*(f)/w)=1/w(s)(L_s(\psi^*(f))/w\cdot(1/{}_s\Omega)\cdot w)$, thus $w\cdot L_s(\psi^*(f)/w)\in\tilde{\mathcal{F}}$. Similarly we can show that $w\cdot R_s(\psi^*(f)/w)\in\tilde{\mathcal{F}}$. To see that $\tilde{T}_{\tilde{\mu}}(\psi^*(f))\in\tilde{\mathcal{F}}$, choose a net $\{s_\alpha\}$ is S such that $\tilde{\epsilon}(s_\alpha)\rightarrow\tilde{\mu}$ in the $\sigma(\tilde{\mathcal{F}}^*,\tilde{\mathcal{F}})$ topology, where $\tilde{\epsilon}:S\rightarrow MM(\tilde{\mathcal{F}})$ denotes the evaluation mapping. (Thus $\epsilon(S_\alpha)\rightarrow\mu$ in $\sigma(\mathcal{F}^*,\mathcal{F})$ topology.) We may assume, without loss of generality, that $x:=\lim_\alpha\psi(s_\alpha)$ exists in X. However

$$\begin{aligned}
\tilde{T}_{\tilde{\mu}}(\psi^*(f))(s)&=w(s)\cdot\mu\left(L_s\left(\frac{\psi^*(f)}{w}\right)\right)=w(s)\cdot\lim_\alpha\epsilon(s_\alpha)\left(L_s\left(\frac{\psi^*(f)}{w}\right)\right)\\
&=w(s)\cdot\lim_\alpha L_s\left(\frac{\psi^*(f)}{w}\right)(s_\alpha)=w(s)\cdot\lim_\alpha\frac{\psi^*(f)}{w}(ss_\alpha)\\
&=w(s)\cdot\lim_\alpha\frac{f(\psi(ss_\alpha))}{w(ss_\alpha)}=w(s)\cdot\lim_\alpha\frac{f(\psi(s)\psi(s_\alpha))}{\overline{w}(\psi(s)\psi(s_\alpha))}\\
&=w(s)\cdot\frac{f(\psi(s)x)}{\overline{w}(\psi(s)x)}=w(s)\cdot R_x\left(\frac{f}{\overline{w}}\right)(\psi(s))\\
&=\overline{w}(\psi(s))\cdot R_x\frac{f}{\overline{w}}(\psi(s))=\left(\overline{w}\cdot R_x\frac{f}{\overline{w}}\right)(\psi(s))=\psi^*\left(\overline{w}\cdot R_x\frac{f}{\overline{w}}\right)(s).
\end{aligned}$$

Thus $\tilde{T}_{\tilde{\mu}}(\psi^*(f))=\psi^*(\overline{w}\cdot R_x(f/w))$, and since $\psi^*(\overline{w}\cdot R_x(f/w))\in\tilde{\mathcal{F}}$, therefore $\tilde{T}_{\tilde{\mu}}(\psi^*(f))\in\tilde{\mathcal{F}}$, as claimed.

Remark 2.15. (i) Let (S,w) be a weighted semitopological semigroup, and let $\tilde{\mathcal{F}}$ be an m-admissible subalgebra of $C(S,w)$, then $MM(\tilde{\mathcal{F}})$ is a compact right topological semigroup (under multiplication $(\tilde{\mu},\tilde{\nu}) \to \tilde{\mu}\tilde{\nu}$), and $\tilde{\epsilon}(S) \subseteq \Lambda(MM(\tilde{\mathcal{F}}))$, thus if there exists a weight function $\overline{w}$ on $MM(\tilde{\mathcal{F}})$ such that $w = \overline{w} \circ \tilde{\epsilon}$, then $(\tilde{\epsilon}, MM(\tilde{\mathcal{F}}), \overline{w})$ is a weighted semigroup compactification of (S,w).
(ii) By [2; 3.1.7] there is a one-to-one correspondence between m-admissible subalgebras of $C(S)$ and compactifications of S, but in the weighted case this is false, since if S is any nontrivial semigroup, with nonconstant weight function w, then $\tilde{\mathcal{F}} = \{\alpha \cdot w : \alpha \in C\}$ is an m-admissible subalgebra of $C(S,w)$, for which $MM(\tilde{\mathcal{F}})$ is the trivial semigroup, thus there is no weight function $\overline{w}$, such that $w = \overline{w} \circ \tilde{\epsilon}$ (since if $\overline{w}$ exists, then w must be constant on S).

Examples 2.16. (i) Let $S = (R = \{0\}, .)$ with the usual topology and let w be defined on S as follows:

$$w(t) = \begin{cases} 2 & \text{if} \quad 0 < |t| < 1 \\ 1 & \text{if} \quad |t| \geq 1. \end{cases}$$

and let $\tilde{\mathcal{F}}$ be any m-admissible subalgebra of $C(S,w)$, which contains $C_0(S,w)$, then we can define $\overline{w}$ on $MM(\tilde{\mathcal{F}})$ as follows:

$$\overline{w}(\tilde{\mu}) = \begin{cases} w(s) & \text{if} \quad \tilde{\mu} = \tilde{\epsilon}(s) \quad \text{for some } s \in S \\ 1 & \text{otherwise.} \end{cases}$$

Note that, Ω is not separately continuous, since ${}_{1/2}\Omega(2) = w(1)/(w(1/2)w(2)) = 1/2$, and if $\{t_\alpha\}$ is an increasing net in R with $1 < t_\alpha < 2$ such that $t_\alpha \to 2$, then ${}_{1/2}\Omega(t_\alpha) = 1$.
(ii) Let $S = N$ with the ordinary multiplication and discrete topology and let w be defined on S as follows:

$$w(s) = \begin{cases} 3 & \text{if } s < 10 \\ 4 & \text{if } s \geq 10, \end{cases}$$

let also $\tilde{\mathcal{F}}$ be any m-admissible subalgebra of $C(S,w)$ which contains $C_0(S,w)$. Then we can define $\overline{w}$ on $MM(\tilde{\mathcal{F}})$ as follows:

$$\overline{w}(\mu) = \begin{cases} w(s) & \text{if } \tilde{\mu} = \tilde{\epsilon} \text{ for some } s \in S \\ 4 & \text{otherwise.} \end{cases}$$

i.e. $(\tilde{\epsilon}, MM(\tilde{\mathcal{F}}), \overline{w})$ is a compactification of (S,w).

Remark 2.17. In [9] another definition for a mean is given, but in that definition f and f/w are both bounded (and w does not need to be less than 1), but in definition (2.1) f need not be bounded (and w can be greater than or equal to 1), but as we will see in the next theorem, when we speak about compactifications, then w must be bounded and not less than 1. Thus in this case these two definitions coincide.

Theorem 2.18. *Let (S,w) be a weighted semitopological semigroup. If*

(i) *w is unbounded or*
(ii) *$w(s) < 1$ for some $s \in S$,*

then, there is no compactification by (s,w)

Proof. Suppose that $(\psi, X, \overline{w})$ is a compactification of (S, w), and (i) holds, so there exists a sequence $\{s_n\}$ in S such that $w(s_n) \to \infty$ (and $w(s_i) \neq w(s_j)$ if $i \neq j$), thus $\{\psi(s_n): n \in N\}$ is a sequence of distinct elements in X, and $\overline{w}(\psi(s_n)) = w(s_n) \to \infty$, which is a contradiction (since X is compact, $\overline{w}$ must be bounded on X).

Now let $(\psi, X, \overline{w})$ be a compactification of (S, w), and $w(s) < 1$ for some $s \in S$. Then as we have seen in Remark (1.2) (i), $\{s^n : n \in N\}$ is a sequence of distinct elements in S, such that $w(s^n) \to 0$, thus $\{\psi(s^n) : n \in N\}$ is a sequence of distinct elements in X such that $\overline{w}(\psi(s_n)) = w(s_n) \to 0$ so $1/(\overline{w}(\psi(s_n))) \to \infty$, which is a contradiction, (since X is compact, so $1/\overline{w}$ must be bounded on X).

COROLLARY 2.19

(i) *Let (S, w) be a weighted semitopological semigroup, and let $\tilde{\mathcal{F}}$ be any m-admissible subalgebra (resp. admissible subspace) of $C(S, w)$. If w is unbounded or $w(s) < 1$ for some $s \in S$, then we have no extension of w to $MM(\tilde{\mathcal{F}})$ (resp. $M(\tilde{\mathcal{F}})$).*

(ii) *If $(\psi, X, \overline{w})$ is a compactification of (S, w), then w must be bounded and not less than 1.*

Theorem 2.20. *Let S be a semitopological semigroup, and let w be a bounded weight function on S, such that $w \geq 1$, then*

(i) *If $w \in \mathcal{LUC}(S, w)$, then $\mathcal{LUC}(S, w)$ is an m-admissible subalgebra of $C(S, w)$.*

(ii) *If w is in $\mathcal{AP}(S, w)$ (resp. $\mathcal{WAP}(S, w)$), then $\mathcal{AP}(S, w)$ (resp. $\mathcal{WAP}(S, w)$) is an m-admissible subalgebra of $C(S, w)$.*

Proof. (i) Suppose that $w \in \mathcal{LUC}(S, w)$, then by Theorem (1.3), $\mathcal{LUC}(S, w)$ is a translation invariant C^*-subalgebra of $C(S, w)$. The map $s \to {}_s\Omega$ from S into $C(S)$ is norm continuous, thus the map $s \to {}_s\Omega^{-1}$ from S to $C(S)$ is norm continuous. But $L_s f/(w(s)w) = L_s(f/w) \cdot {}_s\Omega$, so $f \in \mathcal{LUC}(S, w)$ if and only if $f/w \in \mathcal{LUC}(S)$, thus by [**2**; 4.4.3] $\mathcal{LUC}(S, w)$ is an m-admissible subalgebra of $C(s, w)$.

(ii) Let $w \in \mathcal{AP}(s, w)$, then $A := \{{}_s\Omega : s\}$ is a relatively compact subset of $C(S)$. Since w is bounded and not less than 1, $B := \{{}_s\Omega^{-1} : s \in S\}$ is a relatively (norm) compact subset of $C(S)$. Now let $f \in \mathcal{AP}(S, w)$, and $t \in S$, since $L_s(w \cdot R_t(f/w))/(w(s)w) = {}_s\Omega \cdot R_t(L_s w/(w(s)w) \cdot {}_s\Omega^{-1})$ and $\{{}_s\Omega \cdot R_t(L_s w/(w(s)w)) \cdot {}_s\Omega^{-1} : s \in S\}$, is a relatively (norm) compact subset of $C(S)$, thus $w \cdot R_t(f/w) \in \mathcal{AP}(S, w)$. Similarly, $w \cdot L_t(f/w) \in \mathcal{AP}(S, w)$.

Now let $\tilde{\mu} \in MM(\mathcal{AP}(S, w))$. Since $L_s \tilde{T}_{\tilde{\mu}} f/(w(s)w) = {}_s\Omega \cdot T_\mu(L_s f/(w(s)w) \cdot {}_s\Omega^{-1})$ and $\{{}_s\Omega \cdot T_\mu(L_s f/(w(s)w) \cdot {}_s\Omega^{-1} : s \in S\}$ is a relatively (norm) compact subset of $C(S)$, thus $\tilde{T}_{\tilde{\mu}} f \in \mathcal{AP}(S, w)$.

Similarly we can show that if $w \in \mathcal{WAP}(S, w)$ then $\mathcal{WAP}(S, w)$ is an m-admissible subalgebra of $C(S, w)$.

COROLLARY 2.21

Let (S, w) be a weighted semitopological semigroup such that w is bounded and not less than 1. If w is in $\mathcal{AP}(S, w)$ (resp. $\mathcal{WAP}(S, w)$, $\mathcal{LUC}(s, w)$) and if there exists a weight function $\overline{w}$ on $MM(\mathcal{AP}(S, w))$ (resp. $MM(\mathcal{WAP}(S, w)), MM(\mathcal{LUC}(S, w))$) such that $\overline{w} \circ \tilde{\epsilon} = w$, then $(\tilde{\epsilon}, MM(\mathcal{AP}(S, w)), \overline{w})$ (resp. $(\tilde{\epsilon}, MM(\mathcal{WAP}(S, w)), \overline{w})$, $(\tilde{\epsilon}, MM(\mathcal{LUC}\ (S, w)), \overline{w})$) is a weighted compactification of (S, w), which is universal with respect to topological semigroup property, (resp. semitopological semigroup property, joint continuity property).

Remark 2.22. If S is either a left zero semigroup or a right zero semigroup, with weight function w, then

(i) $\mathcal{AP}(S,w)$ is m-admissible, (in fact $\mathcal{AP}(S,w) = \mathcal{WAP}(S,w) = \mathcal{LUC}(S,w) = C(S,w)$). (Note that in this case we have $L_s f/(w(s)w) = (f(s)/w(s)) \cdot (1/w)$ or $L_s f/(w(s)w) = (1/w(s)) \cdot (f/w)$).

(ii) If there exists a weight function $\overline{w}$ on $MM(\mathcal{AP}(S,w))$, such that $\overline{w} \circ \tilde{\epsilon} = w$, then $(\tilde{\epsilon}, MM(\mathcal{AP}(S,w)), \overline{w})$ is a compactification of (S,w), which is universal with respect to topological semigroup property.

Example 2.23. Let $S = [0,1)$ with left zero multiplication, and usual topology, and let w be a weight function on S defined by $w(s) = 1 + s (s \in S)$, then we can define $\overline{w}$ on $MM(\mathcal{AP}(S,w))$ as follows:

$$\overline{w}(\tilde{\mu}) = \begin{cases} w(s) & \text{if } \tilde{\mu} = \tilde{\epsilon}(s) \text{ for some } s \in S \\ 2 & \text{otherwise.} \end{cases}$$

Thus $(\tilde{\epsilon}, MM(\mathcal{AP}(S,w)), \overline{w})$ is a compactification of (S,w), which is universal with respect to topological semigroup property.

Remark 2.24. (i) If S is a discrete semigroup, with bounded weight function w, such that $w \geq 1$, then $\mathcal{LUC}(S,w) = C(S,w)$ (and $\mathcal{LUC}(S) = C(S)$), thus if there exists a weight function $\overline{w}$ on $MM(\mathcal{LUC}(S,w))$ such that $\overline{w} \circ \tilde{\epsilon} = w$, then $(\tilde{\epsilon}, MM(\mathcal{LUC}(S,w)), \overline{w})$ is a compactification of (S,w) which is universal with respect to joint continuity property.
(ii) Let (S,w) be a weighted semitopological semigroup, then if w is a multiplicative weight function on S, then $\mathcal{AP}(S,w) = \{f \in C(s,w) : f/w \in \mathcal{AP}(S)\}$ (resp. $\mathcal{WAP}(S,w) = \{f \in C(S,w) : f/w \in \mathcal{WAP}(S)\}$, $\mathcal{LUC}(S,w) = \{f \in C(S,w) : f/w \in \mathcal{LUC}(S)\}$), thus if there exists a weight function $\overline{w}$ on $MM(\mathcal{AP}(S,w))$ (resp. $MM(\mathcal{WAP}(S,w))$, $MM(\mathcal{LUC}(S,w))$) such that $\overline{w} \circ \tilde{\epsilon} = w$, then $(\tilde{\epsilon}, MM(\mathcal{AP}(S,w)), \overline{w})$ (resp. $(\tilde{\epsilon}, MM(\mathcal{WAP}(S,w)), \overline{w})$, $(\tilde{\epsilon}, MM(\mathcal{LUC}(S,w)), \overline{w})$), is a compactification of (S,w), which is universal with respect to topological semigroup property, (resp. universal with respect to semi-topological semigroup property, universal with respect to joint continuity.) Note that $w \equiv 1$is the only bounded multiplicative weight function which is not less than 1.
(iii) If $\tilde{\epsilon}$ is one-to-one (for example this happen if $C_0(s,w) \subseteq \tilde{\mathcal{F}}$), then this is a good condition for existence of $\overline{w}$, but this condition is neither necessary nor sufficient (e.g. we may take $w = 1$ or take an unbounded weight function).

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On a conjecture of Hubner

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Abstract. In this paper we show that for a bounded linear operator A on a complex Hilbert space $\mathcal{H}$, the points on the boundary of the numerical range of A with infinite curvature and unique tangent are in the essential spectrum of A, thus positively answering a conjecture raised by Hubner in [3].

Keywords. Spectrum; numerical range; Hilbert space.

1. Introduction and notation

The numerical range $W(A)$ of a bounded linear operator A defined on a complex Hilbert space $\mathcal{H}$ is the set $\{(Au,u) : u \in \mathcal{H} \text{ and } \|u\| = 1\}$, where (Au,u) denotes the inner product of Au and u. The Toeplitz–Hausdorff theorem states that $W(A)$ is a convex set. The closure $\overline{W(A)}$ is also convex, and is a compact set that contains the spectrum $\sigma(A)$ of A. For more results on the numerical range, we refer the reader to [1] and [2]. Hubner in [3] considered the points on the boundary $C(A)$ of $W(A)$ with infinite curvature and proved that they are points in the spectrum of A. He also conjectured that the points in $C(A)$ with infinite curvature and unique tangent are in the essential spectrum of A. To be explicit, we translate and multiply the operator with complex numbers so that $0 \in C(A)$, there is infinite curvature of $C(A)$ at 0 and the real axis is the tangent to $C(A)$ at 0. So Hubner's conjecture may be stated as follows: Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. If the numerical range $W(A)$ of A satisfies the conditions

$$\inf_{\|u\|=1} \operatorname{Im}(Au,u) = 0, \tag{1.1}$$

$$\lim_{\delta\to 0} \inf_{0<|\operatorname{Re}(Au,u)|<\delta, \|u\|=1} \frac{\operatorname{Im}(Au,u)}{(\operatorname{Re}(Au,u))^2} = \infty \tag{1.2}$$

and

$$\lim_{\delta\to 0} \inf_{0<|\operatorname{Re}(Au,u)|<\delta, \|u\|=1} \frac{\operatorname{Im}(Au,u)}{|\operatorname{Re}(Au,u)|} = 0, \tag{1.3}$$

then 0 is in the essential spectrum of A. We settle Hubner's conjecture positively. But first we observe that if A satisfies (1.1) and (1.2), then from the convexity of $\overline{W(A)}$, we have

$$\overline{W(A)} \cap \mathbb{R} = \{0\}, \tag{1.4}$$

where $\mathbb{R}$ denotes the set of real numbers.

For a closed linear operator T in a complex Banach space χ, the nullity $\operatorname{nul}(T)$ of T is the dimension of the kernel $\mathcal{N}(T)$ of T, and the deficiency $\operatorname{def}(T)$ of T is the dimension

of the quotient space $\chi/\mathcal{R}(T)$, where $\mathcal{R}(T)$ denotes the range of T. T is called semi-Fredholm if $\mathcal{R}(T)$ is closed and at least one of $\mathrm{nul}(T)$ and $\mathrm{def}(T)$ is finite. The essential spectrum $\sigma_e(T)$ of T is the complement in the complex plane $\mathbb{C}$ of the set $\{z \in \mathbb{C} : T - zI$ is semi-Fredholm$\}$, where I denotes the identity linear operator (see pages 242 and 243 of [4]. The essential spectrum defined in [4] is a subset of the essential spectrum defined by Wolf [6]).

All Hilbert spaces considered in the paper are over the field of complex numbers $\mathbb{C}$. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, respectively. The imaginary unit is denoted by i and the set of positive integers is denoted by $\mathbb{N}$.

2. The main result

In [3] it is shown that for a bounded linear operator A on a Hilbert space $\mathcal{H}$ that satisfies conditions (1.1) and (1.2), if $\{u_n\}$ is a sequence of unit vectors in $\mathcal{H}$ such that $\{(Au_n, u_n)\}$ is a sequence of nonzero pure imaginary complex numbers converging to 0, then $\|Au_n\| \to 0$ as $n \to \infty$. The following lemma generalizes this result and will be used in Theorem 2.1. We use the same idea used in [3], but the proof of Lemma 2.1 is technically more difficult.

Lemma 2.1. *Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, and assume that A satisfies conditions* (1.1) *and* (1.2). *Suppose that $\{u_n\}$ is a sequence of unit vectors in $\mathcal{H}$ such that* $\lim_{n\to\infty}(Au_n, u_n) = 0$. *Then* $\lim_{n\to\infty} \|Au_n\| = 0$.

Proof. Let $d_n = (Au_n, u_n)$, $\delta_n = \mathrm{Re}\, d_n$ and $\epsilon_n = \mathrm{Im}\, d_n$ for all $n \in \mathbb{N}$. Because of (1.1), we have $\epsilon_n \geq 0$ for all $n \in \mathbb{N}$. Also, the sequences $\{d_n\}$, $\{\delta_n\}$ and $\{\epsilon_n\}$ all converge to 0, and they satisfy the following property:

Property 1. If $\{d'_n\}$ is a subsequence of $\{d_n\}$ such that $\epsilon'_n = \mathrm{Im}\, d'_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \delta'_n/\sqrt{\epsilon'_n} = 0$, where $\delta'_n = \mathrm{Re}\, d'_n$.

Property 1 follows from $\delta_n \to 0$ and (1.2). Since $\{u_n\}$ is a sequence of unit vectors, we can write

$$Au_n = d_n u_n + x_n v_n, \tag{2.1}$$

where $x_n \in \mathbb{C}$, and v_n are unit vectors in $\mathcal{H}$ that satisfy $(u_n, v_n) = 0$ for all $n \in \mathbb{N}$. We prove $\lim_{n\to\infty} x_n = 0$. For all $n \in \mathbb{N}$, consider the perturbed vectors $w_n^{\pm} = u_n \pm z_n v_n$, where

$$z_n = \begin{cases} \sqrt{\epsilon_n} e^{i\phi_n} & \text{if } \epsilon_n \neq 0 \\ e^{i\phi_n}/\sqrt{n} & \text{otherwise,} \end{cases} \tag{2.2}$$

and ϕ_n are to be determined. From (2.1), we have for all $n \in \mathbb{N}$,

$$(Aw_n^{\pm}, w_n^{\pm}) = d_n \pm (z_n y_n + \bar{z}_n x_n) + |z_n|^2 (Av_n, v_n), \tag{2.3}$$

where $y_n = (Av_n, u_n)$. Since $\|w_n^{\pm}\|^2 = 1 + |z_n|^2 > 0$ and $\epsilon_n = \mathrm{Im}\, d_n$, we deduce from (1.1) and (2.3) that

$$\mathrm{Im}\, (Aw_n^{\pm}, w_n^{\pm}) = \epsilon_n \pm \mathrm{Im}\, (z_n y_n + \bar{z}_n x_n) + |z_n|^2 \mathrm{Im}\, (Av_n, v_n) \geq 0 \tag{2.4}$$

for all $n \in \mathbb{N}$. Hence from (2.2), we see that

$$|\, \mathrm{Im}\, (e^{i\phi_n} y_n + e^{-i\phi_n} x_n)| \leq \sqrt{\epsilon_n}(1 + \mathrm{Im}\, (Av_n, v_n)) \text{ if } \epsilon_n \neq 0$$

and

$$|\operatorname{Im}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n)| \le (1/n)^{1/2}\operatorname{Im}(Av_n, v_n) \text{ if } \epsilon_n = 0.$$

Thus from $|\operatorname{Im}(Av_n, v_n)| \le \|A\|$, we obtain

$$|\operatorname{Im}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n)| \le \begin{cases} \sqrt{\epsilon_n}(1+\|A\|) & \text{if } \epsilon_n \ne 0 \\ \|A\|/\sqrt{n} & \text{if } \epsilon_n = 0. \end{cases} \tag{2.5}$$

Since $\lim_{n\to\infty}\epsilon_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$, we deduce from (2.5) that

$$\lim_{n\to\infty} |\operatorname{Im}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n)| = 0. \tag{2.6}$$

Now we prove

$$\lim_{n\to\infty} |\operatorname{Re}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n)| = 0. \tag{2.7}$$

From (2.3) and $\delta_n = \operatorname{Re} d_n$, we have

$$\operatorname{Re}(Aw_n^+, w_n^+) = \delta_n + \operatorname{Re}(z_n y_n + \bar{z}_n x_n) + |z_n|^2 \operatorname{Re}(Av_n, v_n). \tag{2.8}$$

Let $\mathbb{N}_1 = \{n \in \mathbb{N} : \operatorname{Re}(Aw_n^+, w_n^+) \ne 0\}$ and $\mathbb{N}_2 = \mathbb{N} - \mathbb{N}_1$. If $n \in \mathbb{N}_2$, then from (1.4), (2.2) and (2.8) we have

$$\operatorname{Re}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n) = \begin{cases} -\frac{\delta_n}{\sqrt{\epsilon_n}} - \sqrt{\epsilon_n}\operatorname{Re}(Av_n, v_n) & \text{if } \epsilon_n \ne 0 \\ -(1/n)^{1/2}\operatorname{Re}(Av_n, v_n) & \text{if } \epsilon_n = 0. \end{cases} \tag{2.9}$$

Consider the following two cases:

Case 1. $\mathbb{N}_1$ is a finite set. In this case there exists $n_0 \in \mathbb{N}$ such that for all positive integers $n \ge n_0$ we have $n \in \mathbb{N}_2$. Suppose that there exist infinitely many integers n such that $\epsilon_n \ne 0$. Denote the terms d_n of the sequence $\{d_n\}_{n=1}^\infty$ for which $n \ge n_0$ and $\epsilon_n \ne 0$ by the subsequence $\{d_{l_n}\}_{n=1}^\infty$. Since $\lim_{n\to\infty}\epsilon_n = 0$ and $\{(Av_n, v_n)\}$ is bounded, we deduce from Property 1 and the first equation of (2.9) that $\lim_{n\to\infty}\operatorname{Re}(e^{i\phi_{l_n}}y_{l_n} + e^{-i\phi_{l_n}}x_{l_n}) = 0$. Suppose that there exist infinitely many integers n such that $\epsilon_n = 0$. Denote the terms d_n of the sequence $\{d_n\}_{n=1}^\infty$ for which $n \ge n_0$ and $\epsilon_n = 0$ by the subsequence $\{d_{m_n}\}_{n=1}^\infty$. Since $\lim_{n\to\infty} 1/\sqrt{n} = 0$ and $\{(Av_n, v_n)\}$ is bounded, we deduce from the second equation of (2.9) that $\lim_{n\to\infty}\operatorname{Re}(e^{i\phi_{m_n}}y_{m_n} + e^{-i\phi_{m_n}}x_{m_n}) = 0$. This completes the proof of (2.7) in this case.

Case 2. $\mathbb{N}_1$ is an infinite set. Write $\mathbb{N}_1 = \{p_n : n \in \mathbb{N}\}$, where $p_n < p_{n+1}$ for all $n \in \mathbb{N}$. It follows from (1.1) and (1.4) that $\operatorname{Im}(Aw_m^+, w_m^+) > 0$ for all $m \in \mathcal{N}_1$. Since $\lim_{n\to\infty}\delta_n = \lim_{n\to\infty} z_n = 0$, and $\{(1/2)\operatorname{Re}(e^{i\phi_n}y_n + e^{-i\phi_n}x_n)\}$ and $\{\operatorname{Re}(Av_n, v_n)\}$ are both bounded by $\|A\|$, we deduce from (2.2) and (2.8) that $\lim_{n\to\infty}\operatorname{Re}(Aw_n^+, w_n^+) = 0$. Then from $\|w_n^+\|^2 = 1 + |z_n|^2$, we get $\lim_{n\to\infty}(1/1+|z_n|^2)\operatorname{Re}(Aw_n^+, w_n^+) = 0$. Thus from condition (1.2) and $\lim_{n\to\infty} z_n = 0$, we infer that

$$\lim_{n\to\infty} \frac{\operatorname{Im}(Aw_{p_n}^+, w_{p_n}^+)}{(\operatorname{Re}(Aw_{p_n}^+, w_{p_n}^+))^2} = \infty. \tag{2.10}$$

Let $M_n = \operatorname{Im}(Aw_{p_n}^+, w_{p_n}^+)/(\operatorname{Re}(Aw_{p_n}^+, w_{p_n}^+))^2$ for all $n \in \mathbb{N}$. Since $\operatorname{Im}(Aw_n^+, w_n^+) = \epsilon_n + \operatorname{Im}(z_n y_n + \bar{z}_n x_n) + |z_n|^2 \operatorname{Im}(Av_n, v_n)$, we obtain from (2.2), (2.5) and $|\operatorname{Im}(Av_n, v_n)| \le \|A\|$

that

$$\operatorname{Im}(Aw_n^+, w_n^+) \leq \begin{cases} 2\epsilon_n(1+\|A\|) & \text{if } \epsilon_n \neq 0 \\ \frac{2\|A\|}{n} & \text{if } \epsilon_n = 0. \end{cases}$$

Thus from $M_n = \operatorname{Im}(Aw_{p_n}^+, w_{p_n}^+)/(\operatorname{Re}(Aw_{p_n}^+, w_{p_n}^+))^2$ and eqs (2.2), (2.8) and (1.4) ((1.4) is needed to deduce (2.12)), we have for all $n \in \mathbb{N}$,

$$\left(\frac{\delta_{p_n}}{\sqrt{\epsilon_{p_n}}} + \operatorname{Re}(e^{i\phi_{p_n}} y_{p_n} + e^{-i\phi_{p_n}} x_{p_n}) + \sqrt{\epsilon_{p_n}} \operatorname{Re}(Av_{p_n}, v_{p_n})\right)^2 \leq \frac{2(1+\|A\|)}{M_n} \quad \text{if } \epsilon_{p_n} \neq 0 \tag{2.11}$$

and

$$\left(\operatorname{Re}(e^{i\phi_{p_n}} y_{p_n} + e^{-i\phi_{p_n}} x_{p_n}) + \frac{1}{\sqrt{p_n}} \operatorname{Re}(Av_{p_n}, v_{p_n})\right)^2 \leq \frac{2\|A\|}{M_n} \quad \text{if } \epsilon_{p_n} = 0. \tag{2.12}$$

Since $\lim_{n\to\infty} \sqrt{\epsilon_n} = \lim_{n\to\infty} 1/\sqrt{n} = \lim_{n\to\infty} 1/M_n = 0$ and $\{(Av_n, v_n)\}$ is a bounded sequence, we deduce from Property 1 and equations (2.11) and (2.12) that

$$\lim_{n\to\infty} \operatorname{Re}(e^{i\phi_{p_n}} y_{p_n} + e^{-i\phi_{p_n}} x_{p_n}) = 0. \tag{2.13}$$

Thus (2.7) follows in Case 2 if $\mathbb{N}_2$ is finite.

If $\mathbb{N}_2$ is infinite, we write $\mathbb{N}_2 = \{q_1, q_2, \ldots\}$, where $q_n < q_{n+1}$ for all $n \in \mathbb{N}$. Then from (2.9), we have

$$\operatorname{Re}(e^{i\phi_{q_n}} y_{q_n} + e^{-i\phi_{q_n}} x_{q_n}) = \begin{cases} -\frac{\delta_{q_n}}{\sqrt{\epsilon_{q_n}}} - \sqrt{\epsilon_{q_n}} \operatorname{Re}(Av_{q_n}, v_{q_n}) & \text{if } \epsilon_{q_n} \neq 0 \\ (1/q_n)^{1/2} \operatorname{Re}(Av_{q_n}, v_{q_n}) & \text{if } \epsilon_{q_n} = 0. \end{cases} \tag{2.14}$$

Since $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$ and $\{(Av_n, v_n)\}$ is a bounded sequence, we deduce from (2.14) and Property 1 (in a similar way to establishing (2.7) in Case 1) that

$$\lim_{n\to\infty} \operatorname{Re}(e^{i\phi_{q_n}} y_{q_n} + e^{-i\phi_{q_n}} x_{q_n}) = 0. \tag{2.15}$$

Thus (2.7) follows from (2.13) and (2.15) in the case $\mathbb{N}_1$ and $\mathbb{N}_2$ are both infinite sets. Thus (2.7) is proved.

If we choose ϕ_n in (2.6) and (2.7) such that $\arg(e^{i\phi_n} y_n) = \arg(e^{-i\phi_n} x_n)$, where $\arg z$ denotes the argument of the complex number z, then

$$\lim_{n\to\infty} (|x_n| + |y_n|) = \lim_{n\to\infty} \sqrt{(\operatorname{Re}(e^{i\phi_n} y_n + e^{-i\phi_n} x_n))^2 + (\operatorname{Im}(e^{i\phi_n} y_n + e^{-i\phi_n} x_n))^2} = 0.$$

■

Theorem 2.1. *Let A be a nonzero bounded linear operator on a Hilbert space $\mathcal{H}$. If A satisfies conditions* (1.1), (1.2) *and* (1.3), *then* 0 *is in the essential spectrum of each of A and its adjoint A^*.*

Proof. Suppose that A satisfies conditions (1.1), (1.2) and (1.3). We prove that $\mathcal{R}(A)$ is not closed in $\mathcal{H}$.

Assume that $\mathcal{R}(A)$ is closed in $\mathcal{H}$. We construct a unit vector $w \in \mathcal{H}$ satisfying n$(Aw,w) < 0$ (thus contradicting condition (1.1)). This is accomplished through three eps.

tep 1. $0 < \text{nul}(A)$. Assume that $0 = \text{nul}(A)$. Then from the closedness of A and the losedness of $\mathcal{R}(A)$ in $\mathcal{H}$, it follows that A^{-1} is a closed linear operator whose domain $= \mathcal{R}(A))$ is complete. Hence from the closed graph theorem, A^{-1} is bounded. On the ther hand, it follows from Lemma 2.1 that 0 is an approximate eigenvalue of A, which ontradicts that A^{-1} is bounded. This proves $0 < \text{nul}(A)$.

tep 2. There exist a sequence $\{v_n\}$ of unit vectors in the orthogonal complement $\mathcal{N}(A)^{\perp}$ f $\mathcal{N}(A)$ and a sequence $\{u_n\}$ of unit vectors in $\mathcal{N}(A)$ with the following two properties:

roperty 1. There exists $M > 0$ such that $|(Av_n, v_n)| \geq M$ for all but finitely many $n \in \mathbb{N}$.

roperty 2. $\{n \in \mathbb{N} : (Av_n, u_n) \neq 0\}$ is an infinite set. Since A is nonzero and satisfies onditions (1.1), (1.2) and (1.3), there exists a sequence $\{w_n\}$ of unit vectors in $\mathcal{H}$ such hat $\{\text{Re}(Aw_n, w_n)\}$ is a sequence of nonzero reals that converges to 0, and

$$\lim_{n\to\infty} \frac{\text{Im}\,(Aw_n, w_n)}{(\text{Re}\,(Aw_n, w_n))^2} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\text{Im}\,(Aw_n, w_n)}{\text{Re}\,(Aw_n, w_n)} = 0. \tag{2.16}$$

Ve observe, since $0 < \text{nul}(A)$, that the sequence $\{w_n\}$ can be chosen such that the rojection of each of its vectors onto $\mathcal{N}(A)$ is nonzero. Write $w_n = \alpha_n u_n + \beta_n v_n$, where α_n, β_n are complex numbers, $u_n \in \mathcal{N}(A)$ and $v_n \in \mathcal{N}(A)^{\perp}$ are unit vectors for all $n \in \mathbb{N}$. Thus $Aw_n = \beta_n Av_n$ and for all $n \in \mathbb{N}$,

$$(Aw_n, w_n) = \beta_n \overline{\alpha_n}(Av_n, u_n) + |\beta_n|^2 (Av_n, v_n). \tag{2.17}$$

Ve prove that the sequences $\{v_n\}$ and $\{u_n\}$ satisfy properties 1 and 2.

Suppose that property 1 is not true. Then there exists a subsequence $\{v'_n\}$ of $\{v_n\}$ such that $\lim_{n\to\infty}(Av'_n, v'_n) = 0$. Thus, since $\|v'_n\| = 1$, we deduce from Lemma 2.1 that $\|Av'_n\| \to 0$ as $n \to \infty$. Since $\mathcal{R}(A)$ is closed, it follows from Theorem 5.9, p. 217 of [5] that there exists $K > 0$ such that $\|Av'_n\| \geq K \text{dist}(v'_n, \mathcal{N}(A)) = K \inf\{\|v'_n - u\| : u \in \mathcal{N}(A)\}$ for all $n \in \mathbb{N}$. Hence from $\lim_{n\to\infty} \|Av'_n\| = 0$, we get $\lim_{n\to\infty} \text{dist}(v'_n, \mathcal{N}(A)) = 0$. On the other hand, since $\|v_n\| = 1$ and $v_n \in \mathcal{N}(A)^{\perp}$ for all $n \in \mathbb{N}$, we have $\text{dist}(v'_n, \mathcal{N}(A)) \geq 1$ for all $n \in \mathbb{N}$, a contradiction. This proves property 1.

Suppose that property 2 is not true. Then there exists $n_0 \in \mathbb{N}$ such that $(Av_n, u_n) = 0$ for all $n \geq n_0$. Define $v'_n = v_{n+n_0}$ for all $n \in \mathbb{N}$. Thus from $\text{Re}\,(Aw_n, w_n) \neq 0$ for all $n \in \mathbb{N}$, the second equality of (2.16) and equation (2.17), we obtain

$$\lim_{n\to\infty} \frac{\text{Im}\,(Av'_n, v'_n)}{\text{Re}\,(Av'_n, v'_n)} = \lim_{n\to\infty} \frac{\text{Im}\,(Aw_n, w_n)}{\text{Re}\,(Aw_n, w_n)} = 0. \tag{2.18}$$

Since $\{\text{Re}\,(Av'_n, v'_n)\}$ is bounded, we infer from (2.18) and the definition of $\{v'_n\}$ that $\lim_{n\to\infty} \text{Im}\,(Av_n, v_n) = 0$. Then from property 1, there exists $M_1 > 0$ such that $|\text{Re}\,(Av_n, v_n)| \geq M_1$ for all but finitely many n. Hence from $\lim_{n\to\infty} \text{Im}\,(Av_n, v_n) = 0$, the boundedness of $\{\text{Re}\,(Av_n, v_n)\}$ and the local compactness of $\mathbb{C}$, it follows that $\overline{W(A)}$ contains a nonzero real, which contradicts (1.4). This proves property 2.

Step 3. There exists a unit vector $w \in \mathcal{H}$ that satisfies $\mathrm{Im}(Aw, w) < 0$. It follows from step 2 that there exist unit vectors $u_N \in \mathcal{N}(A)$ and $v_N \in \mathcal{N}(A)^{\perp}$, and $M > 0$ such that $|(Av_N, u_N)| \neq 0$ and $|(Av_N, v_N)| \geq M$. We observe that because of (1.1) and (1.4), we have $\mathrm{Im}\,(Av_N, v_N) > 0$. Now let α, β be complex numbers that satisfy the following conditions:

(i) $|\alpha|^2 + |\beta|^2 = 1$,
(ii) β is a positive real,
(iii) $\mathrm{Im}\overline{\alpha}(Av_N, u_N) = -|\alpha||(Av_N, u_N)|$ and
(iv) $|\alpha|/\beta > \mathrm{Im}\,(Av_N, v_N)/(|(Av_N, u_N)|)$.

Let $w = \alpha u_N + \beta v_N$. Then from (i), w is a unit vector. It follows from (ii),

$$(Aw, w) = \beta(\overline{\alpha}(Av_N, u_N) + \beta(Av_N, v_N)).$$

Hence from (ii), (iii) and (iv), we get $\mathrm{Im}\,(Aw, w) < 0$.

Since $W(A^*) = \{\overline{z} \in \mathbb{C} : z \in W(A)\}$, where $\overline{z}$ denotes the complex conjugate of z, we see that A^* satisfies conditions similar to (1.1), (1.2) and (1.3) where the 'inf' is replaced by 'sup', A^* replaces A and ∞ is replaced in (1.2) by $-\infty$. A similar argument to the one used to prove $0 \in \sigma_e(A)$ is applied to prove that $0 \in \sigma_e(A^*)$. ∎

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$\xi - \zeta$ relation

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Abstract. In this note we prove a relation between the Riemann Zeta function, ζ and the ξ function (Krein spectral shift) associated with the harmonic oscillator in one dimension. This gives a new integral representation of the zeta function and also a reformulation of the Riemann hypothesis as a question in $L^1(\mathbb{R})$.

Keywords. Krein spectral shift; Riemann zeta.

1. Introduction

Inverse spectral theory in one dimension involves recovering a Schrödinger operator from the knowledge of spectrum and a spectral function as done by Gelfand–Levitan in the fifties. In recent years there is a great deal of progress achieved in parametrizing iso-spectral classes of potentials (see the reviews of Simon [12], Gesztesy [4], and the papers of Levitan [11], Kotani-Krishna [7], Craig [2] and Sodin-Yuditskii [14] for Schrödinger operators). One of the consequences of a general formulation obtained using the Krein spectral shift function by Gesztesy-Simon [5] is given in this paper.

The Riemann zeta function is a well studied object, for example, Titch-march [15] gives a detailed exposition of this function. There are several expressions for ζ, and in this note we present an integral representation for ζ, that comes from the Krein spectral shift formula of Krein [8, 9]. Recently Gesztesy-Simon [5] generalized the trace formulae for Schrödinger operators using the Krein spectral shift function, which they named the ξ function, as it is central to inverse spectral theories in one dimension and had several important applications in spectral theories of operators in one dimension. This work used the proof of the Krein formula, given in Simon [12], theorem I.10 and its generalizations. A proof of the formula for a slightly larger class is shown in Mohapatra-Sinha [13]. We refer to these papers for the history and other work on the Krein spectral shift function.

Finally we note that the reformulation we obtain for the Riemann hypothesis as a closure problem in the space $L^1(\mathbb{R})$. Though, via the powerful Wiener's theorem, the verification only requires exhibiting a single function g_σ (for each $\sigma \in (1/2, 1)$) to lie in an explicit subspace X_σ.

Beurling provided an equivalent condition earlier in his paper [1], (also see Donoghue [3]) which reformulates the Riemann hypothesis as a completeness problem in $L^2(0, 1)$. Lee Jungseob [10] gave another such reformulation.

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2. ξ function of the harmonic oscillator

In this section we recall the ξ function of the harmonic oscillator (from Gesztesy-Simon [5]).

We consider the harmonic oscillator, $H = 1/2(-(\mathrm{d}^2/\mathrm{d}x^2) + x^2 + 1)$, acting on $L^2(\mathbb{R})$ with the set $C_0^\infty(\mathbb{R})$ its domain of essential self-adjointness and normalized so that its spectrum is the positive integers $\mathbb{Z}^+$. We consider the operator $H_{\infty,x}$ defined on $L^2((-\infty,x)) + L^2((x,\infty))$ as H together with the Dirichlet condition $f(x) = 0$ at x, using the notation of [5, 6, 12]. We denote by H_∞ the operator $H_{\infty,0}$ in the following. Then the spectrum of H_∞ is even integers $2\mathbb{Z}^+$, with uniform multiplicity 2. The Krein spectral shift function $\xi(\lambda)$ for the pair of operators (H, H_∞) is given by

$$\xi(\lambda) = \sum_{m=1}^{\infty} \chi_{[2m-1,2m)}(\lambda),$$

where χ_X denotes the indicator function of X. In terms of the ξ function, Gesztesy-Simon [5], Simon [12] theorem I.10 (case $\alpha = \infty$), and Mohapatra-Sinha theorem 4.2, proved the trace formula,

$$\mathrm{Tr}\,(f(H) - f(H_\infty)) = -\int (f(\lambda))'\xi(\lambda)\,\mathrm{d}\lambda \tag{1}$$

with different smoothness and decay conditions on f. Fix $s = \sigma + it$ and consider the smooth function $f(\lambda) = \lambda^{-s}$, on $[1,\infty)$, and zero in $(-\infty, 0)$, then by functional calculus, it follows that,

$$f(H) = \int f(\lambda)\mathrm{d}E_H(\lambda), \quad \text{and} \quad f(H_\infty) = \int f(\lambda)\mathrm{d}E_{H_\infty}(\lambda)$$

are both trace class for $\sigma > 2$, since

$$\sum_{m=1}^{\infty} m^{-\sigma} \quad \text{and} \quad \sum_{m=1}^{\infty} (2m)^{-\sigma}$$

converge.

Here is the primary relation between the ξ and ζ functions.

Theorem 2.1. *Let $\xi(\lambda)$ denote the Krein spectral shift function for the pair of operators (H, H_∞), defined above. Then the Riemann zeta function $\zeta(s)$ is related to ζ through the relation,*

$$(1 - 2^{(1-s)})\zeta(s) = s\int_1^\infty \lambda^{-s-1}\xi(\lambda)\mathrm{d}\lambda \tag{2}$$

valid for any $s = \sigma + it$, with $\sigma > 0$.

Proof. We consider $s = \sigma + it$ with $\sigma > 2$, then we take $f(x) = x^{(-s)}$. Then by definition $\zeta(s) = \mathrm{Trace}(f(H))$. Now we rewrite this as

$$\zeta(s) = \mathrm{Trace}\,(f(H) - f(H_\infty)) + \mathrm{Trace}\,(f(H_\infty))$$

by the linearity of the trace. Now we notice that since the spectral multiplicity of H_∞ is 2 and the spectrum is the even integers we have

$$\text{Trace}\,(f(H_\infty)) = 2\sum_{n=1}^{\infty}(2n)^{(-s)} = 2^{(1-s)}\sum_{n=1}^{\infty}(n)^{(-s)} = 2^{(1-s)}\zeta(s).$$

Using the above relations, and the trace formula in terms of the Krein spectral shift, we immediately see that for $\sigma > 2$, the theorem is valid, since the function $f(\lambda) = \lambda^{-s}$ is C^2 and satisfies $(1+|\lambda|^2)f^{(j)}(\lambda) \in L^2(\mathbb{R}^+)$, $j = 1, 2$, while its extension to $\sigma > 0$ follows from the analyticity of the left and right hand sides of eq. (2).

A simple change of variables $\ln \lambda = x$ in the expression for ζ given in the above theorem gives the following corollary. In the following we take,

$$\phi(x) = \sum_{n=1}^{\infty} \chi_{[\ln(2n-1), \ln(2n))}(x).$$

COROLLARY 2.2

The zeta function is given, in the region $\sigma > 0, s = \sigma + it$, by

$$\zeta(s) = \frac{s}{(1-2^{1-s})}\int_0^\infty \mathrm{e}^{-sx}\phi(x)\,\mathrm{d}x.$$

Remark 1. Once the above relation is written down it is trivial to prove directly from the definition of the ζ function, using integration by parts, but the connection with the spectral problem is essential for the next part of the remark.

2. We could have considered the Dirichlet operator associated with any point $x \in \mathbb{R}$, in which case we can define a new family of functions $\zeta(x, s)$, given by

$$\zeta(x, s) = \text{Tr}\,(f(H) - f(H_{\infty,x})) = s\int_1^\infty \frac{1}{\lambda^{s+1}}\xi(\lambda, x)\mathrm{d}\lambda,$$

where $\xi(\lambda, x) = \sum_{n=1}^{\infty}\chi_{[n,\mu_n(x)]}(\lambda)$ is the ξ function associated with the pair $H, H_{\infty,x}$ and as before we first define the sum on the right hand side via the integrals for $\sigma > 2$ and then extend them to $\sigma > 0$. The expression for the right hand side agrees with the sum. Then the non-constant points of discontinuity of $\zeta(x, s)$ satisfy a differential equation in x, called the Dubrovin equation (familiar in inverse spectral theory, see [2, 5, 9, 14]),

$$\frac{\mathrm{d}\mu_i(x)}{\mathrm{d}x} = \sigma_i(x)\left(\left.\frac{\partial g_\lambda(x,x)}{\partial\lambda}\right|_{\lambda=\mu_i(x)}\right)^{-1}, \quad i = 1, 2, 3, \ldots$$

where $g_\lambda(x, x) = \lim_{\epsilon\to 0} G(\lambda + i\epsilon, x, x)$, G being the Green function associated with H. This differential equation gives a curve in the space of analytic functions on $\mathbb{C}$ with the zeta function (times the factor $(1 - 2^{1-s})/s$) as the initial value. An explicit analysis of the equation should provide a new tool to study the zeta function.

Once we have the above expression for ζ, we can use the Wiener's L^1 Tauberian theorem (see Wiener [16], § 14, Theorem 9) to obtain the following reformulation of the Riemann hypothesis.

PROPOSITION 2.3

Consider the function $f_\sigma(\lambda) = \mathrm{e}^{-\sigma\lambda}\phi(\lambda)$. Fix, $1 > \sigma > 1/2$. Let X_{f_σ} denote the subspace generated by finite linear combinations of the translates of f_σ. Then, $\hat{f}_\sigma$ is zero free if and only if $X_{f_\sigma} = L^1(\mathbb{R})$.

Proof. The expression for ζ is given by the above corollary as

$$\frac{(1-2^{1-\sigma-it})}{\sigma+it}\zeta(\sigma+it)=\int_0^\infty \mathrm{e}^{-it\lambda} f_\sigma(\lambda)\mathrm{d}\lambda.$$

Since $1/2<\sigma<1$, it is clear by inspection that the right hand side integral vanishes for any t if and only if $\sigma+it$ is a zero of the function ζ. On the other hand for a fixed σ, the right hand side of the above equation is just $\hat{f}_\sigma(t)$. Wiener's theorem gives a precise condition for the Fourier transform of a function to be zero free, which when applied yields the result.

Theorem 2.4. *Let f_σ and X_{f_σ} be as in the above Proposition. Then the Riemann hypothesis is valid if and only if for each $\sigma \in (1/2, 1)$, there is a function g_σ with zero free Fourier transform such that $g_\sigma \in X_{f_\sigma}$.*

Proof. Suppose σ is such that $\hat{f}_\sigma(t_0)=0$, but there is a $g_\sigma \in L^1(\mathbb{R})$ such that $\hat{g}_\sigma(t)\neq 0$ for any $t\in\mathbb{R}$ but $g_\sigma \in X_{f_\sigma}$. Then since $g_\sigma \in X_\sigma$, we can find for any $\epsilon>0$, complex numbers $c_1,\ldots,c_{n(\epsilon)}$ and real numbers $x_1,\ldots,x_{n(\epsilon)}$ such that

$$\left\|\sum_{i=1}^{n} c_n f_\sigma(\cdot - x_i) - g_\sigma(\cdot)\right\|_1 < \epsilon.$$

But

$$|\hat{g}_\sigma(t_0)| = |\hat{g}_\sigma(t_0) - \sum_{i=1}^{n} c_n \mathrm{e}^{it_0x_i}\hat{f}_\sigma(t_0)| \tag{3}$$

$$\leq \left\|\sum_{i=1}^{n} c_n f_\sigma(\cdot - x_i) - g_\sigma(\cdot)\right\|_1 < \epsilon. \tag{4}$$

The epsilon being arbitrary, it follows that $g_\sigma(t_0)=0$, gives a contradiction. Therefore $\hat{f}_\sigma$ is zero free under the assumption of the theorem. Now the equivalence follows from Wiener's Tauberian theorem.

Since the Fourier transform of the convolution of two L^1 functions is the product of their Fourier transforms it is obvious that in the above theorem we could replace f_σ by its convolution with any integrable function of zero free Fourier transform. This we state as a corollary.

COROLLARY 2.5

Let f_σ and X_{f_σ} be defined as in the above theorem. Then the Riemann hypothesis is valid if and only if for each $\sigma \in (1/2,1)$, there is a pair of functions h_σ, g_σ (not necessarily distinct), with zero free Fourier transforms such that $g_\sigma \in X_{f_\sigma * h_\sigma}$.

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nverse theory of Schrödinger matrices

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Abstract. In this note we discuss the inverse spectral theory for Schrödinger matrices, in particular a conjecture of Gesztesy–Simon [1] on the number of distinct iso-spectral Schrödinger matrices. We consider 3×3 matrices and obtain counter examples to their conjecture.

Keywords. Schrödinger matrices; iso-spectral sets; inverse spectral theory.

Introduction

iven a point $B = (b_1, \ldots, b_n)$ in $\mathcal{R}^n$, consider the Schrödinger matrix $H(B)$ as

$$H(B) = \begin{pmatrix} b_1 & 1 & \ldots & \ldots & 0 \\ 1 & b_2 & 1 & \ldots & 0 \\ 0 & 1 & \ddots & 1 & 0 \\ 0 & \ldots & 1 & \ddots & 1 \\ 0 & \ldots & \ldots & 1 & b_n \end{pmatrix}.$$

This matrix has all real and distinct eigenvalues say $t_1, \ldots, t_n$. Define the map $: \mathcal{R}^n \to \mathcal{R}^n$ as $F(B) = (t_1, \ldots, t_n)$ where $t_1 < \cdots < t_n$ are the eigenvalues of $H(B)$. Let $'_n$ denote the range of F, then the following conjecture is posed in [1].

onjecture 1.1 (*Gesztesy–Simon*). Let n be a positive integer and let F and W_n be as ove. Then

W_n is closed set in $\mathcal{R}^n$ whose interior is dense in W_n. For $(t_1, \ldots, t_n)$ in interior of W_n, $F^{-1}(t_1, \ldots, t_n)$ contains $n!$ points and for $(t_1, \ldots, t_n)$ in $\partial W_n, F^{-1}(t_1, \ldots, t_n)$ contains fewer than $n!$ points.

$F^{-1}(W_n^{\text{int}})$ is a disjoint union of $n!$ sets and on each of them F is a diffeomorphism to (W_n^{int}).

We address (1) of the above conjectures. That W_n is closed is already proved in [1] heorem 7.4).

The case $n = 2$ is solved in the paper [1]. Now we will inspect the map F for the case $= 3$. Given a point $B \in \mathcal{R}^3$, consider the characteristic polynomial of $H(B)$, given by $\text{et}(H(B) - tI) = 0$, viz.,

$$t^3 - (b_1 + b_2 + b_3)t^2 + (b_1b_2 + b_2b_3 + b_1b_3 - 2)t + b_1 + b_3 - b_1b_2b_3 = 0.$$

bserve that the characteristic polynomial is not symmetric in b_1, b_2, b_3.

Lemma 1.2. *For* $T = (t_1, t_2, t_3) \in \mathcal{R}^3$, *define* $P_1 = t_1 + t_2 + t_3$; $P_2 = t_1t_2 + t_2t_3 + t_1t_3$; $P_3 = t_1t_2t_3$. *Then* $(t_1, t_2, t_3) = F(b_1, 0, b_3) = F(b_3, 0, b_1)$ *iff* (t_1, t_2, t_3) *satisfy*

$$(P_1^2 - 4(2 + P_2)) \geq 0 \quad \textit{and} \quad P_1 + P_3 = 0. \tag{1}$$

Proof. Consider the characteristic polynomial of $H(B)$ for $B = (b_1, 0, b_3)$. Express the relations between the coefficients of that polynomial and its roots t_1, t_2, t_3. Now the lemma is a straightforward consequence. For the converse part set

$$b_1 = \frac{P_1 \pm \sqrt{P_1^2 - 4(2 + P_2)}}{2}; \quad b_3 = \frac{P_1 \mp \sqrt{P_1^2 - 4(2 + P_2)}}{2}$$

and again verify those relations. □

Theorem 1.3. *For* $(t_1, t_2, t_3) \in \mathcal{R}^3$ *and* P_1, P_2, P_3 *as defined in lemma* (1.2) *we have,* $(t_1, t_2, t_3) \in$ Range (F) *iff the following system of equations has a real solution*

$$\begin{aligned} &y^3 - P_1y^2 + (3 + P_2)y - (P_1 + P_3) = 0 \\ &-3y^2 + 2P_1y + P_1^2 - 4(2 + P_2) \geq 0. \end{aligned} \tag{2}$$

Proof. Observe that $(t_1, t_2, t_3) = F(b_1, b_2, b_3)$ iff

$$(t_1 - b_2, t_2 - b_2, t_3 - b_2) = F(b_1 - b_2, 0, b_3 - b_2). \tag{3}$$

Let

$$\begin{aligned} \bar{P}_1 &= t_1 - b_2 + t_2 - b_2 + t_3 - b_2 = P_1 - 3b_2, \\ \bar{P}_2 &= (t_1 - b_2)(t_2 - b_2) + (t_3 - b_2)(t_2 - b_2) + (t_1 - b_2)(t_3 - b_2) \\ &= P_2 - 2P_1b_2 + 3b_2^2, \\ \bar{P}_3 &= (t_1 - b_2)(t_2 - b_2)(t_3 - b_2) = P_3 - P_2b_2 + P_1b_2^2 - b_2^3. \end{aligned}$$

By lemma (1.2), eq. (3) holds iff (t_1, t_2, t_3) satisfy

$$(\bar{P}_1^2 - 4(2 + \bar{P}_2)) \geq 0 \quad \text{and} \quad \bar{P}_1 + \bar{P}_3 = 0. \tag{4}$$

The conditions (4) for $\bar{P}_1, \bar{P}_2, \bar{P}_3$ translate into

$$b_2^3 - P_1b_2^2 + (3 + P_2)b_2 - (P_1 + P_3) = 0, \tag{5}$$

$$-3b_2^2 + 2P_1b_2 + P_1^2 - 4(2 + P_2) \geq 0. \tag{6}$$

If $(t_1, t_2, t_3) = F(b_1, b_2, b_3)$ then clearly b_2 satisfies eqs (5) and (6) and hence is the solution of the system of equations (2).

Conversely if the system of equations (2) has a solution $y = y_0$, then set $b_2 = y_0$. Therefore $(t_1 - b_2, t_2 - b_2, t_3 - b_2)$ satisfies equations (4). Now set

$$b_1 = b_2 + \frac{\bar{P}_1 - \sqrt{\bar{P}_1^2 - 4(2 + \bar{P}_2)}}{2}$$

and

$$b_3 = b_2 + \frac{\bar{P}_1 - \sqrt{\bar{P}_1^2 - 4(2 + \bar{P}_2)}}{2}.$$

This gives $(t_1, t_2, t_3) = F(b_1, b_2, b_3)$ i.e. $(t_1, t_2, t_3) \in \text{Range}(F)$. □

By observing lemma (1.2) we get that there are two pre-images of (t_1, t_2, t_3) for every value of b_2 satisfying the system of equations (2). We are now interested in finding the conditions under which the system of equations (2) has exactly one real solution.

Consider the general cubic equation with real coefficients

$$x^3 + a_2x^2 + a_1x + a_0 = 0. \tag{7}$$

Let q_1, q_2 denote the complex cube roots of unity, $Q = a_2^2 - 3a_1$, $R = -2a_2^3 + 9a_1a_2 - 27a_0$. Define u, v, w as

$$u = \frac{-a_2}{3}, \quad v = \frac{\sqrt[3]{(R+\sqrt{R^2-4Q^3})/2}}{3}, \quad w = \frac{\sqrt[3]{(R-\sqrt{R^2-4Q^3})/2}}{3}. \tag{8}$$

Then referring to [2] the solutions of eq. (7) are $x_1 = u + v + w$; $x_2 = u + q_1v + q_2w$; $x_3 = u + q_2v + q_1w$.

Lemma 1.4. *The equation* (7) *has exactly* 1 *real root iff* $R^2 - 4Q^3 > 0$.

Proof. Using the expressions given above, one can easily verify that eq. (7) has 3 real roots iff $v = \bar{w}$, which happens iff $R^2 - 4Q^3 \leq 0$. The lemma now follows by negating these statements appropriately. □

For $T = (t_1, t_2, t_3) \in \mathcal{R}^3$, consider the cubic equation (5) given by

$$b_2^3 - P_1b_2^2 + (3 + P_2)b_2 - (P_1 + P_3) = 0.$$

Here

$$\begin{aligned}
&a_2 = -P_1,\ a_1 = 3 + P_2,\ a_0 = -(P_1 + P_3),\\
&Q = a_2^2 - 3a_1 = P_1^2 - 3(3 + P_2)\\
&\quad = t_1^2 + t_2^2 + t_3^2 - (t_1t_2 + t_2t_3 + t_1t_3) - 9, && (9)\\
&R = -2a_2^3 + 9a_1a_2 - 27a_0 = 2P_1^3 - 9P_1P_2 + 27P_3\\
&\quad = 2(t_1^3 + t_2^3 + t_3^3) - 3[t_1^2(t_2 + t_3) + t_2^2(t_1 + t_3) + t_3^2(t_1 + t_2)] + 12t_1t_2t_3. && (10)
\end{aligned}$$

Lemma 1.5. *If Q, R as in eqs* (9), (10) *respectively and v, w are given by eq.* (8), *then the system of equations* (2) *has exactly* 1 *real solution iff* $R^2 - 4Q^3 > 0$ *and* $4P_1^2 - 12(2 + P_2) \geq 9(v + w)^2$.

Proof. By lemma (1.4), the cubic equation (5) has exactly 1 real root iff $R^2 - 4Q^3 > 0$. Moreover this root must satisfy the equation (6) so that $(t_1, t_2, t_3) \in \text{Range}\, F$. In this case the only real root is given by $x_1 = u + v + w = P_1/3 + v + w$. Substituting the value of $b_2 = x_1$ in eq. (6) and simplifying gives

$$4P_1^2 - 12(2 + P_2) \geq 9(v + w)^2. \tag{11}$$

COROLLARY 1.6

Let

$$D = \{T \in \mathcal{R}^3 | R^2 - 4Q^3 > 0\} \cap \{T \in \mathcal{R}^3 | 4P_1^2 - 12(2 + P_2) > 9(v + w)^2\}.$$

Then $D \subset (\text{Range}\, F)^{\text{int}}$.

Proof. Theorem (1.3) and lemma (1.5) implies that $D \subset (\operatorname{Ran} F)$. Moreover D is intersection of 2 open sets and therefore $D \subset (\operatorname{Ran} F)^{\text{int}}$. □

COROLLARY 1.7

Let $C = \{(-\sqrt{2+t^2}, 0, \sqrt{2+t^2}) | t \in (-1,1); t \neq 0\}$. *Then* $C \subset (\operatorname{Ran} F)^{\text{int}}$.

Proof. Let $T = (-\sqrt{2+t^2}, 0, \sqrt{2+t^2}) \in C$, for such points T we have, $P_1 = 0$, $P_2 = -(2+t^2), P_3 = 0, R = 2P_1^3 - 9P_1P_2 + 27P_3 = 0$ and $Q = P_1^2 - 3(3+P_2) = -3(1-t^2)$. Therefore, $R^2 - 4Q^3 = 108(1-t^2)^3 > 0 \; \forall t \in (-1,1)$. Also, $P_1^2 - 12(2+P_2) = 12t^2$ and $9(v+w)^2 = 0$. So by Corollary (1.6), $C \subset D$ and hence $C \subset (\operatorname{Ran} F)^{\text{int}}$. Observe that by lemma (1.5), $F^{-1}(T) = \{(-t,0,t), (t,0,-t)\}$. □

Thus, there are points in the interior of $W_3 = \text{Range}(F)$ which have only two pre-images under the map F, i.e. for each point in the family C there exist only two iso-spectral Schrödinger matrices whose spectrum is that point of C. This gives a counter-example to the Conjecture (1.1) (1).

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A nonstandard definition of finite order ultradistributions

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Abstract. Taking into account that finite-order ultradistributions are inverse Fourier transforms of finite-order distributions a nonstandard representation is obtained for one-dimensional finite-order ultradistributions.

Keywords. Distributions; ultradistributions; nonstandard analysis.

1. Introduction

1.1 *The Silva axioms for finite-order distributions*

Taking the notions of *continuous function* and *derivative* as primitive, the axiomatic definition of (finite-order) distributions on the real line, due to Silva [5], may be given as follows:

DEFINITION 1.1

Distributions on $\mathbb{R}$ may be characterized as the elements of a linear space E for which two linear maps are defined,

$$\iota : \mathcal{C}(\mathbb{R}) \to E \quad \text{and} \quad D : E \to E$$

such that

Axiom 1. ι is the injective inclusion: that is, every function in $\mathcal{C} \equiv \mathcal{C}(\mathbb{R})$ is a distribution on $\mathbb{R}$.

Axiom 2. To each distribution ν on $\mathbb{R}$ there corresponds a distribution $D\nu$, called the derivative of ν, such that if $\nu = \iota(f)$, with $f \in \mathcal{C}^1 \equiv \mathcal{C}^1(\mathbb{R})$, then $D\nu = \iota(f')$.

Axiom 3. If ν is a distribution on $\mathbb{R}$ then there exists a continuous function $f \in \mathcal{C}$ and a natural number $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that we have $\nu = D^r \iota(f)$.

Axiom 4. Given any $f, g \in \mathcal{C}$ and a natural number $r \in \mathbb{N}_0$, the equality $D^r\iota(f) = D^r\iota(g)$ holds if and only if $f - g$ is a polynomial of degree $< r$.

Silva has given an abstract model for these axioms as follows: on the Cartesian product $\mathbb{N}_0 \times \mathcal{C}$ define an equivalence relation, say $\square$, by

$$(r,f)\square(s,g) \Leftrightarrow \exists_{m\in\mathbb{N}_0}[m \geq r, s \wedge (\mathcal{J}_a^{m-r}f - \mathcal{J}_a^{m-s}g) \in \Pi_m],$$

where Π_m denotes the set of all complex-valued polynomials of degree less than m, and $\mathcal{J}_a^k$ is the kth iterated indefinite integral operator with origin at $a \in \mathbb{R}$. If we denote the

quotient set by

$$\mathcal{C}_\infty \equiv \mathcal{C}_\infty(\mathbb{R}) = \mathbb{N}_0 \times \mathcal{C}/\square,$$

then

- $\mathcal{C}_\infty$ is a model for the Silva axioms 1–4, and
- every model for the Silva axioms is isomorphic to $\mathcal{C}_\infty$.

In particular $\mathcal{D}'_{\text{fin}} \equiv \mathcal{D}'_{\text{fin}}(\mathbb{R})$, the space of all Schwartz distributions of finite order is isomorphic to $\mathcal{C}_\infty$. For a detailed account of the theory of Schwartz distributions from this point of view the recent text by Campos Ferreira [1] is recommended.

1.2 *A nonstandard model for the Silva axioms*

In an earlier paper [2] a nonstandard model for the Silva axioms for distributions on the line has been given, using the simple ultrapower model ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\sim$ for the hyperreals (where $\mathbb{N} = \{1, 2, \ldots\}$). For a fuller account, the original paper may be consulted; a general introduction to the concepts and notation peculiar to nonstandard analysis is to be found in, for example, [4]. ${}^*C^\infty(\mathbb{R})$ is the internal set of all infinitely *differentiable functions on ${}^*\mathbb{R}$. Thus

$${}^*C^\infty \equiv {}^*C^\infty(\mathbb{R}) = \{F \equiv [(f_n)_{n\in\mathbb{N}}] : f_n \in C^\infty \text{ for nearly all } n \in \mathbb{N}\}.$$

We denote by ${}^sC^\infty \equiv {}^sC^\infty(\mathbb{R})$ the (external) set of all functions F in ${}^*C^\infty$ which are finite-valued and **S**-continuous at each point of ${}^*\mathbb{R}_b$ (the set of all finite hyperreal numbers). Then an internal function $F \in {}^*C^\infty$ is said to be a *predistribution* if there exists a function $\Phi \in {}^sC^\infty$ and an integer $r \in \mathbb{N}_0$ such that $F = {}^*\mathbf{D}^r\Phi$. The set of all such finite-order *derivatives of functions in ${}^sC^\infty$ is denoted by

$$\begin{aligned} {}^*\mathbf{D}^\infty\{{}^sC^\infty\} &= \cup_{r\geq 0}{}^*\mathbf{D}^r\{{}^sC^\infty\} \\ &= \{F \in {}^*C^\infty : F = {}^*\mathbf{D}^r\Phi, \text{ for some } \Phi \in {}^sC^\infty \text{ and some } r \in \mathbb{N}_0\}. \end{aligned}$$

DEFINITION 1.2

Two functions $F, G \in {}^*\mathbf{D}^\infty\{{}^sC^\infty\}$ are said to be Ξ-equivalent and we write $F \,\Xi\, G$, if and only if there exist internal functions $\Phi, \Psi \in {}^sC^\infty$, and some $k \in \mathbb{N}_0$, such that

(1) $\Phi(x) \approx \Psi(x)$, for all $x \in {}^*\mathbb{R}_b$, and
(2) $F = {}^*\mathbf{D}^k\Phi$, and $G = {}^*\mathbf{D}^k\Psi$.

The members of the quotient ${}^\Xi\mathcal{C}_\infty \equiv {}^\Xi\mathcal{C}_\infty(\mathbb{R}) = {}^*\mathbf{D}^\infty\{{}^sC^\infty\}/\Xi$ are called ${}^\Xi$*distributions* (of finite-order).

Then ${}^\Xi\mathcal{C}_\infty$ is a nonstandard model for the axiomatic definition of finite-order distributions on $\mathbb{R}$ proposed by Silva; ${}^\Xi\mathcal{C}_\infty$ is isomorphic with $\mathcal{C}_\infty$ and hence also with $\mathcal{D}'_{\text{fin}}$.

1.3 *Fourier transforms and $\mathcal{Z}'_{\text{fin}}$*

The Fourier transform maps $\mathcal{D}'_{\text{fin}}$ into $\mathcal{Z}'_{\text{fin}} \equiv \mathcal{Z}'_{\text{fin}}(\mathbb{R})$, the space of all finite-order ultradistributions. Hence, from the known nonstandard representation of finite-order distributions, it should be possible to derive a nonstandard representation for finite-order ultradistributions. In order to achieve this it is convenient to re-phrase slightly the definition of ${}^\Xi\mathcal{C}_\infty$.

As usual $\mathcal{D}$ denotes the Schwartz space of all infinitely differentiable functions of npact support, and ${}^*\mathcal{D}$ is the nonstandard extension of $\mathcal{D}$. We now denote by ${}^s\mathcal{D}$ the $_b$ submodule of ${}^s\mathcal{C}^\infty$ which comprises all infinitely *differentiable functions of percompact support which are finite-valued and **S**-continuous on ${}^*\mathbb{R}_b$.

eorem 1.3. *Every* $^{\Xi}$*distribution in* $^{\Xi}\mathcal{C}_\infty$ *may be represented by an internal function in* ${}^*\mathcal{D}$.

oof. If $\nu \in {}^{\Xi}\mathcal{C}_\infty$ then there exists an integer $r \in \mathbb{N}_0$ and an internal function $F_1 \in {}^s\mathcal{C}^\infty$ ch that $\nu = {}^*\mathbf{D}^r[F_1]$. We need to show that there exists an internal function F in $^\infty \cap {}^*\mathcal{D}$ such that $F \in [F_1]$; then ${}^*\mathbf{D}^r F \in \nu$ will certainly be a function in ${}^*\mathcal{D}$, as asserted.

If $F_1 \in {}^s\mathcal{C}^\infty$, then there exists a (standard) continuous function $f : \mathbb{R} \to \mathbb{C}$ such that $= \mathrm{st}(F_1)$. Now let $T = [(\tau_n)_{n\in\mathbb{N}}]$ be an internal cut-off function defined by a sequence $)_{n\in\mathbb{N}}$ of (standard) functions such that

- $\tau_n(y) = 1$ for all $|y| \leq n$,
- $\tau_n(y) = 0$ for all $|y| \geq n+1$,
- $\tau_n \in C$ for nearly all $n \in \mathbb{N}$.

en

$$T^*f = [(\tau_n f)_{n\in\mathbb{N}}]$$

an internal function in ${}^*\mathcal{C}_0$ which is **S**-continuous on $\mathbb{R}$, has hypercompact support and such that $\mathrm{st}(T^*f) = f$.

Let $\Delta_\theta \in {}^*\mathcal{D}$ be an internal pre-delta function defined by a sequence $(\theta_n)_{n\in\mathbb{N}}$ of andard) functions in $\mathcal{D}$ such that

- there exists a sequence of positive numbers $(a_n)_{n\in\mathbb{N}}$ converging to zero, with $\mathrm{supp}(\theta_n) \subset [-a_n, a_n]$,
- for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} \theta_n(y)\mathrm{d}y = 1.$$

en the internal function

$$F = (T^*f) * \Delta_\theta \equiv [((\tau_n f) * \theta_n)_{n\in\mathbb{N}}]$$

longs to ${}^*\mathcal{D}$, is an internal function of hypercompact support and has a continuous ojection on the real line. Also, $\mathrm{st}(F) = f$, since $(T^*f) * \Delta_\theta = {}^*f * \Delta_\theta - ({}^*f - T^*f) * \Delta_\theta$, ere we have $\mathrm{st}({}^*f * \Delta_\theta) = f$ and $\mathrm{st}({}^*f - T^*f) = 0$. Thus, for each $r \in \mathbb{N}_0$,

$$({}^*\mathbf{D}^r F)\,\Xi\,({}^*\mathbf{D}^r F_1)$$

d so

$$\nu = {}^*\mathbf{D}^r[F_1] = {}^*\mathbf{D}^r[F] = [{}^*\mathbf{D}^r F],$$

ere ${}^*\mathbf{D}^r F \in {}^*\mathcal{D}$. □

follows from this theorem that we can re-define $^{\Xi}\mathcal{C}_\infty$ using only infinitely *differen-ble functions of hypercompact support. That is,

$$^{\Xi}\mathcal{C}_\infty = {}^*\mathbf{D}^\infty({}^s\mathcal{D})/\Xi = \cup_{r\geq 0}\{{}^*\mathbf{D}^r({}^s\mathcal{D})\}/\Xi.$$

2. Nonstandard representation for finite order ultradistributions

2.1 *Differential operators of ∞-order*

In order to avoid the confusion that may eventually arise by the use of the expression *infinite-order operators* to be interpreted in the standard sense we will call them instead ∞-*order operators.* To begin with we therefore recall, very briefly, some facts about ∞-order differential operators acting on the space $\mathcal{Z}$ [6]. Let $\mathcal{H}(\mathbb{C})$ denote the space of all standard complex-valued functions which can be extended into the complex plane as entire functions. For every entire function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}(\mathbb{C})$ define the operator $\mathbf{A} : \mathcal{Z} \to \mathcal{Z}$ by setting, for each $\varphi \in \mathcal{Z}$,

$$\mathbf{A}[\varphi](t) = \sum_{n=0}^{\infty} a_n(-i\mathbf{D})^n \varphi(t) = \sum_{n=0}^{\infty} (-i)^n a_n \varphi^{(n)}(t).$$

Since each $\varphi \in \mathcal{Z}$ is the inverse Fourier transform of a unique function $\hat{\varphi} \in \mathcal{D}$ we have

$$\mathbf{A}[\varphi](t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(y)\hat{\varphi}(y)\exp(iyt)\mathrm{d}y = \mathcal{F}^{-1}[A\hat{\varphi}](t),$$

where the tacit interchange of order of integration and summation is easily justified by the fact that $\hat{\varphi}$ has compact support.

2.2 *The inverse Fourier transform in* $^{\Xi}\mathcal{C}_{\infty}$

Given an internal function $\hat{F} \equiv [(\hat{f}_n)_{n\in\mathbb{N}}] \in {}^*\mathcal{D}$, its inverse Fourier transform $F = {}^*\mathcal{F}^{-1}[\hat{F}]$ is defined in the obvious way by

$$F \equiv [(f_n)_{n\in\mathbb{N}}] = [(\mathcal{F}^{-1}[\hat{f}_n])_{n\in\mathbb{N}}],$$

where $\hat{f}_n \in \mathcal{D}$ for (nearly) all $n \in \mathbb{N}$, and so $f_n \in \mathcal{Z}$ for (nearly) all $n \in \mathbb{N}$. Hence it follows that we have

$$\hat{F} \in {}^*\mathcal{D} \Leftrightarrow F \equiv {}^*\mathcal{F}^{-1}[\hat{F}] \in {}^*\mathcal{Z}.$$

However, not all internal functions in ${}^*\mathcal{D}$ represent $^{\Xi}$distributions and so, correspondingly, not all internal functions in ${}^*\mathcal{Z}$ will represent $^{\Xi}$ultradistributions. We require now to characterize those internal functions in ${}^*\mathcal{Z}$ which do represent inverse Fourier transforms of $^{\Xi}$distributions of finite order in $^{\Xi}\mathcal{C}_{\infty}$.

If $\hat{F} \equiv [(\hat{f}_n)_{n\in\mathbb{N}}]$ belongs to ${}^s\mathcal{D}$ then there certainly exists a standard continuous function $\hat{f} \in \mathcal{C}$ such that $\hat{f} = \mathrm{st}(\hat{F})$. We consider first the case where the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ belonging to $\hat{F}$ is such that

$$\hat{f}_n \to \hat{f} \text{ in } \mathbf{L}^1(\mathbb{R})$$

and there exists some $C \in \mathbb{R}^+$ for which

$$\|\hat{f}_n\|_{\mathbf{L}^1} \leq C, \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $\hat{f}$ belongs to $\mathbf{L}^1$, the classical inverse Fourier transform of $\hat{f}$ exists, and

$$\mathcal{F}^{-1}[\hat{f}_n] \equiv f_n \to f \equiv \mathcal{F}^{-1}[\hat{f}] \quad \text{as} \quad n \to \infty, \quad \text{almost uniformly on } \mathbb{R}.$$

Hence, $F \equiv {}^*\mathcal{F}^{-1}[\hat{F}]$ is finite-valued and $\mathbf{S}$-continuous at every standard point and satisfies $|F(t)| \leq C$ for all $t \in {}^*\mathbb{R}_b$.

Next, consider the case of an internal function ${}^*\mathbf{D}^r\hat{F} \in {}^*\mathcal{D}$ where $r \in \mathbb{N}_0$ and $\hat{F}$ satisfies the above constraints. Then

$$ {}^*\mathcal{F}^{-1}[{}^*\mathbf{D}^r\hat{F}](t) = (-it)^r F(t), $$

where the restriction of $(-it)^r F(t)$ to $\mathbb{R}$ is a continuous function of polynomial growth.

To deal finally with the general case when $\hat{f} = \mathrm{st}(\hat{F})$ is a continuous function of arbitrary growth on $\mathbb{R}$, and not necessarily an $\mathbf{L}^1$ function, we need to recall a classical result on bounds for functions $f : \mathbb{R} \to \mathbb{C}$.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be locally bounded. Then there exists an entire function $A : \mathbb{C} \to \mathbb{C}$ such that $A(t) \geq |f(t)|$ for all $t \geq 0$. Moreover we may choose the function $A \in \mathcal{H}(\mathbb{C})$ to have no zeros.*

Proof. Without loss of generality we may assume that f is non-negative and monotone increasing on $[0, +\infty)$. (Otherwise we consider instead the function $g(t) = \sup_{-t \leq x \leq t} |f(x)|$, $t \geq 0$.)

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two increasing sequences of positive reals such that $(a_n/b_n) > 1$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} b_n = +\infty$. For each $n \in \mathbb{N}$ there will exist $m_n \in \mathbb{N}$ such that

$$ \left(\frac{a_n}{b_n}\right)^{m_n} \geq f(a_{n+1}). $$

Again without loss of generality we may assume that the sequence $(m_n)_{n \in \mathbb{N}}$ is strictly increasing and tends to $+\infty$ with n. Then the power series $\sum_{n=1}^{\infty} (z/b_n)^{m_n}$ will have infinite radius of convergence and the function defined by

$$ \Psi(z) = f(a_1) + \sum_{n=1}^{\infty} \left(\frac{z}{b_n}\right)^{m_n}, \quad z \in \mathbb{C} $$

belongs to $\mathcal{H}(\mathbb{C})$, is non-negative and monotone increasing on the interval $[0, +\infty)$, and satisfies $\Psi(t) \geq |f(t)|$ for all $t \geq 0$.

Now consider the function

$$ g(t) = \log\left[1 + \sup_{0 \leq x \leq t} f(x)\right], \quad t \geq 0. $$

Let $\Psi \in \mathcal{H}(\mathbb{C})$ be a function, as defined above, such that $\Psi(t) \geq g(t)$ for all $t \geq 0$. Then we have only to define

$$ A(z) = \exp[\Psi(z)] $$

for all $z \in \mathbb{C}$. □

COROLLARY 2.2

If $f : \mathbb{R} \to \mathbb{C}$ is locally bounded then there exists an entire function $A : \mathbb{C} \to \mathbb{C}$, with no zeros, such that $A(t) \geq |f(t)|$ for all $t \in \mathbb{R}$.

Proof. Considering the function defined by

$$ f_1(x) = \begin{cases} \sup_{-x \leq t \leq x} |f(t)| & \text{if } x > 0 \\ \sup_{x \leq t \leq -x} |f(t)| & \text{if } x < 0 \end{cases} $$

the result follows immediately from the lemma. □

In particular, if $\hat{F} \equiv [(\hat{f}_n)_{n\in\mathbb{N}}] \in {}^s\mathcal{D}$ is such that $\hat{f} = \mathrm{st}(\hat{F})$ is a continuous function of arbitrary growth on $\mathbb{R}$, and not necessarily in $\mathbf{L}^1$, then it follows immediately that there exists some function $\Phi \in \mathcal{H}$, without zeros, such that

$$|\hat{f}(y)| \leq \Phi(y) \quad \text{for all} \quad y \in \mathbb{R}.$$

Hence we may assert that there exists an entire function $A \in \mathcal{H}(\mathbb{C})$, with no real zeros such that $\hat{f}/A$ belongs to $\mathbf{L}^1$: it is enough to take

$$A(\lambda) = (1+\lambda^2)\Phi(\lambda), \quad \lambda \in \mathbb{C}.$$

Then the functions $\hat{g}_{A,n} = \hat{f}_n/A$, for $n = 1, 2, \ldots$ and $\hat{g}_A = \hat{f}/A$ satisfy the conditions

$$\hat{g}_{A,n} \to \hat{g}_A$$

and there exists some $C \in \mathbb{R}^+$ such that $\|\hat{g}_{A,n}\|_1 \leq C$, for all $n \in \mathbb{N}$. Hence, defining in the usual (standard) sense

$$g_{A,n} = \mathcal{F}^{-1}[\hat{g}_{A,n}] \quad \text{and} \quad g_A = \mathcal{F}^{-1}[\hat{g}_A]$$

it follows that

$$g_{A,n} \to g_A \quad \text{as} \quad n \to \infty$$

almost uniformly on $\mathbb{R}$. Therefore,

$$G_A = [(g_{A,n})_{n\in\mathbb{N}}] = {}^*\mathcal{F}^{-1}[\hat{F}/A]$$

is an internal function in ${}^*\mathcal{Z}$ such that $g_A = \mathrm{st}(G_A)$ is a bounded continuous function on $\mathbb{R}$ which tends to zero as $|t|$ tends to infinity. Further, since

$$\hat{F} = {}^*A\left(\frac{\hat{F}}{{}^*A}\right) = {}^*A\hat{G}_A$$

we obtain

$${}^*\mathcal{F}^{-1}[\hat{F}] = \mathbf{A}[G_A].$$

That is to say, the inverse Fourier transform of $\hat{F} \in {}^s\mathcal{D}$ is a derivative of ∞-order of an internal function in ${}^*Z$ which is S-continuous and bounded on $\mathbb{R}$. Moreover, for any $r \in \mathbb{N}_0$,

$${}^*\mathcal{F}^{-1}[{}^*\mathbf{D}^r\hat{F}](t) = (-it)^{r*}\mathcal{F}^{-1}[\hat{F}](t) = (-it)^r\mathbf{A}[G_A](t). \tag{1}$$

We now prove the central result of this paper:

Theorem 2.3. *The inverse Fourier transform of a* ${}^{\Xi}$*distribution in* ${}^{\Xi}\mathcal{C}_\infty$ *is representable as a finite sum of (standard)* ∞*-order derivatives of internal functions in* ${}^*\mathcal{Z}$ *whose standard parts are continuous functions of polynomial growth.*

Proof. The proof proceeds by (finite) induction on $r \in \mathbb{N}_0$ to show that for given $A \in \mathcal{H}(\mathbb{C})$, we have

$$(-it)^r\mathbf{A}[G_A](t) = \sum_{j=0}^{r} \binom{r}{j} \mathbf{A}^{(j)}\{(-it)^{r-j}G_A(t)\}, \tag{2}$$

where $\mathbf{A}^{(j)}$, $j = 0, 1, 2, \ldots, r$ is the ∞-order operator associated with the jth derivative of the function $A \in \mathcal{H}(\mathbb{C})$.

Equation (2) trivially holds for $r = 0$. Now, if $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then we have

$$\begin{aligned}
\mathbf{A}\{(-it)G_A(t)\} &= \sum_{n=0}^{\infty} a_n(-i^*\mathbf{D})^n\{(-it)G_A(t)\} \\
&= a_0(-it)G_A(t) + \sum_{n=1}^{\infty} a_n(-i)^n \left\{ \sum_{j=0}^{n} \binom{n}{j} ({}^*\mathbf{D}^j(-it))({}^*\mathbf{D}^{n-j}G_A(t)) \right\} \\
&= a_0(-it)G_A(t) + \sum_{n=1}^{\infty} a_n(-i)^n\{(-it)({}^*\mathbf{D}^n G_A(t)) + n(-i)({}^*\mathbf{D}^{n-1}G_A(t))\} \\
&= (-it)\mathbf{A}[G_A](t) - \mathbf{A}'[G_A](t).
\end{aligned}$$

Hence,

$$(-it)\mathbf{A}[G_A](t) = \mathbf{A}\{(-it)G_A(t)\} + \mathbf{A}'[G_A](t)$$

so that eq. (2) holds for $r = 1$. Suppose now that (2) holds for arbitrary $r \in \mathbb{N}_0$. Then, after some manipulations,

$$\begin{aligned}
(-it)^{r+1}\mathbf{A}[G_A](t) &= (-it)\{(-it)^r\mathbf{A}[G_A](t)\} \\
&= \sum_{j=0}^{r} \binom{r}{j} (-it)\mathbf{A}^{(j)}\{(-it)^{r-j}G_A(t)\} \\
&= \sum_{j=0}^{r} \binom{r}{j} [\mathbf{A}^{(j)}\{(-it)^{r+1-j}G_A(t)\} + \mathbf{A}^{(j+1)}\{(-it)^{r-j}G_A(t)\}] \\
&= \sum_{j=0}^{r+1} \binom{r+1}{j} \mathbf{A}^{(j)}\{(-it)^{(r+1)-j}G_A(t)\}
\end{aligned}$$

which shows that (2) holds for $r + 1$. □

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oments of escape times of random walk

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Abstract. Extending an idea of Spitzer [2], a way to compute the moments of the time of escape from $(-N, L)$ by a symmetric simple random walk is exhibited. It is shown that all these moments depend polynomially on L and N.

Keywords. Random walk; escape times.

t $M = \{M_n : n \geq 0\}$ denote a symmetric simple random walk starting at 0. For integers $N \geq 1$, let $\tau_{L,N}$ denote the random time of escape from $(-N, L)$ by M, that is,

$$\tau_{L,N} = \inf\{n \geq 1 : M_n = -N \text{ or } L\}.$$

ie of the elementary facts known about the $\tau_{L,N}$'s is that for each L and N, $\tau_{L,N} < \infty$ th probability 1. The exact distribution of the $\tau_{L,N}$'s is, however, extremely complited, even when $L = N$ (see [2], pp. 242–243, for example). In the sequel, we describe s situation as the 'symmetric boundaries' case. In this case, Spitzer ([2], pp. 237–243) ed an elegant technique to compute the mean of $\tau_{N,N}$ without using its exact distribun, and showed that $\mathbf{E}\tau_{N,N} = N^2$. This, of course, is also an immediate consequence of oob's optional sampling theorem applied to the martingale $M_n^2 - n$. In this note, we ow that the ideas of Spitzer [2] extend towards computation of all the higher order oments of $\tau_{L,N}$ as well. In particular, we show that they all depend polynomially on L d N.

At this juncture, we point out that while the moments of $\tau_{L,N}$ can in principle be termined from the known (see, e.g., [1], pp. 351) probability generating function

$$\mathbf{E}s^{\tau_{L,N}} = \frac{\lambda_1^L(s) + \lambda_1^N(s) - (\lambda_2^L(s) - \lambda_2^N(s))}{\lambda_1^{L+N}(s) - \lambda_2^{L+N}(s)}, \quad s \in (0,1),$$

iere $\lambda_1(s) = (1 + \sqrt{1 - s^2})/s$ and $\lambda_2(s) = (1 - \sqrt{1 - s^2})/s$, it appears by no means sy to prove from this that these are polynomials in L and N.

The main idea in this work is the following. For a fixed N, the random variable $\tau_{L,N}$ has e same distribution as that of the escape time T_{L+N} from the set $\{0, 1, \ldots, L+N\}$ for a ndom walk $\{W_n^N\}$ starting at N. We now introduce some notations, chiefly adapted from], to be used in the sequel. For a fixed K, let Q_K denote the submatrix of the transition atrix P of the symmetric simple random walk corresponding to the states $0, 1, \ldots, K$.

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More explicitly, Q_K is the square matrix of order $K+1$ defined as

$$Q_K(x,y) = P(x,y) \quad \text{for } 0 \le x, y \le K.$$

Let $Q_K^n, n \ge 0$ be the successive powers of Q_K with Q_K^0 obviously denoting the identity matrix I of order $K+1$. It is not difficult to see that for $0 \le x, y \le K, Q_K^n(x,y)$ actually represents the probability of the event that a symmetric simple random walk starting at x reaches state y at time n without having exited the set $\{0, 1, \ldots, K\}$ by that time. Let g_K denote the matrix $(I - Q_K)^{-1} = \sum_{n=0}^{\infty} Q_K^n$. We are now ready for the main result.

Theorem 1. *For each $k \ge 1$, $\mathbf{E}\tau_{L,N}^k$ is a polynomial in L and N.*

Proof. By virtue of what has been said in the preceding paragraph, it is enough to prove that under the distribution $\mathbf{P}_N$ of the symmetric simple random walk W^N starting at N, each moment of T_{L+N}, the escape time from $\{0, 1, \ldots, L+N\}$, is a polynomial in L and N, or equivalently, that each factorial moment $\mathbf{E}_N((T_{L+N})_k)$ is so. Here, $(x)_k$ denotes $x(x-1)\cdots(x-k+1)$.

Now,

$$\begin{aligned}
\mathbf{E}_N((T_{L+N})_k) &= \sum_{n=k}^{\infty} (n)_k \mathbf{P}_N(T_{L+N} = n) \\
&= \sum_{n=k}^{\infty} (n)_k [\mathbf{P}_N(T_{L+N} > n-1) - \mathbf{P}_N(T_{L+N} > n)] \\
&= \sum_{n=k}^{\infty} [(n+1)_k - (n)_k] \mathbf{P}_N(T_{L+N} > n) \\
&= \sum_{n=k}^{\infty} k(n)_{k-1} \mathbf{P}_N(T_{L+N} > n) \\
&= k \sum_{n=k}^{\infty} (n)_{k-1} \sum_{x=0}^{L+N} \mathbf{P}_N(T_{L+N} > n,\, W_n^N = x) \\
&= k \sum_{n=k}^{\infty} (n)_{k-1} \sum_{x=0}^{L+N} Q_{L+N}^n(N,x) \\
&= k \sum_{x=0}^{L+N} \sum_{n=k}^{\infty} (n)_{k-1} Q_{L+N}^n(N,x) \\
&= k \sum_{x=0}^{L+N} \left[\left\{ \sum_{n=k}^{\infty} (n)_{k-1} Q_{L+N}^{n-k+1} \right\} \cdot Q_{L+N}^{k-1} \right](N,x) \\
&= k(k-1)! \sum_{x=0}^{L+N} [(I - Q_{L+N})^{-k} Q_{L+N}^{k-1}](N,x) \\
&= k(k-1)! \sum_{x=0}^{L+N} g_{L+N}^k Q_{L+N}^{k-1}(N,x),
\end{aligned}$$

the penultimate step being justified by the fact that each eigenvalue of $I - Q_K$ is strictly between 0 and 1.

n [2] (Proposition P4, page 242) g_K is computed as

$$g_K(x,y) = 2[(K+2)\min(x+1,y+1) - (x+1)(y+1)].$$

now make the following claims:

im 1. For every $k \geq 1, \exists$ a three-variable polynomial q_k such that

$$\begin{aligned} g_K^k(x,y) &= q_k(K,x,y) \quad \text{for } x \leq y, \text{ and} \\ &= q_k(K,y,x) \quad \text{for } x > y. \end{aligned}$$

im 2. For every $k \leq 1$ and $0 \leq m < k, \exists$ a polynomial $r_{k,m}$ such that

$$\begin{aligned} g_K^k Q_K^m(x,y) &= r_{k,m}(K,x,y) \quad \text{for } x \leq y, \quad \text{and} \\ &= r_{k,m}(K,y,x) \quad \text{for } x > y. \end{aligned}$$

aim 1 is proved by simple induction. Firstly, $g_K^j(y,x) = g_K^j(x,y)$ for every k since g_K is nmetric. Next, the assertion is trivial for $k=1$; and for $x \leq y$,

$$\begin{aligned} g_K^k(x,y) &= \sum_{z=0}^{K} g_K^{k-1}(x,z) g_K(z,y) \\ &= \sum_{z=0}^{x-1} q_{k-1}(z,x) q(z,y) + \sum_{z=x}^{y} q_{k-1}(x,z) q(z,y) \\ &\quad + \sum_{z=y+1}^{K} q_{k-1}(x,z) q(y,z) \quad \text{by induction hypothesis.} \end{aligned}$$

is last expression is a polynomial in (K,x,y), with the understanding that null sums are ated as 0.

As for Claim 2, we likewise observe firstly the symmetry; and secondly that $\forall k \geq 1$ d $0 \leq m < k$,

$$g_K^{k-m} = g_K^k (I - Q_K)^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} g_K^k Q_K^i,$$

that

$$g_K^k Q_K^m = (-1)^m g_K^{k-m} - \sum_{i=0}^{m-1} (-1)^{m+i} \binom{m}{i} g_K^k Q_K^i,$$

d the Claim follows by induction on m and Claim 1.

Finally, the Proposition is an easy consequence of Claim 2 since from previous lculations,

$$\begin{aligned} \mathbf{E}_N((T_{L+N})_k) &= k(k-1)! \sum_{x=0}^{L+N} g_{L+N}^k Q_{L+N}^{k-1}(N,x), \\ &= k(k-1)! \left[\sum_{x=0}^{N} r_{k,k-1}(L+N,x,N) + \sum_{x=N+1}^{L+N} r_{k,k-1}(L+N,N,x) \right], \end{aligned}$$

nich, of course, is a polynomial in L and N. ∎

As a concluding remark we point out that unlike the case of first moment, we have not found appropriate martingales which, through optional sampling theorem, will yield our result for the higher moments, even for symmetric boundaries.

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Explicit generalized solutions to a system of conservation laws

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Abstract. This paper studies a special 3 by 3 system of conservation laws which cannot be solved in the classical distributional sense. By adding a viscosity term and writing the system in the form of a matrix Burgers equation an explicit formula is found for the solution of the pure initial value problem. These regularized solutions are used to construct solutions for the conservation laws with initial conditions, in the algebra of generalized functions of Colombeau. Special cases of this system were studied previously by many authors.

Keywords. Conservation laws; Colombeau algebra; generalized solutions.

1. Introduction

In this paper we consider a system of partial differential equations of the form

$$
\begin{aligned}
&u_t + \left(\frac{u^2}{2}\right)_x = 0,\\
&v_t + (uv)_x = 0,\\
&w_t + \left(\frac{v^2}{2} + uw\right)_x = 0,
\end{aligned}
\tag{1.1}
$$

in $-\infty < x < \infty, t > 0$, supplemented with an initial condition at $t = 0$,

$$
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{1.2}
$$

where $u_0(x), v_0(x); w_0(x)$ are bounded measurable functions. The system is not strictly hyperbolic. In fact the eigenvalues of the Jacobian matrix of $F(u,v,w) = ((u^2/2), uv, (v^2/2) + uw)$ are equal, namely u and classical theory of conservation laws does not apply. Even for Riemann initial data (1.1) and (1.2) cannot be solved in the class of classical simple waves. For example when the initial data,

$$
\begin{aligned}
&u(x,0) = 1, x < 0, u(x,0) = -1, x > 0,\\
&v(x,0) = v_l, x < 0, v(x,0) = v_r, x > 0,\\
&w(x,0) = w_l, x < 0, w(x,0) = w_r, x > 0,
\end{aligned}
\tag{1.3}
$$

where v_l, v_r, w_l, w_r are constants, a simple wave solution for (1.1) with initial data (1.3) can be found if and only if $v_l + v_r = 0$ and $w_l + w_r = 0$. This follows easily from the observation that the entropy weak solution of the first equation of (1.1) with the above data is $u(x,t) = 1$ if $x < 0$ and -1 if $x > 0$; a shock wave with speed $s = 0$. If $v_l + v_r$ is

not equal to zero, then the v component contains a δ measure along $x=0$, see Joseph [9]. Though the product, uv, does not make sense in the classical theory of distributions, this product can be defined in the sense of Dalmaso–LeFloch–Murat [6], but not, v^2, a square of δ measure. To overcome such difficulties, Colombeau [2] introduced a new notion of generalized functions. In recent works [1,3–5,10,13] and in many other references there, it is recognized by many authors that the generalized functions of Colombeau is a convenient setup to seek global solutions, where such difficulties arise. Further, this approach takes into account the microscopic structure of the shocks in the solutions. To do this we study (1.1) and (1.2) by the vanishing viscosity method. We obtain an explicit formula for the solution with viscous terms in the equation. We study the limit of these solutions as ϵ tends to 0 and construct a solution in the algebra of the generalized functions of Colombeau. Similar results based also on the viscous approximation to Riemann problem for a different system were recently published by Hu [8].

This paper is organized in the following way. In § 2, we recall the definition of the algebra of the generalized functions of Colombeau and the definition of association. In § 3 we get explicit formula for the solution with the viscous term and in § 4 we show that it is indeed in the algebra of generalized functions and is solution in the sense of association of Colombeau. The paper concludes with some remarks on a general system with viscous terms.

2. Colombeau algebras

We take the domain $\Omega = (x,t), -\infty < x < \infty, t > 0$. Consider $C^\infty(\Omega)$, the class of infinitely differentiable functions in Ω and take the infinite product $\varepsilon(\Omega) = [C^\infty(\Omega)]^{(0,1)}$. Thus any element u of $\varepsilon(\Omega)$ is a map from (0,1) to $C^\infty(\Omega)$ and is denoted by $u = (u^\epsilon)_{0<\epsilon<1}$. We take a subclass $\varepsilon_{\mathcal{M}}(\Omega)$, called the moderate elements of $\varepsilon(\Omega)$. An element $u = (u^\epsilon)_{0<\epsilon<1}$ is called moderate if given a compact subset K of Ω and j and l nonnegative integers, there exists $N > 0$ such that

$$\|\partial_t^j \partial_x^l u^\epsilon\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{-\mathcal{N}}) \tag{2.1}$$

as ϵ tends to 0. An element $u = (u^\epsilon)_{0<\epsilon<1}$ is called null if for all compact subsets K of Ω and for all nonnegative integers j and l and for all $M > 0$,

$$\|\partial_t^j \partial_x^l u^\epsilon\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{\mathcal{M}}), \tag{2.2}$$

as ϵ goes to 0. The set of null elements is denoted by $\mathcal{N}(\Omega)$. It is easy to see that $\varepsilon_{\mathcal{M}}(\Omega)$ is an algebra with partial derivatives, the operations being defined pointwise on representatives and $\mathcal{N}(\Omega)$ is an ideal which is closed under differentiation. The quotient space denoted by

$$\mathcal{G}(\Omega) = \frac{\varepsilon_{\mathcal{M}}(\Omega)}{\mathcal{N}(\Omega)}$$

is an algebra with partial derivatives, the operations being defined on representatives. The algebra $\mathcal{G}(\Omega)$ is called the algebra of generalized functions of Colombeau. Two elements u and v in $\mathcal{G}(\Omega)$ are said to be associated, if for some (and hence all) representatives $(u^\epsilon)_{0<\epsilon<1}$ and $(v^\epsilon)_{0<\epsilon<1}$, of u and v, $u_\epsilon - v_\epsilon$ goes to 0 as ϵ tends to 0, in the sense of distribution and is denoted by $u \approx v$. Here we remark that this notion is different from the

notion of equality in $\mathcal{G}(\Omega)$, which means that $u - v \in \mathcal{N}(\Omega)$, or in other words,

$$\|\partial_t^j \partial_x^l (u^\epsilon - v^\epsilon)\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{\mathcal{M}})$$

for all M, for all j, l nonnegative integers and for all compact subsets K of Ω.

3. Explicit formula with viscous term

In this section we consider the system

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= \frac{\epsilon}{2} u_{xx}, \\ v_t + (uv)_x &= \frac{\epsilon}{2} v_{xx}, \\ w_t + \left(\frac{v^2}{2} + uw\right)_x &= \frac{\epsilon}{2} w_{xx}, \end{aligned} \tag{3.1}$$

in $-\infty < x < \infty, t > 0$ with initial conditions

$$u(x,0) = u_0^\epsilon(x), v(x,0) = v_0^\epsilon(x), w(x,0) = w_0^\epsilon(x), \tag{3.2}$$

at $t = 0$. Let us denote

$$F^\epsilon(x,y,t) = U_0^\epsilon(y) + \frac{(x-y)^2}{2t}, \tag{3.3}$$

where

$$U_0^\epsilon(x) = \int_0^x u_0^\epsilon(y)\mathrm{d}y, \tag{3.4}$$

then we have the following theorem.

Theorem 3.1. *Let $u_0^\epsilon(x)$, $v_0^\epsilon(x)$ and $w_0^\epsilon(x)$ be bounded measurable functions on R^1 for each ϵ positive, then*

$$u^\epsilon(x,t) = \partial_x U^\epsilon(x,t), v^\epsilon(x,t) = \partial_x V^\epsilon(x,t), w^\epsilon(x,t) = \partial_x W^\epsilon(x,t). \tag{3.5}$$

where U^ϵ, V^ϵ and W^ϵ are given by

$$U^\epsilon(x,t) = -\epsilon \log\left[\frac{1}{(2\pi t\epsilon)(1/2)} \int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y\right], \tag{3.6}$$

$$V^\epsilon(x,t) = \frac{\int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}, \tag{3.7}$$

$$\begin{aligned} W^\epsilon(x,t) = {} & \frac{\int_{-\infty}^{+\infty} \left(W_0^\epsilon(y) - \frac{V_0^\epsilon(y)^2}{2\epsilon}\right) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y} \\ & + \frac{1}{2\epsilon}\left[\frac{\int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}\right]^2 \end{aligned} \tag{3.8}$$

is a solution for (3.1)–(3.2).

Proof. To prove the theorem first we note that this system (3.1) can be written as a matrix Burgers equation

$$A_t + \left(\frac{A^2}{2}\right)_x = \frac{\epsilon}{2} A_{xx}, \tag{3.9}$$

where A is the lower triangular matrix of the form

$$A = \begin{pmatrix} u & 0 & 0 \\ v & u & 0 \\ w & v & u \end{pmatrix} \tag{3.10}$$

with initial condition

$$A(x,0) = \begin{pmatrix} u_0^\epsilon(x) & 0 & 0 \\ v_0^\epsilon(x) & u_0^\epsilon(x) & 0 \\ w_0^\epsilon(x) & v_0^\epsilon(x) & u_0^\epsilon(x) \end{pmatrix}. \tag{3.11}$$

Now we use Hopf–Cole transformation, see [7], generalized to matrix equations (3.9), where we use the fact $AA_t = A_tA, AA_x = A_xA$,

$$C = \exp\left(-\frac{B}{\epsilon}\right), \tag{3.12}$$

with

$$B = \begin{pmatrix} U & 0 & 0 \\ V & U & 0 \\ W & V & U \end{pmatrix}, \tag{3.13}$$

where U, V, W are given by

$$U(x,t) = \int_0^x u(y,t)\mathrm{d}y, V(x,t) = \int_0^x v(y,t)\mathrm{d}y, W(x,t) = \int_0^x w(y,t)\mathrm{d}y. \tag{3.14}$$

Using (3.12)–(3.14) in (3.9)–(3.10), we see that C satisfies the equation

$$C_t = \frac{\epsilon}{2} C_{xx},$$
$$C(x,0) = C_0(x),$$

where $C_0(x)$ is the matrix

$$C_0(x) = \begin{pmatrix} a_0(x) & 0 & 0 \\ b_0(x) & a_0(x) & 0 \\ c_0(x) & b_0(x) & a_0(x) \end{pmatrix}$$

with

$$a_0(x) = \exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right),$$
$$b_0(x) = -\frac{V_0^\epsilon(x)}{\epsilon}\exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right),$$
$$c_0(x) = \left(\frac{V_0^\epsilon(x)^2}{2\epsilon^2} - \frac{W_0^\epsilon(x)}{\epsilon}\right)\exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right).$$

On solving this explicitly we find that its solution takes the form of the following lower triangular matrix

$$C^\epsilon = \begin{pmatrix} a^\epsilon & 0 & 0 \\ b^\epsilon & a^\epsilon & 0 \\ c^\epsilon & b^\epsilon & a^\epsilon \end{pmatrix},$$

where $a^\epsilon, b^\epsilon; c^\epsilon$ are given by

$$a^\epsilon = \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y, \tag{3.15}$$

$$b^\epsilon = \frac{-1}{\epsilon} \cdot \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y, \tag{3.16}$$

$$c^\epsilon = \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} \left[\frac{V_0^\epsilon(y)^2}{2\epsilon^2} - \frac{W_0^\epsilon(y)}{\epsilon}\right] \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y. \tag{3.17}$$

Now to get back U^ϵ, V^ϵ and W^ϵ we use (3.12), namely

$$B^\epsilon = -\epsilon \log(C^\epsilon).$$

An easy calculation gives

$$B^\epsilon = \begin{pmatrix} U^\epsilon & 0 & 0 \\ V^\epsilon & U^\epsilon & 0 \\ W^\epsilon & V^\epsilon & U^\epsilon \end{pmatrix},$$

where

$$U^\epsilon = -\epsilon \log(a^\epsilon),\ V^\epsilon = -\epsilon \frac{b^\epsilon}{a^\epsilon},\ W^\epsilon = \epsilon\left(-\frac{c^\epsilon}{a^\epsilon} + \frac{(b^\epsilon)^2}{2(a^\epsilon)^2}\right). \tag{3.18}$$

Now substituting the expressions (3.15)–(3.17) for a^ϵ, b^ϵ and c^ϵ in (3.18) we get the formulas (3.6)–(3.8) for U^ϵ, V^ϵ, W^ϵ. Now it follows from (3.14) that

$$u^\epsilon = U_x^\epsilon,\ v^\epsilon = V_x^\epsilon,\ w^\epsilon = W_x^\epsilon.$$

The proof of the theorem is complete.

In order to study the limit of the functions $U^\epsilon, V^\epsilon, W^\epsilon$ given by (3.6)–(3.8) as ϵ goes to 0 we use the following result in the spirit of Hopf [7] and Lax [11].

PROPOSITION 3.2

Let $u_0(x)$ be bounded measurable and $p(x)$ Lipshitz continuous and both independent of ϵ. Let $F(x,y,t) = U_0(y) + ((x-y)^2/2t)$, where $U_0(x) = \int_0^x u_0(y)\mathrm{d}y$, then

(1) *For each $t > 0$ and $-\infty < x < \infty$, there exists at most a finite number of minimizers $y_0(x,t)$ for*

$$\min_{-\infty<y<+\infty} F(x,y,t). \tag{3.19}$$

For each (x,t) define maximum and minimum of these minimizers $y_0(x,t)$

$$y_0^+(x,t) = \max[y_0(x,t)],$$

$$y_0^-(x,t) = \min[y_0(x,t)],$$

then for each $t > 0$, except for a countable set of x, $y_0^+(x,t) = y_0^-(x,t)$.

(2) *For each $t > 0$, the limit,*

$$\lim_{\epsilon \to 0} \frac{\int_{-\infty}^{+\infty} p(y) \exp(-F(x,y,t)/\epsilon) \mathrm{d}y}{\int_{-\infty}^{+\infty} \exp(-F(x,y,t)/\epsilon) \, \mathrm{d}y} = p(y_0(x,t)),$$

exists except for a countable set of x. Also at every point $(x,t), p(y_0(x+,t))$ and $p(y_0(x-,t))$ exist.

Remark. Since the speed of propagation of the inviscid case is bounded by the L^∞ norm of u_0 we can restrict to initial data with compact support. We restrict to special initial data and using the above Proposition and from the explicit formula for U^ϵ, V^ϵ and W^ϵ given in theorem (3.1), the following is immediate.

PROPOSITION 3.3

Let $u_0^\epsilon(x) = (u_0 * \phi^\epsilon)(x)$, $v_0^\epsilon(x) = (v_0 * \phi^\epsilon)(x)$, $w_0^\epsilon(x) = (w_0 * \phi^\epsilon)(x)$ where u_0, v_0 and w_0 are bounded measurable functions with compact support and ϕ^ϵ is the usual Friedrichs mollifier with U^ϵ, V^ϵ and W^ϵ are as given by (3.6)–(3.8), *then for each $t > 0$, the limits* $\lim_{\epsilon \to 0} \epsilon u^\epsilon$, $\lim_{\epsilon \to 0} V^\epsilon$ *and* $\lim_{\epsilon \to 0} \epsilon W^\epsilon$ *exists except for a countable x and is given by*

$$\lim_{\epsilon \to 0} \epsilon u^\epsilon = 0, \quad \lim_{\epsilon \to 0} V^\epsilon = V_0(y_0(x,t)), \quad \lim_{\epsilon \to 0} \epsilon W^\epsilon = 0,$$

where $y_0(x,t)$ is a minimizer in (3.19).

Proof. First we notice that since

$$\lim_{\epsilon \to 0} \int_0^x u_0^\epsilon(y) \mathrm{d}y = \int_0^x u_0(y) \mathrm{d}y, \lim_{\epsilon \to 0} \int_0^x v_0^\epsilon(y) \mathrm{d}y$$
$$= \int_0^x v_0(y) \mathrm{d}y, \lim_{\epsilon \to 0} \int_0^x u_0^\epsilon(y) \mathrm{d}y = \int_0^x w_0(y) \mathrm{d}y,$$

uniformly on R^1 as ϵ goes to zero, the conclusions follow from the expressions (3.6)–(3.8) and proposition (3.2).

4. Generalized solutions for (1.1) and (1.2)

In this section we solve the problem

$$\begin{aligned} &u_t + \left(\frac{u^2}{2}\right)_x \approx 0, \\ &v_t + (uv)_x \approx 0, \\ &w_t + \left(\frac{v^2}{2} + vw\right)_x \approx 0, \end{aligned} \tag{4.1}$$

with initial conditions

$$u(x,0) = u_0, v(x,0) = v_0, w(x,0) = w_0, \tag{4.2}$$

where $u_0 = (u_0^\epsilon(x))_{0<\epsilon<1}$, $v_0 = (v_0^\epsilon(x))_{0<\epsilon<1}$ and $w_0 = (w_0^\epsilon(x))_{0<\epsilon<1}$ are in $\mathcal{G}(R^1)$, the algebra of generalized functions. Here we assume that $u_0^\epsilon(x)$, $v_0^\epsilon(x)$ and $w_0^\epsilon(x)$ are obtained

mollifying compactly supported bounded measurable functions $u_0(x)$, $v_0(x)$ and $w_0(x)$ pectively with Friedrichs mollifiers so that we have the following estimates

$$
\begin{aligned}
&\|\partial_x^l u_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}),\\
&\|\partial_x^l v_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}),\\
&\|\partial_x^l w_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}).
\end{aligned} \tag{4.3}
$$

w we state our main existence result.

eorem 4.1. *Let $u = (u^\epsilon)_{0<\epsilon<1}$, $v = (v^\epsilon)_{0<\epsilon<1}$ and $w = (w^\epsilon)_{0<\epsilon<1}$, where u^ϵ, v^ϵ and w^ϵ given by* (3.5)–(3.8) *with $u_0^\epsilon(x)$, $v_0^\epsilon(x)$, and $w_0^\epsilon(x)$ are as described above, then u, v and are in the algebra of generalized functions of Colombeau, $\mathcal{G}(\Omega)$ and solve the problem* 1)–(4.2).

oof. First we show that $u = (u^\epsilon)$, $v = (v^\epsilon)$ and $w = (w^\epsilon)$ are in $\mathcal{G}(\Omega)$. For this we have verify the estimate (2.1), for $(u^\epsilon)_{0<\epsilon<1}$, $(v^\epsilon)_{0<\epsilon<1}$ and $(w^\epsilon)_{0<\epsilon<1}$. It is clear from the mulas (3.5)–(3.8) for u^ϵ, v^ϵ and w^ϵ that, they are $C^\infty(\Omega)$. Further a typical term in the pressions of these functions is of the form $\epsilon^{-k}H^\epsilon(x,t)$ for $k = 0, 1, 2$ with

$$
H^\epsilon(x,t) = \frac{H_1^\epsilon(x,t)}{H_2^\epsilon(x,t)}
$$

d $H_1^\epsilon(x,t)$ and $H_2^\epsilon(x,t)$ taking the form

$$
\begin{aligned}
H_1^\epsilon(x,t) &= \int_{-\infty}^{+\infty} H_0^\epsilon(y)\exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y,\\
H_2^\epsilon(x,t) &= \int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y,
\end{aligned}
$$

d H_0^ϵ satisfying estimates of the form (4.3). Now by Leibinitz's rule $\partial_x^{j_0}H^\epsilon$ is a finite ear combination of elements of the form

$$
\frac{\partial_x^{j_k}H_1^\epsilon}{H_2^\epsilon}\cdot\frac{\partial_x^{(j_{k-1}-j_k)}H_2^\epsilon}{H_2^\epsilon}\cdots\frac{\partial_x^{j_0-j_1}H_2^\epsilon}{H_2^\epsilon},\quad j_k < j_{k-1} < j_1 < j_0,\ k = 0, 1, \ldots, j_0.
$$

w making a change of variable $y = x - \sqrt{2t\epsilon}z$ in the integrals of H_1^ϵ and H_2^ϵ and using 3) we get,

$$
\left\|\frac{\partial_x^j H_1^\epsilon}{H_2^\epsilon}\right\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}),\quad \left\|\frac{\partial_x^j H_2^\epsilon}{H_2^\epsilon}\right\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}).
$$

ese estimates together with our earlier observation on the form of u^ϵ, v^ϵ and w^ϵ leads to estimates

$$
\|\partial_x^j u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}), \|\partial_x^j v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+1)}), \|\partial_x^j w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+2)}). \tag{4.4}
$$

w from the PDE (3.1) and (4.4) we get

$$
\|\partial_t u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-1}), \|\partial_t v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2}), \|\partial_t w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-3}). \tag{4.5}
$$

Now applying the differential operator $\partial_t^j \partial_x^l$ on both sides of (3.1), first $l = 1, j = 0, 1, 2, \ldots$ and then $l = 2, j = 0, 1, 2, \ldots$ etc; proceeding successively we get the following estimate. For each j and l nonnegative integers,

$$\|\partial_t^j \partial_x^l u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l)}),$$
$$\|\partial_t^j \partial_x^l v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l+1)}),$$
$$\|\partial_t^j \partial_x^l w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l+2)}).$$

These estimates show that u, v and w are in $\mathcal{G}(\Omega)$. Now to show that u, v, and w satisfy eq. (1.1) in the sense of association we multiply (3.1) by a test function ϕ and integrate by parts to get

$$-\int_0^\infty \int_{-\infty}^\infty \left(u^\epsilon \phi_t + \frac{(u^\epsilon)^2}{2}\phi_x\right) \mathrm{d}x\, \mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_{xx} \mathrm{d}x\, \mathrm{d}t,$$
$$\int_0^\infty \int_{-\infty}^\infty (v^\epsilon \phi_t + (u^\epsilon v^\epsilon)\phi_x) \mathrm{d}x\, \mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty V^\epsilon \phi_{xxx} \mathrm{d}x\, \mathrm{d}t,$$
$$\int_0^\infty \int_{-\infty}^\infty \left(w^\epsilon \phi_t + \left(\frac{(v^\epsilon)^2}{2} + u^\epsilon v^\epsilon\right)\phi_x\right) \mathrm{d}x\, \mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty W^\epsilon \phi_{xxx} \mathrm{d}x\, \mathrm{d}t.$$

It follows from the asumption (4.3) on the initial data and the formulas (3.5)–(3.8) for u^ϵ and V^ϵ and ϵW^ϵ that these are uniformly bounded. Further by Proposition (3.3) and an application of dominated convergence theorem it follows that the right hand side of each of the above equations goes to 0 as ϵ goes to 0. This completes the proof of the theorem.

5. Concluding remarks

In general we could use Hopf–Cole transformation to find explicit solutions for any system of equations of the form

$$A_t + \frac{(A^2)_x}{2} = \frac{\epsilon}{2} A_{xx}, \tag{5.1}$$

with initial condition at $t = 0$,

$$A(x, 0) = A_0(x), \tag{5.2}$$

where A is a lower triangular matrix of the form

$$A = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_1 & 0 & 0 & 0 & 0 \\ u_3 & u_2 & u_1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & u_1 & 0 \\ u_n & u_{n-1} & \cdot & \cdot & u_2 & u_1 \end{pmatrix}. \tag{5.3}$$

Component wise (5.1) and (5.3) gives a system of n equation for the unknowns $u_1, u_2, \ldots, u_n$ namely

$$(u_j)_t + \sum_{i=1}^{j} \left(\frac{u_i u_{j-i+1}}{2}\right)_x = \frac{\epsilon}{2}(u_j)_{xx}, \tag{5.4}$$

for $j = 1, 2, \ldots, n$. The case, $n = 1$, in (5.4) is the standard Burgers equation and Hopf [7], used the Hopf–Cole transformation to construct solutions for $\epsilon > 0$ and obtained global solution for the inviscid Burgers equation with bounded measurable initial data by passing to the limit ϵ tends to 0. The limit function remains to be bounded and hence the solutions are understood in the standard theory of distributions. If one considers more general initial data such as bounded Borel measures, standard distribution theory does not work and so Biagoni and Oberguggenberger [1] constructed global solutions, in the algebra of generalized functions of Colombeau, for the case of more general initial data. In the case, $n = 2$ even for Riemann data the limit function contains δ-measures and this case was treated by Joseph [9], see also LeFloch [12], for a more general 2 by 2 system where the theory of DalMaso–LeFloch–Murat [6], applies. The explicit solutions of the initial value problem for the system (5.4) is complicated for general n, however the general feature of the solution for $n = 3, 4, \ldots$ remains the same. In fact, as ϵ tends to zero, the order of 'singularities' in the solution of (5.4), increases as n increases and we cannot use the method of [6], to define some of the products which appear in the equation and get global solutions for the inviscid case. In the present paper we have shown that we can use Colombeau's theory to get a global existence result for the inviscid case for $n = 3$.

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Boundary stabilization of a hybrid Euler–Bernoulli beam

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Abstract. We consider a problem of boundary stabilization of small flexural vibrations of a flexible structure modeled by an Euler–Bernoulli beam which is held by a rigid hub at one end and totally free at the other. The hub dynamics leads to a hybrid system of equations. By incorporating a condition of small rate of change of the deflection with respect to x as well as t, over the length of the beam, for appropriate initial conditions, uniform exponential decay of energy is established when a viscous boundary damping is present at the hub end.

Keywords. Boundary stabilization; Euler–Bernoulli beam equation; hybrid system; small deflection; exponential energy decay.

1. Introduction and mathematical formulation

We study the boundary stabilization of an Euler–Bernoulli beam of length l with a rigid hub of mass m^{H} capable of lateral motion at one end and the other end totally free, as in the case of a solar cell array, shown in figure 1 ([4, 5]). The objective here, is to study uniform stability of the overall system under suitable stabilizing force $Q(t)$ at the rigid-hub end only. Such a system is equivalent to a flexible space structure hoisted from a rigid hub. For small vibrations of the system, let $y^{\mathrm{H}}(t)$ be the transverse displacement of the hub and $y^{\mathrm{B}}(x,t)$ that of the beam at the position x relative to the hub at time t, then the total transverse deflection $y(x,t) = y^{\mathrm{H}}(t) + y^{\mathrm{B}}(x,t)$ satisfies the Euler–Bernoulli beam equation

$$my_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 \le x \le l, \quad t \ge 0, \tag{1}$$

under the assumptions $|y(x,t)| \ll l$ and $|y_x(x,t)| \ll 1$. The constants EI and m are the flexural rigidity and mass per unit length of the beam respectively, and subscripts in y denote partial derivatives with respect to the corresponding variables.

The equation of motion of the hub on which the stabilizing force $Q(t)$ is assumed to act, yields the differential equation ([4, 5])

$$m^{\mathrm{H}}y^{\mathrm{H}}_{tt}(t) + EIy^{\mathrm{B}}_{xxx}(0,t) + Q(t) = 0.$$

The exact controlability of a similar problem has been investigated recently in Gorain and Bose [6]. To study boundary stabilization, we assume that $Q(t)$ is proportional to $y_t(0,t)$ say, $Q(t) = by_t(0,t)$ i.e., a viscous boundary damping (stabilizer) is present at the hub end, the constant $b > 0$ being the viscous damping parameter. Also $y(0,t) = y^{\mathrm{H}}(t)$ and

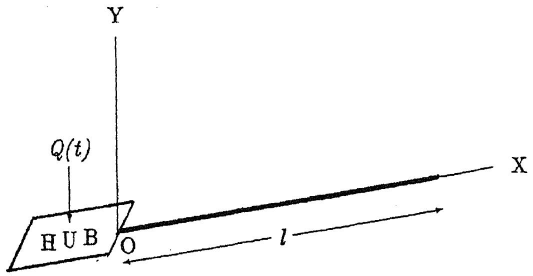

Figure 1. Schematic of the rigid hub and the beam.

$y_x(x,t) = y_x^{\mathrm{B}}(x,t)$, hence the above yields the hybrid boundary condition

$$y_{xxx}(0,t) + \alpha y_{tt}(0,t) + \lambda b y_t(0,t) = 0, \quad t \geq 0, \tag{2}$$

where $\alpha = m^{\mathrm{H}}/EI$ and $\lambda = 1/EI$. Assuming at $x = 0$, the beam is built-in position with the hub, we have

$$y_x(0,t) = 0, \quad t \geq 0. \tag{3}$$

At the free end of the beam

$$y_{xx}(l,t) = 0 \quad \text{and} \quad y_{xxx}(l,t) = 0, \quad t \geq 0, \tag{4}$$

and initially the beam is set to vibrations with

$$y(x,0) = y^0(x) \quad \text{and} \quad y_t(x,0) = y^1(x), \quad 0 \leq x \leq l. \tag{5}$$

The boundary stabilization for Euler–Bernoulli beam equation has been studied by Chen and Zhou [1], Chen *et al* [2], Littman and Markus [8], Morgül [9] and Rao [10]. All their investigations have shown the controlability and stabilization of Euler–Bernoulli beam equation, clamped at one end and feedback damping or control force (viscous damping) on the other end. Littman and Markus [8], and Chen and Zhou [1] in particular, have shown by calculating the eigenvalues of certain hybrid system that uniform stabilization is not possible because of the inclusion of infinitely large wave number k, during the passage of a wave along the length of the beam. Rao [10] concludes the same by semigroup theory.

The difficulty in proving uniform stability, appears to stem from not imposing any restriction that the beam remains approximately straight during vibration ([3, 11]). Motivated by this consideration, the rate of change in both x and t from the equilibrium position of the displacement $y(x,t)$ remains small, that is to say, $|y_{xt}(x,t)|$ remains small. The implication is that the time rate of variation of small slope remains small and also the gradient of the velocity along the length of the beam remains small. Therefore considering the totality along the length of the beam, we impose the restriction that $\int_0^l y_{xt}^2 \mathrm{d}x$ remains small. If we compare this quantity with a similar one, $\int_0^l y_{xx}^2 \mathrm{d}x$ which is actually $2/EI$ times the potential energy of bending of the beam and is thus finite, then accordingly the restriction on vibrations satisfying (1), is assumed to be governed by

$$\int_0^l y_{xt}^2 \mathrm{d}x \leq \frac{EI}{ml^2} \int_0^l y_{xx}^2 \mathrm{d}x \quad t > t_0, \tag{6}$$

for appropriate $y^0(x)$ and $y^1(x)$. Here EI/ml^2 is a dimensionality constant. For our purpose we have assumed (6) to hold for time $t > t_0$, where t_0 is finite but may be as large as we please.

In practice, it is important to translate the condition (6) in terms of initial data $\{y^0, y^1\}$. As far as our knowledge goes, this remains an open problem.

2. Energy of the system

Associated with each solution of (1)–(5), the total energy at time t is defined by the functional

$$E(t) = \frac{1}{2}\int_0^l (my_t^2 + EIy_{xx}^2)\mathrm{d}x + \frac{1}{2}m^{\mathrm{H}}y_t^2(0,t). \tag{7}$$

Now differentiating (7) with respect to t and replacing my_{tt} by $-EIy_{xxxx}$, we obtain

$$\dot{E}(t) = EI\int_0^l \frac{\partial}{\partial x}(y_{tx}y_{xx} - y_t y_{xxx})\mathrm{d}x + m^{\mathrm{H}}y_t(0,t)y_{tt}(0,t),$$

where dot represents the time derivative. Applying the boundary conditions (2)–(4), we get

$$\begin{aligned}\dot{E}(t) &= -EI\{\alpha y_{tt}(0,t) + \lambda b y_t(0,t)\}y_t(0,t) + m^{\mathrm{H}}y_t(0,t)y_{tt}(0,t)\\ &= -by_t^2(0,t) \le 0,\end{aligned} \tag{8}$$

for all $t \ge 0$, since $\alpha = m^{\mathrm{H}}/EI, \lambda = 1/EI$. This implies

$$E(t) \le E(0) \quad \text{for all } t \ge 0. \tag{9}$$

Hence the energy $E(t)$ is non-increasing with time and the system (1)–(5) is energy dissipating due to boundary damping at the hub end.

As the energy decays, our main interest is to obtain explicitly the uniform exponential energy decay estimate for the solution of (1)–(5), that is to establish the result of the form

$$E(t) \le M\mathrm{e}^{-\mu t}E(0), \quad t \ge 0 \tag{10}$$

for some reals $\mu > 0$ and $M \ge 1$.

3. Uniform stability result

Theorem 1. *Let $y(x,t)$ be a solution of the system (1)–(5) corresponding to the initial conditions $\{y^0, y^1\}$ for which (6) holds and $E(0) < \infty$. Then $E(t)$ satisfies the relation (10) for some reals $\mu > 0$ and $M \ge 1$.*

Proof. Proceeding as in Komornik [7], when $0 \le t \le t_0$, where t_0 (may be large enough) is a finite number such that (6) holds, we have

$$\mathrm{e}^{1-t/t_0} \ge 1.$$

Evidently, we can write from (9) that

$$E(t) \le E(0) \le \mathrm{e}^{1-t/t_0}E(0) = M_1\mathrm{e}^{-\mu_1 t}E(0) \quad \text{for } 0 \le t \le t_0, \tag{11}$$

where $M_1 = e$ and $\mu_1 = 1/t_0$.

For the case $t > t_0$, the proof is as in the following: Let $\epsilon > 0$ be a fixed small constant. We define the scalar-valued function V as

$$V(t) = E(t) + \epsilon\rho(t) \tag{12}$$

for all $t \geq t_0$, where

$$\rho(t) = 2m \int_0^l x y_t y_x \mathrm{d}x. \tag{13}$$

Since $y_x(0,t) = 0$, by Wirtinger's inequality [12], we have

$$\int_0^l y_x^2 \mathrm{d}x \leq \frac{4l^2}{\pi^2} \int_0^l y_{xx}^2 \mathrm{d}x, \tag{14}$$

and also it can be easily established that

$$y_t^2(l,t) \leq 2\left(y_t^2(0,t) + l \int_0^l y_{xt}^2 \mathrm{d}x\right). \tag{15}$$

Now from (13) we can estimate $\rho(t)$ as

$$\begin{aligned} |\rho(t)| &\leq \frac{4l^2}{\pi}\sqrt{\frac{m}{EI}} \int_0^l \left|\sqrt{m} y_t\right| \left|\frac{\pi}{2l}\sqrt{EI} y_x\right| \mathrm{d}x \\ &\leq \frac{2l^2}{\pi}\sqrt{\frac{m}{EI}} \int_0^l \left(m y_t^2 + \frac{\pi^2}{4l^2} EI y_x^2\right) \mathrm{d}x \leq \mu_0 E(t) \end{aligned} \tag{16}$$

by (14) and the energy equation (7), where

$$\mu_0 = \frac{4l^2}{\pi}\sqrt{\frac{m}{EI}}. \tag{17}$$

Thus from (12), we find

$$(1 - \epsilon\mu_0)E(t) \leq V(t) \leq (1 + \epsilon\mu_0)E(t). \tag{18}$$

Now differentiating (13) with respect to t, integrating by parts and applying the system of equations (1)–(4), it becomes

$$\begin{aligned} \dot{\rho}(t) &= \int_0^l x \frac{\partial}{\partial x}(m y_t^2 + EI y_{xx}^2) \mathrm{d}x + 2EI \int_0^l y_x y_{xxx} \mathrm{d}x \\ &= m^{\mathrm{H}} y_t^2(0,t) + ml y_t^2(l,t) - 2EI \int_0^l y_{xx}^2 \mathrm{d}x - 2E(t). \end{aligned} \tag{19}$$

Inserting the inequality (15) into (19), we obtain

$$\dot{\rho}(t) \leq (m^{\mathrm{H}} + 2ml) y_t^2(0,t) + 2\left(ml^2 \int_0^l y_{xt}^2 \mathrm{d}x - EI \int_0^l y_{xx}^2 \mathrm{d}x\right) - 2E(t),$$

and by the use of (6), we ultimately have

$$\dot{\rho}(t) \leq (m^{\mathrm{H}} + 2ml) y_t^2(0,t) - 2E(t). \tag{20}$$

Again differentiating (12) with respect to t, and inserting (8) and (20), we obtain the differential inequality

$$\dot{V}(t) \leq -2\epsilon E(t) - (b - \epsilon(m^{\mathrm{H}} + 2ml)) y_t^2(0,t). \tag{21}$$

If we choose $\epsilon \leq \epsilon_0$, where

$$\epsilon_0 = \min\{b/(m^{\mathrm{H}} + 2ml),\ 1/2\mu_0\}, \tag{22}$$

n from (21), it follows that for all $t > t_0$,

$$\dot{V}(t) + 2\epsilon E(t) \leq 0, \tag{23}$$

l at the same time we have from (18)

$$\mu_0 \epsilon E(t) \leq V(t) \leq (1 + \epsilon\mu_0)E(t). \tag{24}$$

th the help of (24), (23) yields

$$\dot{V}(t) + \mu_2 V(t) \leq 0, \tag{25}$$

ere

$$\mu_2 = \frac{2\epsilon}{1 + \epsilon\mu_0} > 0. \tag{26}$$

w multiplying (25) by $e^{\mu_2 t}$ and integrating over t_0 to t, we obtain

$$V(t) \leq e^{-\mu_2(t-t_0)} V(t_0). \tag{27}$$

en finally, inserting (24), it follows from (27) that for $t > t_0$,

$$E(t) \leq \frac{1 + \epsilon\mu_0}{\mu_0 \epsilon} e^{-\mu_2(t-t_0)} E(t_0) \leq M_2 e^{-\mu_2 t} E(0), \tag{28}$$

virtue of (9), where

$$M_2 = \frac{1 + \epsilon\mu_0}{\mu_0 \epsilon} e^{\mu_2 t_0}.$$

From (11) and (28), we conclude the result (10) for some reals $M = \max\{M_1, M_2\}$ and $= \min\{\mu_1, \mu_2\}$.

mark. It follows from (26) that exponential energy decay rate μ after passage of the ie t_0 will be maximum for largest admissible value of ϵ, i.e., for $\epsilon = \epsilon_0$. Choosing ϵ_0 ıal to $b/(m^{\mathrm{H}} + 2ml)$ or $1/2\mu_0$ according to (22), the maximum decay rate μ will be ıal to either $2b(m^{\mathrm{H}} + 2ml + b\mu_0)^{-1}$ or $2/3\mu_0$, and since as in (17), μ_0 is proportional ^{2}, the maximum energy decay rate μ decreases quadratically with increasing l after the pse of the time t_0. Hence it appears, that the decay of the solution of the system will be wer for a longer beam, which is very significant to our problem as one end of the beam otally free.

Conclusions

re we have established uniform boundary stabilization of small flexural vibrations of a xible Euler–Bernoulli beam attached to a movable rigid hub at one end and free at the er, and obtained a uniform exponential energy decay rate for the solution of this hybrid tem by taking into account a natural restriction for small vibrations [11] of the beam. e motivation of considering this type of hybrid system arises from many practical tems which consists of two parts: coupled elastic part and rigid part, constituting the brid system such as solar cell array, space craft with flexible attachments, robot with xible links and parts of many mechanical system. For these systems the situation ıerally occurs when it is very difficult or undesirable to apply the boundary control at

the free end of the elastic part where as, it is easier to apply it on the rigid part to obtain a good performance of the overall system. For initial conditions $y^0(x)$ and $y^1(x)$, when the energy and the motion decay with time following (8) and the beam approaches its straight position, we have assumed (6) to hold at the stages of vibration after elapse of some time t_0, however large. Our discussion here, has significantly covered the cases of uniform stability of such type of small vibration problem from mathematical point of view.

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A note on the dispersion of Love waves in layered monoclinic elastic media

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Abstract. The dispersion equation for Love waves in a monoclinic elastic layer of uniform thickness overlying a monoclinic elastic half-space is derived by applying the traction-free boundary condition at the surface and continuity conditions at the interface. The dispersion curves showing the effect of anisotropy on the calculated phase velocity are presented. The special cases of orthotropic and transversely isotropic media are also considered. It is shown that the well-known dispersion equation for Love waves in an isotropic layer overlying an isotropic half-space follows as a particular case.

Keywords. Dispersion; half-space; Love waves; monoclinic media.

1. Introduction

The study of surface wave dispersion in an isotropic half-space containing anisotropic layers is important in seismology for determining the presence or absence of anisotropic layers within the Earth. Such studies play a significant role in in-seam seismic exploration as well. The propagation of surface waves in an anisotropic half-space has been considered by many investigators. Wave propagation in a half-space with cubic symmetry has been discussed by Buchwald and Davies [3], and with orthorhombic symmetry by Stoneley [10]. Elastic wave propagation in transversely isotropic media has been reviewed by Payton [8]. Van der Hijden [12] discussed in great detail the propagation of transient elastic waves in stratified anisotropic media. Recent investigations on the propagation of elastic waves through anisotropic media include, among others, papers by Mench and Rasolofosaon [7], Savers [9] and Thomsen [11].

In an isotropic medium, SH type motion is decoupled from the P-SV type motion [6]. Surface waves of the SH type are known as Love waves and surface waves of the P-SV type are known as Rayleigh waves. The dispersion relation for Love waves in an isotropic elastic layer of uniform thickness H overlying an isotropic elastic half-space can be written in the form ([2], Sec. 3.6.2)

$$\tan\left[kH\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2}\right]=\frac{\mu_2}{\mu_1}\cdot\frac{(1-c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2-1)^{1/2}},\tag{1}$$

where k denotes the wave number, c the phase velocity, μ_1 and μ_2 the rigidities of the layer and the half-space, respectively, and β_1 and β_2 the shear-wave velocities of the layer and the half-space, respectively ($\beta_1 < c < \beta_2$). Equation (1) is also the dispersion relation

for Love waves in an isotropic elastic layer of uniform thickness $2H$ sandwiched between two isotropic elastic half-spaces of identical properties [5]. The purpose of the present study is to derive the corresponding dispersion relation when the media are anisotropic of the monoclinic type possessing one plane of elastic symmetry.

2. Love waves in a monoclinic layer overlying a monoclinic half-space

We consider the propagation of Love waves in a monoclinic elastic layer of uniform thickness H overlying a monoclinic elastic half-space. The layer $(0 \leq x_3 \leq H)$ is designated as medium (1) with displacement $u_1^{(1)}$ (x_2, x_3, t), density ρ_1 and elastic constants c_{ij} [6] and the half-space $(x_3 \geq H)$ is designated as medium (2) with displacement $u_1^{(2)}(x_2, x_3, t)$, density ρ_2 and elastic constants d_{ij}. A monoclinic medium has one plane of elastic symmetry [4]. We assume that the plane of symmetry is parallel to the x_2x_3-plane. For Love waves propagating in the positive x_2-direction with phase velocity c, we assume

$$u_1^{(1)} = f(x_3)\exp[ik(x_2 - ct)]. \tag{2}$$

The horizontal displacement $u_1^{(1)}$ satisfies the equation

$$c_{66}\frac{\partial^2 u_1}{\partial x_2^2} + 2c_{56}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + c_{55}\frac{\partial^2 u_1}{\partial x_3^2} = \rho_1 \frac{\partial^2 u_1}{\partial t^2}. \tag{3}$$

Equations (2) and (3) yield

$$c_{55}f''(x_3) + 2c_{56}ikf'(x_3) - k^2(c_{66} - \rho_1 c^2)f(x_3) = 0. \tag{4}$$

The general solution of eq. (4) is

$$f(x_3) = A_1\exp(ikb_1x_3) + B_1\exp(-ikb_2x_3),$$

where A_1, B_1 are arbitrary constants and

$$\begin{aligned} b_1 &= (\sqrt{A} - c_{56})/c_{55}, \quad b_2 = (\sqrt{A} + c_{56})/c_{55}, \\ A &= c_{55}(\rho_1 c^2 - c_{66}) + c_{56}^2. \end{aligned} \tag{5}$$

We thus have

$$u_1^{(1)} = (A_1 e^{ikb_1x_3} + B_1 e^{-ikb_2x_3})e^{ik(x_2-ct)}. \tag{6}$$

For surface waves

$$u_1^{(2)} \to 0 \text{ as } x_3 \to \infty.$$

Therefore, we assume

$$u_1^{(2)} = A_2 e^{ikb_3x_3} \cdot e^{ik(x_2-ct)}, \tag{7}$$

with

$$\begin{aligned} b_3 &= (i\sqrt{B} - d_{56})/d_{55}, \\ B &= d_{55}(d_{66} - \rho_2 c^2) - d_{56}^2, \end{aligned} \tag{8}$$

$\operatorname{Im} b_3 > 0$ (i.e. $B > 0$). The boundary conditions are

$$\tau_{13}^{(1)} = 0 \text{ at } x_3 = 0, \tag{9}$$

$$u_1^{(1)} = u_1^{(2)} \text{ at } x_3 = H, \tag{10}$$

$$\tau_{13}^{(1)} = \tau_{13}^{(2)} \text{ at } x_3 = H. \tag{11}$$

But

$$\tau_{13}^{(1)} = c_{55}\frac{\partial u_1^{(1)}}{\partial x_3} + c_{56}\frac{\partial u_1^{(1)}}{\partial x_2}, \tau_{13}^{(2)} = d_{55}\frac{\partial u_1^{(2)}}{\partial x_3} + d_{56}\frac{\partial u_1^{(2)}}{\partial x_2}. \tag{12}$$

Equations (5) to (12) yield

$$\begin{aligned} &A_1 - B_1 = 0, \\ &A_1 \exp(ikb_1H) + B_1 \exp(-ikb_2H) = A_2 \exp(ikb_3H), \\ &A_1 \exp(ikb_1H) - B_1 \exp(-ikb_2H) = i(B/A)^{1/2} A_2 \exp(ikb_3H). \end{aligned} \tag{13}$$

Eliminating A_1, B_1 and A_2, we obtain

$$\tan\theta = (B/A)^{1/2}, \tag{14}$$

where

$$\theta = (b_1 + b_2)kH/2 = (\sqrt{A}/c_{55})kH. \tag{15}$$

Equation (14) is the dispersion equation (frequency equation) for Love waves. From eq. (8) we note that $B > 0$. As in the case of isotropic media [2], it can be shown that (14) has no relevant solution if $A < 0$. Therefore, for the existence of Love waves, we must have $A > 0$, $B > 0$. Using (5) and (8), the dispersion equation (14) may be written in the form

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2} - 1 + \varepsilon_1\right)^{1/2} kH\right] = \left(\frac{d_{55}d_{66}}{c_{55}c_{66}}\right)^{1/2} \frac{(1 - \varepsilon_2 - c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2 - 1 + \varepsilon_1)^{1/2}}, \tag{16}$$

where

$$\begin{aligned} &\beta_1^2 = c_{66}/\rho_1, \quad \beta_2^2 = d_{66}/\rho_2, \\ &\varepsilon_1 = c_{56}^2/(c_{55}c_{66}), \varepsilon_2 = d_{56}^2/(d_{55}d_{66}), \\ &\gamma_1^2 = c_{66}/c_{55}. \end{aligned} \tag{17}$$

The conditions $A > 0, B > 0$ imply

$$(1 - \varepsilon_1)^{1/2}\beta_1 < c < (1 - \varepsilon_2)^{1/2}\beta_2.$$

Equation (16) is the dispersion relation for Love waves in a monoclinic layer of thickness H overlying a monoclinic half-space. It is also the dispersion relation for a monoclinic layer of thickness $2H$ sandwiched between two monoclinic half-spaces of identical elastic properties.

From (16), we note that the dispersion relation for Love waves in a free monoclinic plate of thickness H reduces to

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2} - 1 + \varepsilon_1\right)^{1/2} kH\right] = 0,$$

which implies $c > (1-\varepsilon_1)^{1/2}\beta_1$ and

$$\gamma_1\left(\frac{c^2}{\beta_1^2}-1+\varepsilon_1\right)^{1/2} kH = n\pi, \quad n = 0,1,2,\dots. \tag{18}$$

This relation also holds for a monoclinic layer in contact with a fluid layer on one or both sides.

3. Particular cases

3.1 *Orthotropic media*

For orthotropic media, $c_{56} = d_{56} = 0$. Therefore, $\varepsilon_1 = \varepsilon_2 = 0$ and the dispersion equation (16) reduces to

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2} kH\right] = \left(\frac{d_{55}d_{66}}{c_{55}c_{66}}\right)^{1/2}\frac{(1-c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2-1)^{1/2}}, \quad (\beta_1 < c < \beta_2). \tag{19}$$

The dispersion relation (18) for Love waves in a free plate of thickness H becomes

$$\gamma_1\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2} kH = n\pi, \quad n = 0,1,2,\dots. \tag{20}$$

The corresponding equation given by Stoneley [10] is in error.

3.2 *Transversely isotropic media*

The dispersion equations (19) and (20) are also valid when the two media are transversely isotropic. These coincide with the dispersion equations given by Anderson [1] for transversely isotropic media.

3.3 *Isotropic media*

For isotropic media,

$$c_{55} = c_{66} = \mu_1, \quad d_{55} = d_{66} = \mu_2, \quad c_{56} = d_{56} = 0, \quad \varepsilon_1 = \varepsilon_2 = 0, \quad \gamma_1 = \gamma_2 = 1.$$

Using these relations, the dispersion eq. (16) reduces to the form (1) valid for isotropic media.

4. Numerical results and discussion

Equation (16) is the dispersion equation for Love waves propagating in the plane of symmetry of a monoclinic elastic layer of thickness H overlying a monoclinic half-space. This is also the dispersion equation for Love waves propagating in the plane of symmetry of a monoclinic elastic layer of thickness $2H$ sandwiched between two monoclinic elastic half-spaces of identical properties. For computing the dispersion curves, we assume

$$c_{55} = c_{66}, \quad d_{55} = d_{66}, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon, \quad d_{55}/c_{55} = a, \quad \beta_2/\beta_1 = b. \tag{21}$$

The dispersion equation (16) can now be written as

$$\tan[(C^2-1+\varepsilon)^{1/2}K] = a\frac{(1-\varepsilon-C^2/b^2)^{1/2}}{(C^2-1+\varepsilon)^{1/2}}, \tag{22}$$

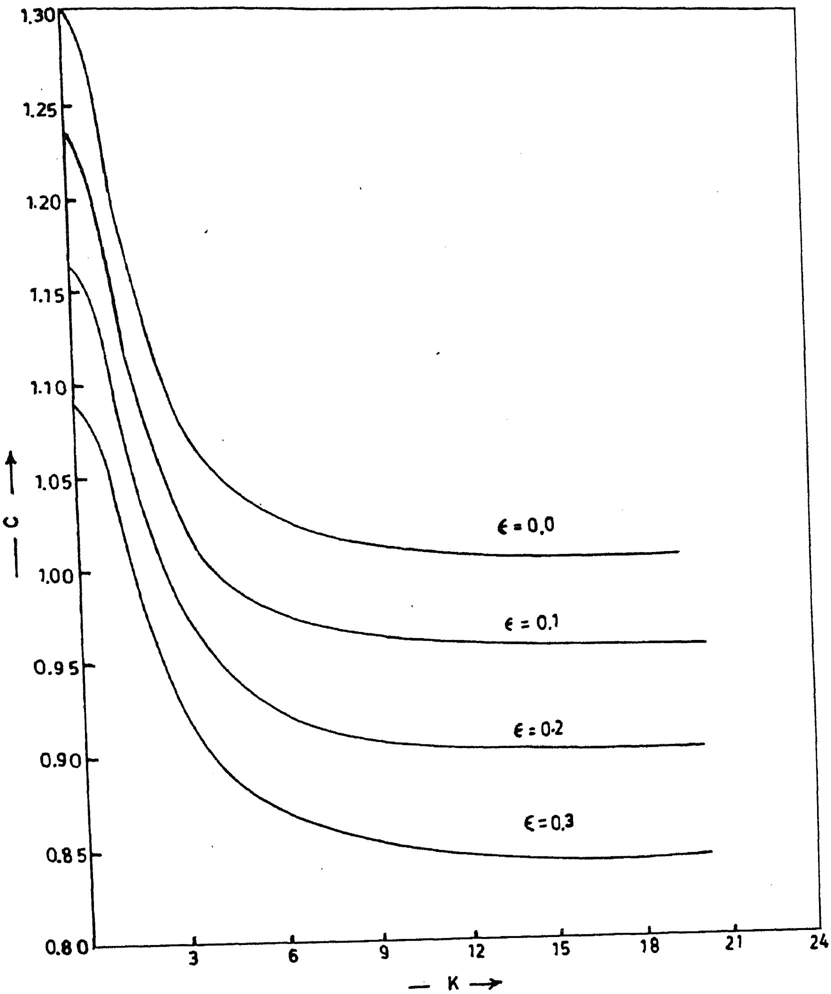

Figure 1. Variation of the dimensionless phase velocity $C = c/\beta_1$ with dimensionless wave number $K = kH$ for the fundamental Love mode for various values of the anisotropy parameter ε. The curve corresponding to $\varepsilon = 0$ is for isotropic media. The phase velocity lies in the range $(1-\varepsilon)^{1/2} < C < 1.3(1-\varepsilon)^{1/2}$. $K = 0$ for $C = (1-\varepsilon)^{1/2}$; $K \to \infty$ as $C \to 1.3(1-\varepsilon)^{1/2}$.

where $C = c/\beta_1$ is the dimensionless phase velocity, $K = kH$ is the dimensionless wave number and

$$(1-\varepsilon)^{1/2} < C < b(1-\varepsilon)^{1/2}. \tag{23}$$

Equation (22) has been used to obtain the dispersion curves showing the variation of the phase velocity with wave number for various values of the anisotropy parameter ε assuming $a = 2$, $b = 1.3$ (figure 1). $\varepsilon = 0$ corresponds to the case when both the layer and the half-space are isotropic. From condition (23) we note that the effect of anisotropy is to reduce the range of the phase velocity from

$$\beta_1 < c < \beta_2,$$

Table 1. Cut-off period (in seconds) of the first three overtones ($n = 1, 2, 3$) for various values of the anisotropy parameter ε_1 of the layer.

n	Isotropic $\gamma_1 = 1, \varepsilon_1 = 0$	Monoclinic, $\gamma_1 = 1.1$ $\varepsilon_1 = 0.1$	0.2	0.3
1	12.57	14.85	15.80	16.70
2	6.29	7.42	7.90	8.35
3	4.19	4.95	5.27	5.57

valid for isotropic media, to

$$(1-\varepsilon)^{1/2}\beta_1 < c < (1-\varepsilon)^{1/2}\beta_2.$$

For a given k, the phase velocity decreases as the value of the anisotropy parameter ε increases.

From (16), we find that the cut-off period for the nth overtone is given by

$$T_n = \frac{2\gamma_1 H}{n\beta_1}\left[1 - \left(\frac{1-\varepsilon_1}{1-\varepsilon_2}\right)\left(\frac{\beta_1}{\beta_2}\right)^2\right]^{1/2}. \tag{24}$$

For studying the effect of the anisotropy on the cut-off period, we assume that the half-space is isotropic ($\varepsilon_2 = 0$) and $H = 35\,\text{km}$, $\beta_1 = 3.5\,\text{km/s}$, $\beta_2 = 4.5\,\text{km/s}$.

Table 1 gives the values of the cut-off period for the first three overtones for various values of the anisotropy parameter ε_1 of the layer. We note that the cut-off period increases as the value of the anisotropy parameter increases.

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astic wave propagation in a cylindrical bore situated a micropolar elastic medium with stretch

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Abstract. Propagation of surface elastic waves in a cylindrical bore through a micropolar elastic medium with stretch is analysed in two cases. In the first case, the cylindrical bore is considered empty while in the second case, the bore is filled with homogeneous inviscid liquid. In both the problems, period equations are obtained in closed form. The problem of Banerji and Sengupta [2,3] has been reduced as a special case. Numerical calculations have been performed for a particular model and results obtained are presented graphically. It is noticed that the effect of micropolarity on dispersion curve is significant while the effect of micro-stretch on dispersion curve is not appreciable.

Keywords. Surface waves; cylindrical bore; micropolarity; stretch; phase velocity; dispersion.

Introduction

ngen and his coworker [4, 9] developed the theory of simple microelastic solids in ich the 'micro' deformations and rotations of the material particles contained in a crovolume element with respect to its centriod are taken into account in an average se. Material affected by such micromotion and micro deformations are called micro-rphic materials. Later, Eringen [5] developed theory for a subclass of micromorphic terials which are called 'micropolar solids'. These solids represent the materials that made up of dipole atoms or dumbbell type molecules and are subjected to surface and ly couples. The deformation in these materials is characterized not only by classical nslational degrees of freedom represented by the deformation vector field $\vec{u}(\vec{x},t)$, but by the rotation vector $\vec{\omega}(\vec{x},t)$. In fact, micropolar elastic solids are special class of croelastic solids in which micro-stretch is ignored compared to micro-rotation. Eringen further extended his work to include the effect of axial stretch during the rotation of lecules and presented a theory of 'micropolar elastic solids with stretch'. These terials can be thought of being composed of a large number of short springs which can sess average inertia moments and can deform in axial direction. The deformation in se solids is characterized by three quantities, i.e., the intrinsic axial stretch $\phi^*(\vec{x},t)$, ich is a scalar quantity, in addition to the deformation field vectors $\vec{u}(\vec{x},t)$, and $\vec{\omega}(\vec{x},t)$ micropolar elasticity.

The problem of wave propagation along the surface of a cylindrical borehole in an stic medium of infinite extent had been first studied by Biot [1]. He calculated the ehole guided wave, the pseudo Rayleigh wave, and the Stoneley wave dispersion ves. Banerji and Sengupta [2, 3] studied the corresponding problem in micropolar

elastic medium. It is believed that the soil is very close to micromorphic medium and the problems of long well or borehole are useful in exploration of chemicals, oils, water etc. beneath the earth surface through wave propagation technique. Since the micropolar elastic material with stretch is very close to the micromorphic media e.g. soil. Our research studies the motivation of this type of physical situation. In order to determine the dispersive nature of the elastic waves in the bore hole, we have considered the problem of wave propagation in a cylindrical bore through micropolar elastic solid with stretch. The frequency equation of axial symmetric waves propagating at the surface of the cylindrical bore is obtained in two cases. In Case I, the cylindrical bore passing through the micropolar elastic medium with stretch and of infinite extent is assumed to be empty, while in the Case II, it is assumed to be filled with homogeneous inviscid liquid.

2. Propagation of waves in cylindrical empty bore

Case I

Consider a cylindrical empty bore of circular cross section having diameter $2a$, in a homogeneous, isotropic, micropolar elastic medium with stretch and of infinite extent. The waves of axial symmetry and pure sinusoidal along the axial direction are considered. Taking cylindrical polar co-ordinates (r, θ, z) with z-axis pointing upward. In the absence of body forces and body moments, the governing equations (ref. [6], eqs (6.3) to (6.5), pp. 12) in the medium considered are:

$$(\mu+K)\nabla^2\vec{u}+(\lambda+\mu)\nabla(\nabla\cdot\vec{u})+K\nabla\times\vec{\omega}=\rho\frac{\partial^2\vec{u}}{\partial t^2}, \tag{1}$$

$$\gamma\nabla^2\vec{\omega}+(\alpha+\beta)\nabla(\nabla\cdot\vec{\omega})-2K\vec{\omega}+K\nabla\times\vec{u}=\rho j\frac{\partial^2\vec{\omega}}{\partial t^2}, \tag{2}$$

$$\alpha_0\nabla^2\phi^*-\eta_0\phi^*=\frac{\rho j}{2}\frac{\partial^2\phi^*}{\partial t^2}, \tag{3}$$

where $\lambda, \mu, K, \alpha, \beta, \gamma, \alpha_0, \eta_0$ are material moduli, $\vec{u}(x_i,t)$, $\vec{\omega}(x_i,t)$ denote the displacement and rotation vectors respectively, while ϕ^* is the scalar microstretch, ρ is the density and j is the inertia of the medium considered. The constitutive relations expressing the components of stress tensor t_{ji}, couple stress tensor m_{ji} and stress moment tensor λ_j^* are given by ([6], eqs (6.6) to (6.8), pp. 12),

$$t_{ji}=\lambda u_{k,k}\delta_{ij}+\mu(u_{i,j}+u_{j,i})+K(u_{i,j}-\varepsilon_{kji}\,\omega_k), \tag{4}$$

$$m_{ji}=\beta_0\,\varepsilon_{kji}\phi^*_{,k}+\alpha\omega_{k,k}\delta_{ij}+\beta\omega_{j,i}+\gamma\omega_{i,j}, \tag{5}$$

$$\lambda_j^*=\alpha_0\phi^*_{,j}+\frac{\beta_0}{3}\varepsilon_{kji}\omega_{k,i}, \tag{6}$$

where the various symbols have their usual meanings.

Since we are considering axi-symmetric problem, so we take $u_k=(u_r,0,u_z)$ and $\omega_k=(0,\omega_\theta,0)$, and the quantities would remain independent of θ. This consideration would lead (1)–(3) to the following ones,

$$(\mu+K)\left(\nabla^2-\frac{1}{r^2}\right)u_r+(\lambda+\mu)\frac{\partial e}{\partial r}-K\frac{\partial\omega_\theta}{\partial z}=\rho\frac{\partial^2u_r}{\partial t^2}, \tag{7}$$

$$(\mu+K)\nabla^2u_z+(\lambda+\mu)\frac{\partial e}{\partial z}+\frac{K}{r}\frac{\partial}{\partial r}(r\omega_\theta)=\rho\frac{\partial^2u_z}{\partial t^2}, \tag{8}$$

$$\left[\gamma\left(\nabla^2 - \frac{1}{r^2}\right) - 2K\right]\omega_\theta + K\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] = \rho j\,\frac{\partial^2 \omega_\theta}{\partial t^2}, \tag{9}$$

$$\alpha_0 \nabla^2 \phi^* - \eta_0 \phi^* = \frac{\rho j}{2}\frac{\partial^2 \phi^*}{\partial t^2}, \tag{10}$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z}, \tag{10a}$$
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

Introducing the potentials ϕ, ψ and Γ in the following forms

$$u_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}, \tag{11}$$

$$u_z = \frac{\partial \phi}{\partial z} - \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\psi, \tag{12}$$

$$\omega_\theta = -\frac{\partial \Gamma}{\partial r}. \tag{13}$$

The solutions of eqs (7) to (10) for unattenuated waves propagating in the z-direction are

$$\begin{aligned}
\phi &= A_0 K_0(\sigma r)\cos(kz - \omega t),\\
\psi &= [A_1 K_0(\lambda_1 r) + A_1' K_0(\lambda_2 r)]\sin(kz - \omega t),\\
\Gamma &= [A_2 K_0(\lambda_1 r) + A_2' K_0(\lambda_2 r)]\sin(kz - \omega t),\\
\phi^* &= A_3 K_0(\xi r)\cos(kz - \omega t),
\end{aligned} \tag{14}$$

where A_i and A_i' are constants and

$$\begin{aligned}
&\sigma^2 = k^2 - \sigma_1^2, \quad \sigma_n^2 = \frac{\omega^2}{c_n^2}, \quad (n = 1,2,4),\\
&\sigma_5^2 = \frac{\rho J \omega^2}{2\alpha_0} - \frac{\eta_0}{\alpha_0}, \quad \xi^2 = k^2 - \sigma_5^2,\\
&c_1^2 = \frac{\lambda + 2\mu + K}{\rho}, \quad c_2^2 = \frac{\mu + K}{\rho}, \quad c_4^2 = \frac{\gamma}{\rho j},\\
&\nu^2 = \frac{2K}{\gamma}, \quad \eta^2 = \frac{K^2}{\gamma(\mu + K)},
\end{aligned} \tag{15}$$

$$\lambda_{1,2}^2 = k^2 - \frac{1}{2}[\sigma_4^2 + \sigma_2^2 + \eta^2 - \nu^2 \mp \{(\sigma_4^2 - \sigma_2^2 - \nu^2 + \eta^2)^2 + 4\eta^2\sigma_2^2\}^{1/2}]. \tag{16}$$

The phase velocity of waves in the z-direction is denoted by $c\,(=\omega/k)$, k is the wave number and ω being the circular frequency. $K_0(\ )$ is the modified Bessel function of second kind and zero order. Substituting the values of ψ and Γ from (14) in (8) and (9) through (11) to (13), we obtain

$$A_2 = b_1 A_1, \quad A_2' = b_2 A_1', \tag{17}$$

where

$$b_i = (k^2 - \sigma_2^2 - \lambda_i^2)/p, \quad (i = 1,2) \quad p = K/(\mu + K). \tag{18}$$

Boundary conditions. The boundary conditions are the vanishing of stresses on the surface of the cylindrical bore, i.e. at $r = a$,

$$t_{rr} = t_{rz} = m_{r\theta} = \lambda_r^* = 0. \tag{19}$$

It is easy to calculate the requisite components of stresses from (4) to (6) in terms of potentials and using these expressions along with eqs (14) and (17) in the above boundary conditions (19), we shall obtain four homogeneous equations in four unknowns namely A_0, A_1, A_1' and A_3. The condition for a non-trivial solution for these unknowns would give the following frequency equation as

$$\left[(2\mu+K)\sigma^2\left(\frac{K_0(\sigma a)}{K_1(\sigma a)}+\frac{1}{\sigma a}\right)+\lambda(\sigma^2-k^2)\frac{K_0(\sigma a)}{K_1(\sigma a)}\right](a_1D_2-a_2D_1)$$
$$-(2\mu+K)^2k^2\sigma\left[\lambda_1\left(\frac{K_0(\lambda_1 a)}{K_1(\lambda_1 a)}+\frac{1}{\lambda_1 a}\right)D_2-\lambda_2\left(\frac{K_0(\lambda_2 a)}{K_1(\lambda_2 a)}+\frac{1}{\lambda_2 a}\right)D_1\right]=0, \tag{20}$$

where

$$a_i = (2\mu+K)k^2 + (\mu+K)(\lambda_i^2-k^2) + Kb_i,$$
$$D_i = \lambda_i b_i\left[\frac{\beta+\gamma}{\lambda_i a}+\gamma\frac{K_0(\lambda_i a)}{K_1(\lambda_i a)}\right]\alpha_0\xi - \frac{\beta_0}{3}k^2 b_i\frac{K_0(\xi a)}{K_1(\xi a)}, \quad i = 1,2. \tag{21}$$

Equation (20) determines the phase velocity of axial symmetric surface wave as a function of dimensionless wave number ka, indicating that these waves are dispersive in nature.

Physically, it must be expected that for every short wave length, i.e., for large values of ka, the dispersion equation (20) should coincide with the dispersion equation for the Rayleigh type waves at a plane boundary of a micropolar elastic half-space with stretch. This can be done by putting $ka = \infty$ in (20). For large value of the argument, the asymptotic value of modified Bessel functions is ([10], eq. (7.101), pp. 318)

$$K_0(u) = K_1(u) = \sqrt{\frac{\pi}{2u}}\cdot \mathrm{e}^{-u}, \quad (u \to \infty). \tag{22}$$

Hence equation (20) reduces to

$$[(\lambda+2\mu+K)\sigma^2-\lambda k^2](a_1D_2'-a_2D_1')-(2\mu+K)^2k^2\sigma(\lambda_1D_2'-\lambda_2D_1')=0. \tag{23}$$

where

$$D_i' = b_i\left[\lambda_i\gamma\alpha_0\xi-\frac{\beta_0}{3}k^2\right], \quad i=1,2. \tag{24}$$

Equation (23) is the equation for Rayleigh waves at a plane boundary of a micropolar elastic medium with stretch.

Special cases. (I) If we neglect the stretch effect i.e. putting $\beta_0 = 0$ in (20) and (23), we obtain the following dispersion equations

$$\left[(2\mu+K)\sigma^2\left(\frac{K_0(\sigma a)}{K_1(\sigma a)}+\frac{1}{\sigma a}\right)+\lambda(\sigma^2-k^2)\frac{K_0(\sigma a)}{K_1(\sigma a)}\right]$$

$$\left[a_1\lambda_2 b_2\left\{\frac{\beta+\gamma}{\lambda_2 a}+\gamma\frac{K_0(\lambda_2 a)}{K_1(\lambda_2 a)}\right\}-a_2\,\lambda_1 b_1\left\{\frac{\beta+\gamma}{\lambda_1 a}+\gamma\frac{K_0(\lambda_1 a)}{K_1(\lambda_1 a)}\right\}\right]$$

$$-(2\mu+K)^2k^2\sigma\lambda_1\lambda_2\left[b_2\left(\frac{K_0(\lambda_1 a)}{K_1(\lambda_1 a)}+\frac{1}{\lambda_1 a}\right)\left\{\frac{\beta+\gamma}{\lambda_2 a}+\gamma\frac{K_0(\lambda_2 a)}{K_1(\lambda_2 a)}\right\}\right.$$

$$\left.-b_1\left(\frac{K_0(\lambda_2 a)}{K_1(\lambda_2 a)}+\frac{1}{\lambda_2 a}\right)\left\{\frac{\beta+\gamma}{\lambda_2 a}+\gamma\frac{K_0(\lambda_1 a)}{K_1(\lambda_1 a)}\right\}\right]=0, \tag{25}$$

$$[(\lambda+2\mu+K)\sigma^2-\lambda k^2]\left(a_1b_2-\frac{\lambda_1}{\lambda_2}a_2b_1\right)-(2\mu+K)^2k^2\sigma\lambda_1(b_2-b_1)=0, \tag{26}$$

and

$$\alpha_0\xi=0. \tag{27}$$

Equation (25) is the dispersion equation for the propagation of elastic waves in a cylindrical bore, through micropolar elastic medium of infinite extent and coincides with the equation (2.24) of Banerji and Sengupta [2] with the appropriate change in notations, that is, replacing μ, K, γ, β and α by $\mu-\alpha$, 2α, $\gamma+\varepsilon$, $\gamma-\varepsilon$ and β respectively. Equation (26) coincides with the equation of Rayleigh waves at a plane boundary of micropolar medium obtained by Kaliski *et al* [8] and eq. (27) is due to the solution of equation (3) by using the boundary condition $\lambda_r^*=0$ (with $\beta_0=0$) and represents to a hypothetical medium in which only microstretch may exist.

(II) If we neglect the micropolar and stretch effects together in the medium considered i.e. putting $\alpha=0$ and $\beta_0=0$ in (20), one can obtain the eqs (2.26) and (2.27) of ref. [2], pp. 262(646) and represent the dispersion equation obtained by Biot [1] for the propagation of elastic waves in borehole.

3. Propagation of waves in a cylindrical bore filled with liquid

Case II

Here, we consider that the cylindrical bore is filled with homogeneous inviscid liquid. For axially symmetric waves, the displacement potential ϕ^0 in the liquid satisfies the wave equation

$$\nabla^2\phi^0=\frac{1}{c_2^2}\frac{\partial^2\phi^0}{\partial t^2}, \tag{28}$$

where $c_2=(\lambda_0/\rho_0)^{1/2}$ is the velocity of dilatational wave in liquid, ρ_0, λ_0 are the density and bulk modulus of the liquid respectively and Laplacian operator is defined in (10a). The solution of equation (28) corresponding to the surface wave may be written as

$$\phi^0=A_4I_0(r\zeta)e^{i(kz-\omega t)},\quad c<c_2 \tag{29}$$

where

$$\zeta^2=k^2-\frac{\omega^2}{c_2^2},$$

and $I_0(\ )$ is the modified Bessel function of first kind and order zero. A_4 is an unknown constant. Notating $y = c/c_2$, then the liquid pressure and radial displacement respectively are (for $y < 1$)

$$p = \rho_0\omega^2 A_4 I_0[rk(1-y^2)^{1/2}]e^{i(kz-\omega t)}, \tag{30}$$

$$q_r = k(1-y^2)^{1/2} A_4 I_1[rk(1-y^2)^{1/2}]e^{i(kz-\omega t)}. \tag{31}$$

Boundary conditions. The appropriate boundary conditions at the interface $r = a$ between liquid and micropolar medium with stretch are

$$\begin{aligned} q_r &= u_r, \quad -p = t_{rr}, \\ 0 &= t_{rz} = m_{r\theta} = \lambda_r^*. \end{aligned} \tag{32}$$

The first two boundary conditions can be written as

$$-\frac{p}{q_r} = \frac{t_{rr}}{u_r}, \tag{33}$$

which is a composite boundary condition to the matching of mechanical impedance. The last three boundary conditions are same as in Case I. These three boundary conditions and boundary condition (33) with the use of (4) to (6), (14), (17), (30) and (31), lead to four equations in four unknowns namely A_0, A_1, A_1' and A_4. These four homogeneous equations will have non-trivial solution if the determinant of the coefficients of the unknowns is zero, which yields (for $y < 1$)

$$\begin{aligned} &\left[(2\mu+K)\sigma^2\left(\frac{K_0(\sigma a)}{K_1(\sigma a)} + \frac{1}{\sigma a}\right) + \lambda(\sigma^2 - k^2)\frac{K_0(\sigma a)}{K_1(\sigma a)} - M\sigma\right](a_1 D_2 - a_2 D_1) \\ &\quad - (2\mu+K)^2 k^2 \sigma\left[\left\{\lambda_1\left(\frac{K_0(\lambda_1 a)}{K_1(\lambda_1 a)} + \frac{1}{\lambda_1 a}\right) - \frac{M}{2\mu+K}\right\}D_2\right. \\ &\quad \left. - \left\{\lambda_2\left(\frac{K_0(\lambda_2 a)}{K_1(\lambda_2 a)} + \frac{1}{\lambda_2 a}\right) - \frac{M}{2\mu+K}\right\}D_1\right] = 0, \end{aligned} \tag{34}$$

where

$$M = \frac{\rho\omega^2}{k\sqrt{1-y^2}} \frac{I_0(ak\sqrt{1-y^2})}{I_1(ak\sqrt{1-y^2})} \tag{35}$$

and D_i's are defined earlier in (21). If β_0 approaches to zero then it can be seen that dispersion equation (34) reduces to that obtained by Banerji and Sengupta ([3], eq. (2.10), pp. 649] for the relevant problem in micropolar elastic medium.

4. Numerical results and conclusion

Equation (20) and (34) are the dispersion equations for surface wave propagation near borehole and express the implicit relationship between phase velocity of wave propagation and wave number. In order to study the problem numerically, we have calculated the normalized phase velocity (c/c_1) for different values of normalized wave number (ka), in the case of empty bore. For numerical data of micropolar elastic medium with stretch, we have taken the following values of relevant parameters ([7], pp. 457), in non-dimensional

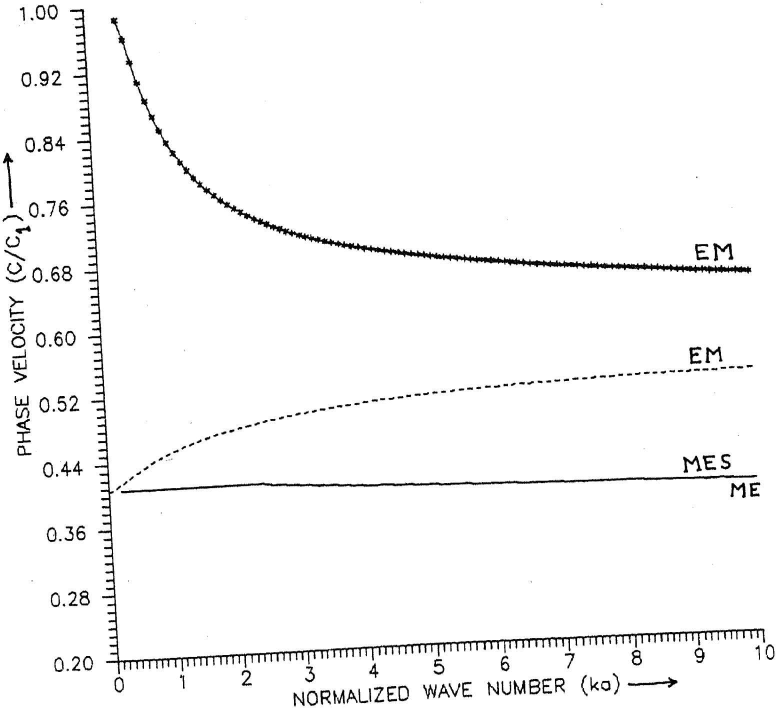

Figure 1. Variation of phase velocity with wave number.

form as

$$\frac{\lambda}{2\mu+K}=2.01,\quad \frac{K}{2\mu+K}=0.004,\quad \frac{\beta+\gamma}{2\beta_0 a^2}=131.5,$$
$$\frac{\gamma}{\beta_0 a^2}=131.505,\quad \frac{\gamma}{\alpha_0}=3.288,\quad \frac{\eta_0 a^2}{\alpha_0}=0.00625,$$
$$\rho=2.192\,\mathrm{gm/cm}^3,\quad J=0.196\,\mathrm{mm}^{-2}.$$

Figure 1 shows the variation of c/c_1 with ka. We note that the solid curve without center symbol represents mode of propagation of waves in bore through micropolar elastic medium with stretch (MES). The value of c/c_1 is 0.407943 at $ka = 0.13$, it increases to a very small extent as ka increases beyond 0.13 and gets the value 0.408343 at $ka = 1.0$. Thereafter the value of c/c_1 decreases slowly with the further increase in the value of ka and approaches to the value 0.389546 as ka approaches to the value 10. This is clearly shown in figure 2.

The same curve having minute difference at large values of wave number is obtained for micropolar elastic medium (ME). This difference is not visible in figure 1 and has been shown in figure 2. Two curves, one dashed and other with center symbols represent the two modes of wave propagation in classical elastic medium (EM). The value of c/c_1 is 0.407543 at $ka = 0.01$: it increases slowly with increase in ka and approaches to the value 0.524542 as ka approaches to the value 10. This is shown by dashed curve in figure 1. For the remaining mode of propagation which is shown by solid curve with center symbol, the value of c/c_1 exists at $ka = 0.50$ and is equal to 0.985412, it decreases and

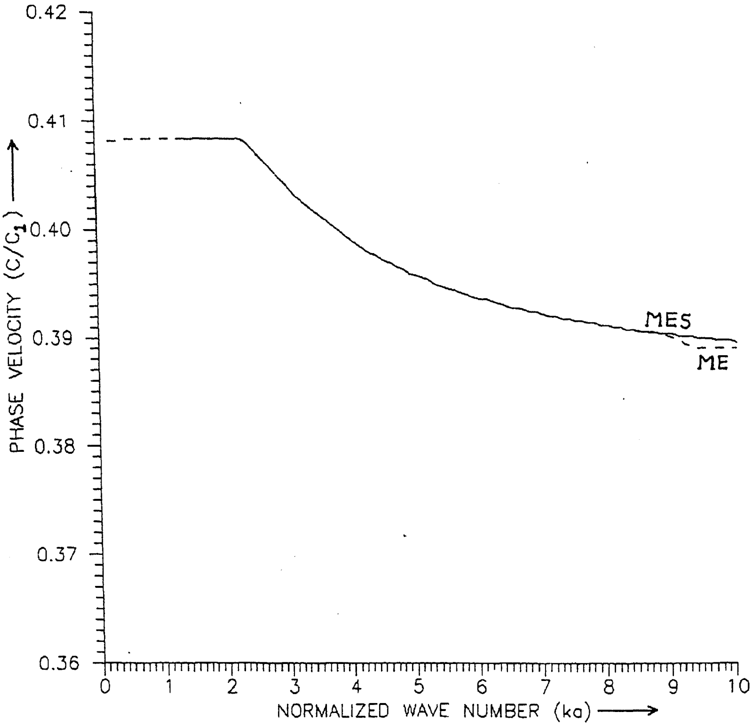

Figure 2. Variation of phase velocity with wave number.

goes to the value 0.643555 as the value of ka goes to 10. It is observed that no value of phase velocity exists in the range $0 < ka < 0.12$ for MES and ME, while this does not happen in case of EM, i.e., phase velocity exists for all values of ka. Also the effect of micropolarity on propagation of surface elastic waves in cylindrical bore is significant. This effect can be seen on comparing dashed and solid curves in figure 1, whereas stretch property has negligible effect on micropolarity. In MES and ME the phase velocity decreases with increase of wave number and for large wave numbers it approaches to the value of velocity of Rayleigh velocity at the free plane surface, which is approximately equal to 0.386146. This phenomenon has different value in EM.

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lutions of one dimensional steady flow of dusty gas an anholonomic co-ordinate system

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Abstract. We study the geometry of one-dimensional (i.e. unidirectional) incompressible steady dusty gas flow in Frenet frame field system (anholonomic co-ordinate system) by assuming the paths of velocities of dust and fluid phases to be in the same direction. The intrinsic decompositions of the basic equation are carried out and solutions for velocity of fluid phase u, velocity of dust phase v and pressure of the fluid are obtained in terms of spin coefficients, i.e. geometrical parameters like curvatures and torsions of the streamline when the flow is

(i) parallel straight line i.e. $k_s = 0$
(ii) parallel and $k_s \neq 0$, under the assumption that, the sum of the deformations at a point of the fluid surface along the stream line, its principal normal and binormal is constant.

Further, we have proved a result, which is an extension of Barron, and a graph of p against s is plotted (figure 1).

Keywords. Frenet frame field; dusty gas; velocities of dusty gas and fluid phases.

Introduction

study two phase flow of a dusty gas because it is useful in lunar ash flow, which plains many features of lunar soil. By ash flow, we mean the flow of a mixture of gas l small particles (ash or fine powder) such that the ash particles are fluidized and they have like a pseudo fluid. Thus, from a fluid dynamical point of view, the ash flow is a o phase flow of a gas and a pseudo fluid of solid particles. We consider gas flow in ich spherical solid dusty particles are uniformly distributed. The study of such flow is interest in a wide variety of areas of technical importance like environmental pollution, mation of raindrops and blood flow etc.

The introduction of geometric theories in the study of fluid flows simplifies the thematical complexities to a great extent and furnishes information regarding flow lds in a more general way. During the second part of the 20th century some authors e Truesdell [10], Kanwal [5, 6], Indrasena [4, 8], Purushotham [7, 8] and Bagewadi l Prasanna Kumar [1, 2] etc., have tried to study fluid flow by solving partial differential uations without using boundary conditions. They have prescribed some analytic l geometric conditions, which are valid not only on the boundary but also in the ole region. It seems that not much work has been done on the lines of the above hors.

Barron [3] has obtained solutions in steady plane flow of a viscous dusty fluid with parallel velocity fields in orthogonal curvilinear coordinates. i.e. by taking co-ordinate axes as fluid streamlines η = constant and their orthogonal trajectories ξ = constant as co-ordinate curves. He has proved the following theorems:

Theorem A. *In the steady plane flow of an incompressible viscous dusty fluid, if the velocity of the dust particles is everywhere parallel to the velocity of the fluid particles, then the flow must be radial or in parallel straight lines.*

Theorem B. *No steadyplane radial flow of a viscous incompressible dusty fluid exists for which N is constant on streamlines or constant throughout the flow.*

In the present paper we study and obtain solutions for one-dimensional steady incompressible viscous dusty fluid flow in Frenet frame field system and establish the following results:

Theorem 1.1. *Consider steady incompressible viscous dusty fluid flow in the Frenet frame field system. If the directions of paths of dust and fluid particles are same then prove either the flow is parallel straight line flow or no radial flow exists for which N is constant on streamlines.*

Theorem 1.1 is a generalization of theorems A and B of Barron [3].

2. Steady incompressible dusty gas flow

The steady flow of a dusty incompressible and viscous gas is governed by the following system of equations [9].

For fluid phase

Continuity equation:

$$\nabla \cdot \vec{u} = 0. \tag{2.1}$$

Conservation of linear momentum:

$$\rho(\vec{u} \cdot \nabla)\vec{u} + \nabla p = \mu\nabla^2\vec{u} + kN(\vec{v} - \vec{u}). \tag{2.2}$$

For dust phase

Continuity equation:

$$\nabla \cdot (N\vec{v}) = 0. \tag{2.3}$$

Conservation of linear momentum:

$$(\vec{v} \cdot \nabla)\vec{v} = \frac{k}{m}(\vec{u} - \vec{v}), \tag{2.4}$$

where u is the fluid phase velocity, v the dust phase velocity, m the mass of the dust particle, k the Stoke's resistance (drag) coefficient, ρ the density of the gas, p the pressure of the gas, μ the viscosity of the gas and N is the number density of the dust particles.

Let s, n, b be triply orthogonal unit vectors, tangent, principal normal and binormal respectively to the spatial curves of congruences formed by the fluid phase velocity and dust phase velocity lines respectively. We have the following geometrical relations by

net formulae [11]

$$\vec{u} = u\vec{s}, \quad \vec{v} = v\vec{s},$$

$$\frac{\partial \vec{s}}{\partial s} = k_s\vec{n}; \quad \frac{\partial \vec{n}}{\partial s} = \tau_s\vec{b} - k_s\vec{s}; \quad \frac{\partial \vec{b}}{\partial s} = -\tau_s\vec{n},$$

$$) \quad \frac{\partial \vec{n}}{\partial n} = k'_n\vec{s}; \quad \frac{\partial \vec{b}}{\partial n} = -\sigma'_n\vec{s}; \quad \frac{\partial \vec{s}}{\partial n} = \sigma'_n\vec{b} - k'_n\vec{n},$$

$$) \quad \frac{\partial \vec{b}}{\partial b} = k''_b\vec{s}; \quad \frac{\partial \vec{n}}{\partial b} = -\sigma''_b\vec{s}; \quad \frac{\partial \vec{s}}{\partial b} = \sigma''_b\vec{n} - k''_b\vec{b},$$

$$\nabla \cdot \vec{s} = \theta_{ns} + \theta_{bs}; \quad \nabla \cdot \vec{n} = \theta_{bn} - k_s; \quad \nabla \cdot \vec{b} = \theta_{nb}, \tag{2.5}$$

ere $\partial/\partial s$, $\partial/\partial n$ and $\partial/\partial b$ are the intrinsic differential operators along fluid phase ocity (or dust phase velocity) lines, principal, normal and binormal in an anholonomic ce. (k_s, k'_n, k''_b) and $(\tau_s, \sigma'_n, \sigma''_b)$ are the curvatures and torsions of the above curves. θ_{ns} l θ_{bs} are the normal deformations of these curves along their principal normals and ormals respectively.

of of Theorem 1.1. From eqs (2.5) (i) and (v) in eq. (2.1) we have

$$\frac{\partial u}{\partial s} + u(\theta_{ns} + \theta_{bs}) = 0 \quad \text{i.e} \quad \frac{\partial}{\partial s}\log u + (\theta_{ns} + \theta_{bs}) = 0. \tag{2.6}$$

o from eqs (2.5) (i) and (v) in (2.3), we have

$$\frac{\partial (Nv)}{\partial s} + Nv(\theta_{ns} + \theta_{bs}) = 0$$

$$\frac{\partial}{\partial s}\log N + \frac{\partial}{\partial s}\log v + (\theta_{ns} + \theta_{bs}) = 0. \tag{2.7}$$

uations (2.6) and (2.7) are intrinsic decompositions following eqs (2.1) and (2.3) in the net frame field system.

Jsing eqs (2.6) and (2.7) we have,

$$\frac{\partial}{\partial s}(\log N + \log v - \log u) = 0,$$

$$\frac{\partial}{\partial s}\log\frac{Nv}{u} = 0 \quad \text{i.e.,} \quad \frac{Nv}{u} = \alpha \tag{2.8}$$

ich is constant along fluid streamlines. i.e. $\alpha = \alpha(n, b)$ and thus from the above $= N(n, b)$. i.e. constant along streamlines. Therefore u is parallel to v. Using eqs (2.5) (ii) and (2.4) we have,

$$v\left(\frac{\partial v}{\partial s}\vec{s} + vk_s\vec{n}\right) = \frac{k}{m}\left(\frac{N}{\alpha} - 1\right)v\vec{s}.$$

e above equation gives

$$\frac{\partial v}{\partial s} = \frac{k}{m}\left(\frac{N}{\alpha} - 1\right) \quad \text{and} \quad vk_s = 0. \tag{2.9}$$

Equations (2.8) are the intrinsic decompositions of (2.4) in Frenet frame field system. From (2.8), $vk_s = 0 \Rightarrow v = 0$ or $k_s = 0$. By virtue of eq. (2.8), we obtain solutions for $v = 0$ implies $u = 0$. Thus $u = v = 0$. i.e. the flow does not exist. Hence v cannot be zero, we should have $k_s = 0$, i.e. curvature of dust phase is zero and hence dusty phase is a straight line.

If $v = 0$ and $k_s \neq 0$, then there is no dusty flow and since curvature $k_s \neq 0$ the flow must be radial. Thus, no steady radial flow of a dusty fluid exists. Now we obtain solutions for u, v and p as follows:

By virtue of eq. (2.8), we have $N = N(n, b)$ and $\alpha = \alpha(n, b)$ and (2.9) gives the magnitude of dust phase velocity $v = k/m((N/\alpha) - 1)s$.

Using this v and (2.8) we obtain fluid phase velocity

$$u = \frac{k}{m}\frac{N}{\alpha}\left(\frac{N}{\alpha} - 1\right)s.$$

From eqs (2.5) (i) to (v), we can decompose (2.2) into three equations as

$$\rho u \frac{\partial u}{\partial s} + \frac{\partial p}{\partial s} = \mu\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial n^2} + \frac{\partial^2 u}{\partial b^2} - u(\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2)\right] + kN(v - u), \tag{2.10}$$

$$\frac{\partial p}{\partial n} = \mu\left[\sigma_b'' \frac{\partial u}{\partial b} + \frac{\partial (u\sigma_b'')}{\partial b} - k_n' \frac{\partial u}{\partial n} - \frac{\partial (uk_n')}{\partial n}\right], \tag{2.11}$$

$$\frac{\partial p}{\partial b} = \mu\left[\sigma_n' \frac{\partial u}{\partial n} + \frac{\partial (u\sigma_n')}{\partial n} - k_b'' \frac{\partial u}{\partial b} - \frac{\partial (uk_b'')}{\partial b}\right]. \tag{2.12}$$

The above are intrinsic decompositions of eqs (2.2) in Frenet frame field system. Also, these equations have not assumed unidirectional motion. Suppose $\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2$ is independent of s i.e. a function of n and b. Using $u = (k/m)(N/\alpha)((N/\alpha) - 1)s$ and (2.10), we have

$$\frac{\partial p}{\partial s} = A(n, b)s, \tag{2.13}$$

where $A(n, b)$ is given by

$$A(n, b) = \frac{k}{m}\frac{N}{\alpha}\left(\frac{N}{\alpha} - 1\right)\left[\left(\frac{\alpha}{N} - 1\right)kN - \mu(\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2) + \rho\frac{k}{m}\frac{N}{\alpha}\left(\frac{N}{\alpha} - 1\right)\right],$$

$$\frac{\partial p}{\partial n} = \mu u\left[\frac{\partial \sigma_b''}{\partial b} - \frac{\partial k_n'}{\partial n}\right], \tag{2.14}$$

$$\frac{\partial p}{\partial b} = \mu u\left[\frac{\partial \sigma_n'}{\partial n} - \frac{\partial k_b''}{\partial b}\right]. \tag{2.15}$$

Integrating (2.13), we have

$$p = A(n, b)s^2 + f(n, b),$$

where $f(n, b)$ is an arbitrary function of n, b. By virtue of (2.14) and (2.15), $f(n, b)$ reduces to a constant. Choosing it as zero, p is given by $p = A(s^2/2)$. If $(N/\alpha) =$ constant

and $\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2 =$ constant, $A(n, b)$ reduces to constant and finally u, v and p are given by $u = c_1 s$, $v = c_2 s$ and $p = c_3 s^2$. These shows that paths of fluid and dust velocities are straight lines, whereas the pressure vary quadratically along the streamline.

We shall obtain solutions for steady-incompressible viscous dusty fluid flow if the sum of the deformations at a point of the fluid surface along streamlines and its principal and bi-normal is constant. i.e. when $(\theta_{ns} + \theta_{bs}) =$ constant.

Integrating (2.6) and (2.7), we have by virtue of (2.8)

$$u = u_0 \mathrm{e}^{-\int(\theta_{ns}+\theta_{bs})\mathrm{d}s}, \tag{2.16}$$

$$v = \frac{\alpha}{N} u_0 \mathrm{e}^{-\int(\theta_{ns}+\theta_{bs})\mathrm{d}s}. \tag{2.17}$$

Suppose $(\theta_{ns} + \theta_{bs}) =$ constant in the above equations, then u and v are given by $u = u_0 \mathrm{e}^{-(\theta_{ns}+\theta_{bs})s}$ and $v = (\alpha/N) u_0 \mathrm{e}^{-(\theta_{ns}+\theta_{bs})s}$. We know that (2.10), (2.11) and (2.12) are the intrinsic decompositions of (2.2) in Frenet frame field system and these equations have not assumed unidirectional motion.

Using $u = u_0\, \mathrm{e}^{-(\theta_{ns}+\theta_{bs})s}$ and (2.10), we have

$$\frac{\partial p}{\partial s} = \left[\mu[(\theta_{ns}+\theta_{bs})^2 - (\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2)] + kN\left(\frac{\alpha}{N} - 1\right)\right] u + \rho u^2(\theta_{ns} + \theta_{bs}) \tag{2.18}$$

and $\partial p/\partial n$, $\partial p/\partial b$ are similar to (2.14) and (2.15).

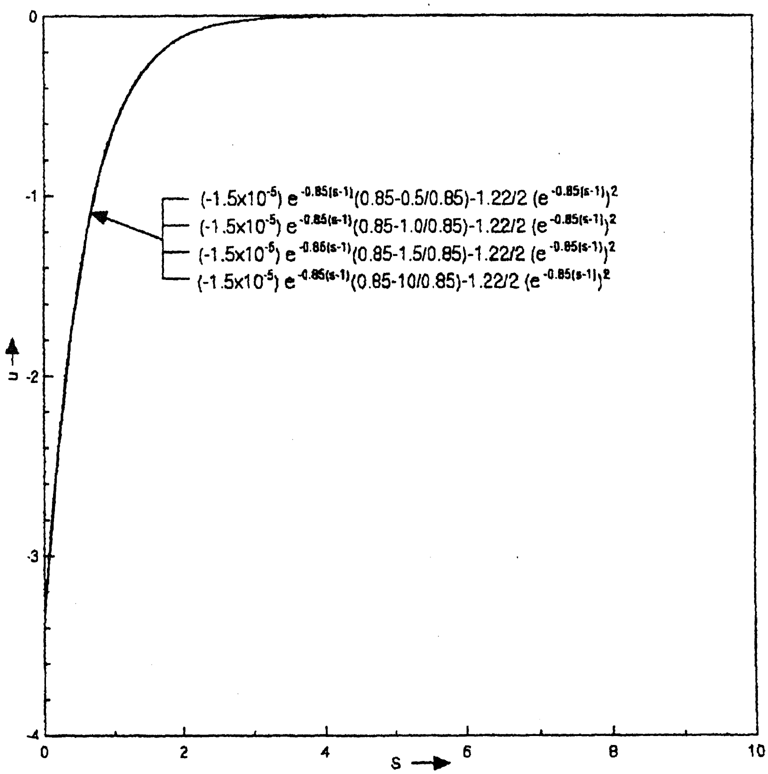

Figure 1. Variation of pressure with respect to s when $(\alpha/N) = 1$ for various values of λ.

Suppose $\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2$ is independent of s and integrating (2.18), we have

$$p = -\left[\mu[(\theta_{ns} + \theta_{bs})^2 - (\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2)] + kN\left(\frac{\alpha}{N} - 1\right)\right]$$
$$\times \frac{u}{(\theta_{ns} + \theta_{bs})} - \frac{\rho u^2}{2} + g(n,b),$$

where $g(n,b)$ is an arbitrary function of n,b. By virtue of (2.14) and (2.15), $g(n,b)$ reduces to a constant. Choosing it as zero, p is given by

$$p = -\left[\mu(\theta_{ns} + \theta_{bs})^2 - (\sigma_n'^2 + k_n'^2 + \sigma_b''^2 + k_b''^2) + kN\left(\frac{\alpha}{N} - 1\right)\right]$$
$$\times \frac{u}{(\theta_{ns} + \theta_{bs})} - \frac{1}{2}\rho u^2.$$

3. Applications

Sucking phenomena is included in the study of unidirectional flow. Asthma patients use inhalers to suck medicine (which is of dusty gas type) into the lungs. It is known that the particles present in the medicine move parallely which reaching the mouth and then the medicine (mixture of particles and gas) moves in one direction and reaches the lungs. Similarly, one can consider vacuum cleaners used to clean the house.

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Stokes drag on axially symmetric bodies: a new approach

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Abstract. In this paper a new approach to evaluate the drag force in a simple way on a restricted axially symmetric body placed in a uniform stream (i) parallel to its axis, (ii) transverse to its axis, is advanced when the flow is governed by the Stokes equations. The method exploits the well-known integral for evaluating the drag on a sphere. The method not only provides the value of the drag on prolate and oblate spheroids and a deformed sphere in axial flow which already exists in literature but also new results for a cycloidal body, an egg shaped body and a deformed sphere in transverse flow. The salient results are exhibited graphically. The limitations imposed on the analysis because of the lack of fore and aft symmetry in the case of an egg-shaped body is also indicated. It is also seen that the analysis can be extended to calculate the couple on a body rotating about its axis of symmetry.

Keywords. Stokes drag; spheroid; cyclical body moment.

1. Introduction

The drag force on a sphere placed in a slow uniform stream of viscous fluid was first determined by Stokes [5] in the year 1851 and is commonly known as Stokes drag. Because of their growing importance, particularly in the fields of biomechanics and chemical engineering slow flow problems have started engaging considerable attention in recent years. These problems are governed by the so called Stokes equations and the reader is referred to the book by Happel and Brennel [4] for some classical solutions. The basic singularity of this equation is the Stokeslet, a name proposed by Hancock [3] in 1953. It was utilized by Chawng and Wu [1] to generate solutions for a prolate spheroid in a variety of flows. It is worth noting that the most sought after quantity in such flow problems is the drag force rather than a detailed description of the flow field. In this study, we have proposed a new method for estimating the drag force on an axially symmetric body under some limitations. The method exploits the well-known drag formula for a sphere and the steps used in its derivation [p. 122, 4]. Using this method, first a simple formula is obtained for evaluating the drag force on an axially symmetric body, with continuously turning tangent, placed in an uniform stream along the axis of symmetry, and then the method is extended to the transverse flow situation. The formula, thus obtained, is used to estimate the values of the drag force on prolate and oblate spheroids and are seen to agree with known results [2, 4, 6] existing in the literature.

Having thus tested the formula, we use it to obtain the drag force for

(i) a deformed sphere,
(ii) an axis symmetric body obtained by rotation of cycloid and
(iii) an egg shaped body.

Further, this analysis is extended to calculate the couple on a body rotating about its axis of symmetry. The authors have not come across such results in the existing literature.

2. The method

Axial Flow: Let us consider the axially symmetric body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of viscous fluid of density ρ and kinematic viscosity ν. When the Reynolds number UL/ν is small, the motion is governed by Stokes equation [4],

$$0 = -(1/\rho)\,\mathrm{grad}\,p + \nu\nabla^2\mathbf{u},\ \mathrm{div}\,\mathbf{u} = 0, \tag{2.1}$$

subject to the no slip boundary condition. For the case of a sphere of radius R, the solution is easily obtained and on evaluating the stress, the drag force F comes out as [4]

$$F = (9/2)\pi\mu U \int_0^{\pi} R\sin^3\alpha\,\mathrm{d}\alpha = \lambda R, \tag{2.2}$$

where

$$\lambda = 6\pi\mu U. \tag{2.3}$$

This shows that the drag force increases linearly with the radius of the sphere. In other words, the difference between drag force on two spheres of radii y and $y + \mathrm{d}y$ is given by

$$\mathrm{d}F = \lambda \mathrm{d}y. \tag{2.4}$$

A sphere of radius b is obtained by rotating the curve $x = b\cos t, y = b\sin t\ (0 \le t \le \pi)$ about the x-axis and the force $F = \lambda b$ is obtained from (2.4) as $\int_0^b \lambda\,\mathrm{d}y$ exhibiting that the force system $\mathrm{d}F$ may be considered as lying in the xy plane.

The element force $\mathrm{d}F$ may be decomposed into two parts $(1/2)\,\mathrm{d}F$, each acting over the upper half and lower half; $(1/2)\,\mathrm{d}F$ on the upper half acts at a height y (say) above the x-axis. The total force $F/2$ on the upper half, may be considered as made up of these differential forces $\mathrm{d}F/2$ acting over elements corresponding to a system of half spheres of radii increasing from 0 to b and spread over from A to A' (figure 1). The moment of this force system (taken to be in the xy plane) about O, now provides

$$h.(F/2) = M = (1/2)\int_0^b y\,\mathrm{d}F = (1/2)\lambda\int_0^b y\,\mathrm{d}y = (1/4)\lambda b^2,$$

or

$$F = (1/2)(\lambda b^2)/h, \tag{2.5}$$

where h is the height of centroid of the force system. In the case of a sphere of radius b we have $F = \lambda b$, and so we get from (2.5), $h = b/2$ as it should be.

Next, we can express (2.2) also as

$$F = \int_{\alpha=0}^{\pi} \mathrm{d}f, \tag{2.6}$$

where

$$\mathrm{d}f = (3/4)\lambda R\sin^3\alpha\,\mathrm{d}\alpha \tag{2.7}$$

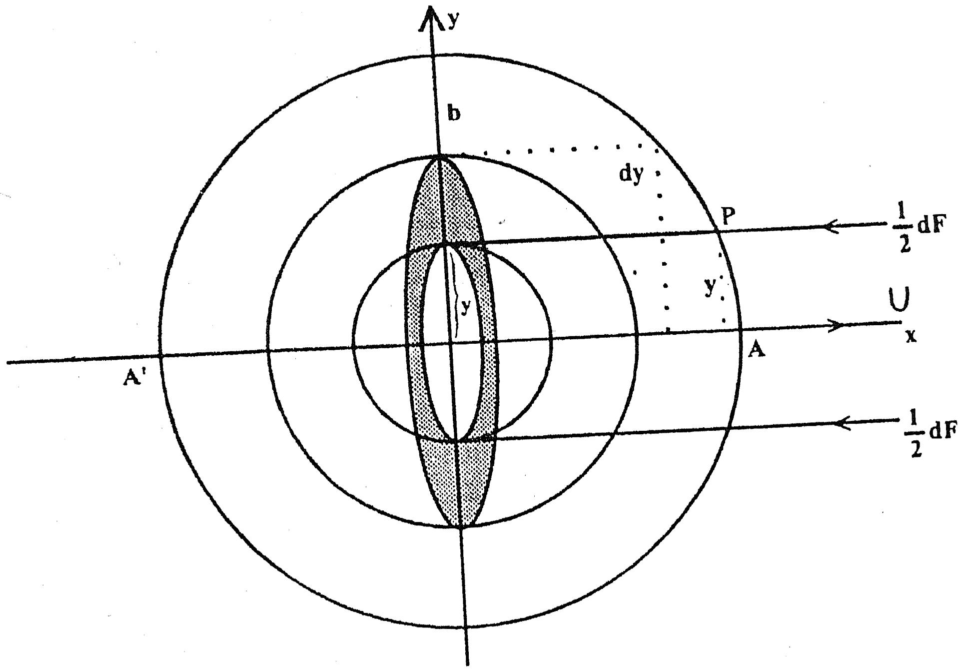

Figure 1. Elemental force system on the sphere.

is the elemental force on a circular ring element P (figure 2) [4, p. 122]. For the purpose of calculating $F/2$, the force on upper half, $(1/2)\mathrm{d}f$ may be taken to be acting at height η (say), above x-axis, given by

$$\begin{aligned} h &= \int \eta((1/2)\mathrm{d}f) \Big/ \int (1/2)\mathrm{d}f \\ &= \int_0^{\pi} \eta((3/4)\lambda R \sin^3\alpha \,\mathrm{d}\alpha) \Big/ \int_0^{\pi} (3/4)\lambda R \sin^3\alpha \,\mathrm{d}\alpha \\ &= (3/4) \int_0^{\pi} \eta \sin^3\alpha \,\mathrm{d}\alpha. \end{aligned}$$

Taking $\eta = R/2$, the result is seen to correspond to the value $h = b/2$ confirmed earlier. Thus we have

$$h = (3/8) \int_0^{\pi} R \sin^3\alpha \,\mathrm{d}\alpha. \tag{2.8}$$

It is proposed that the formula (2.8) holds good for an axially symmetric body also, when R is interpreted as the normal distance PM between the point P on the body and the point of intersection M of the normal at P with axis of symmetry and α as its slope (figure 2). On inserting the value of h from (2.8) in (2.5), we finally obtained

$$F_x = (1/2)(\lambda b^2)/h = (4/3)(\lambda b^2) \Big/ \int_0^{\pi} R \sin^3\alpha \,\mathrm{d}\alpha, \tag{2.9}$$

where the suffix x has been introduced to assert that the force is in the axial direction.

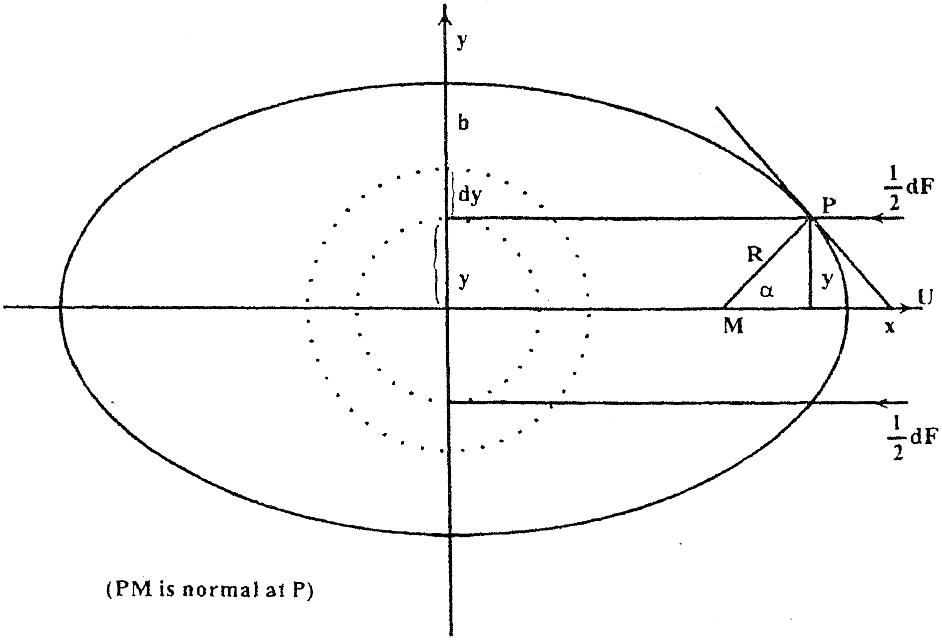

Figure 2. Force system on axially symmetric body.

While using (2.9) it should be kept in mind that b denotes intercept between the meridian curve and the axis of the normal perpendicular to the axis i.e., $b = R$ at $\alpha = \pi/2$.

Transverse flow: We set up a polar coordinate system (R, β, γ) with β as the polar angle with the y-axis and γ the azimuthal angle in $z - x$ plane. Since the y-axis is not the axis of symmetry for the body we cannot make use of circular ring elemental force $(3/4)\lambda R \sin^3\beta \, d\beta$ corresponding to (2.7). But we can easily write down the elemental force on the element $R^2 \sin\beta d\beta \, d\gamma$ as

$$\delta f = (3\lambda R/8\pi) \sin^3\beta d\beta \, d\gamma.$$

Transforming the above to the polar coordinates (R, θ, ϕ) with the x-axis as the polar axis, we have

$$\delta f = (3\lambda R/8\pi)(1 - \sin^3\alpha \cos^2\phi) \sin\alpha \, d\alpha \, d\phi,$$

as the force on the element $R^2 \sin\alpha \, d\alpha \, d\phi$. On integrating over ϕ from 0 to 2π, we get

$$df_y = (3\lambda R/8)(2 - \sin^2\alpha) \sin\alpha \, d\alpha, \tag{2.10}$$

where the suffix y has been placed to designate the force due to the external flow along the y-axis, the transverse direction.

Integrating df_y over the surface of the sphere, we get

$$F_y = (3\lambda/8) \int_0^{\pi} (2\sin\alpha - \sin^3\alpha) d\alpha = \lambda R, \tag{2.11}$$

agreeing with the correct value. This suggests we can take the force df_y as given by (2.10) as the element force on the circular ring element at P. Although the force F_y is along the y

direction, we have reduced it to elemental forces on a system of spheres centered on the x-axis. Since F_y and df_y themselves are scalar quantities, on comparing (2.10) with (2.7), we can use the analysis as in the axial flow case with h replaced by

$$h_y = (3/16)\int_0^{\pi} R(2\sin\alpha - \sin^3\alpha)d\alpha. \tag{2.12}$$

Thus, we get from (2.5)

$$F_y = (\lambda b^2)/(2h_y). \tag{2.13}$$

In addition, it seems that formula (2.5) should be valid only for a curve with continuously varying tangent, but not for a curve with edges or other kind of nodes.

We are working to improve our proposed conjecture and results will be presented in a future problem so that above stated limitations could be removed and results would have been true for general axi-symmetric bodies.

3. Flow past a spheroid

Prolate spheroid: A prolate spheroid is generated by the rotation of an ellipse about the x-axis. The parametric equation of the ellipse may be taken as

$$x = a\cos t, \quad y = b\sin t. \tag{3.1}$$

This provides the following values:

$$\sin\alpha = (b\sin t)/R, \cos\alpha = ((b^2/a)\cos t)/R, \;\; R = (b/a)(b^2\cos^2 t + a^2\sin^2 t)^{1/2}. \tag{3.2}$$

Therefore, for the axial flow, using (2.8), we get

$$h = (3/8)(b^2/a)\int_0^{\pi} \sin^3 t/(1 - e^2\cos^2 t)dt \tag{3.3}$$

$$= (3b^2/16ae^3)[(1+e^2)L - 2e], \tag{3.4}$$

where $L = \log\{(1+e)/(1-e)\}$. Substituting the above value of h in (2.9) and setting $\lambda = 6\pi\mu U$, we get the well-known results [2]

$$F_x = 16\pi\mu U a e^3[(1+e^2)L - 2e]^{-1}. \tag{3.5}$$

When $e \to 0$, it may be confirmed that we get the classical result $6\pi\mu Ua$ for a sphere placed in a uniform stream U. Next, using the values (3.2) in (2.12), we get

$$h_y = (3/32)(b^2/ae^3)[(3e^2 - 1)L + 2e] \tag{3.6}$$

and hence from (2.13), we have for the transverse flow

$$F_y = 32\pi\mu U a e^3[2e + (3e^2 - 1)L]^{-1}. \tag{3.7}$$

The above result is again seen to be in agreement with that obtained by Chwang and Wu [2]. When $e \to 0$, we get the classical result $6\pi\mu Ua$, for a sphere.

Oblate spheroid: The oblate spheroid is obtained by a rotation about the x-axis of the ellipse with parametric equations

$$x = b\cos t, \quad y = a\sin t. \tag{3.8}$$

Notice that the cross-section at O is now a circle of radius a and not b. This provides the following values:

$$\sin\alpha = (a\cos t)/R, \cos\alpha = (a^2/b)(\sin t)/R, R = (a/b)(a^2\sin^2 t + b^2\cos^2 t)^{1/2}. \tag{3.9}$$

Next, using the above values in (2.9), we get

$$h = (3/8)(a/e^3)[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1} e],$$

on substituting the above value of h in (2.9) with b replaced by a we get the axial drag force as

$$F_x = (1/2)(\lambda a^2/h) = (8\pi\mu U a e^3)[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1} e]^{-1}, \tag{3.10}$$

when $e \to 0$, it may be confirmed that we get the classical result $F_x = 6\pi\mu U a$ for a sphere. For the transverse flow we have from (3.6)

$$h_y = (3/16)(a/e^3)[(1+2e^2)\sin^{-1} e - e(1-e^2)^{1/2}],$$

and so from (2.12) with b replaced by a

$$F_y = (\lambda a^2/2h_y) = 16\pi\mu U a e^3/[(1+2e^2)\sin^{-1} e - e(1-e^2)^{1/2}]. \tag{3.11}$$

The values (3.10) and (3.11) are seen to agree with known results [4].

4. Flow past a deformed sphere

Consider the axially symmetric body defined by

$$r = a\left[1 + \epsilon\left\{d_0 + d_2P_2(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P_{2k+1}(\mu)\right\}\right], \quad \mu = \cos\theta, \tag{4.1}$$

where (r, θ) are spherical polar coordinates and $P_n(\mu)$ are Legendre functions. For a small value of parameter ϵ, this equation (4.1) represents a spheroid of small eccentricity $e = \sqrt{(3\epsilon d_2)}$ when $d_n = 0\,(n \geq 3)$. This provides the following values (figure 2):

$$\sin\alpha = \sin\theta + \epsilon\left\{d_2P'_2(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P'_{2k+1}(\mu)\right\}\sin\theta\cos\theta + 0(\epsilon^2), \tag{4.2}$$

$$\cos\alpha = \cos\theta - \epsilon\left\{d_2P'2(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P'_{2k+1}(\mu)\right\}\sin^2\theta + 0(\epsilon^2), \tag{4.3}$$

$$R = (r\cdot\sin\theta)/\sin\alpha. \tag{4.4}$$

Now, we have

$$b = R_{\alpha=\pi/2} = a[1 + \epsilon(d_0 - d_2/2) + \epsilon\{d_3.P_3(0) + d_4.P_4(0) + d_5.P_5(0) + \ldots\} + 0(\epsilon^2)],$$

if it is to be same as that for a spheroid, we must have

$$d_3P_3(0) + d_4P_4(0) + d_5P_5(0) + \cdots = 0,$$

and then

$$b = a[1 + \epsilon(d_0 - d_2/2)]. \tag{4.5}$$

Therefore, for the axial flow, using (2.8), we get

$$h = (a/2)[1 + \epsilon(d_0 - (4/5)d_2)].$$

Substituting the above value of h in (2.7), we get the result derived by Usha and Nigam mentioned in (4), viz.,

$$F_x = 6\pi\mu Ua\{1 + \epsilon(d_0 - d_2/5) + 0(\epsilon^2)\}. \tag{4.6}$$

As $\epsilon \longrightarrow 0$, we get the classical result for a sphere with radius a placed in a uniform stream U. Next, for the transverse flow, using the values (4.2), (4.3), (4.4) in (2.12), we get

$$h_y = (a/2)[1 + \epsilon\{d_0 - (11/10)d_2\}], \tag{4.7}$$

providing

$$F_y = (\lambda b^2)/(2h_y) = 6\pi\mu Ua[1 + \epsilon\{d_0 - (1/10)d_2\}]. \tag{4.8}$$

For $\epsilon \longrightarrow 0$, we again get the classical result for a sphere. It is interesting to note that so long the transverse length b remains unaltered, the drag values F_x and F_y do not differ, from those of a spheroid.

5. Flow past a cycloidal body of revolution

Case I: Let us take the inverted cycloid

$$x = a(t + \sin t), y = a(1 + \cos t), -\pi \leq t \leq \pi, \tag{5.1}$$

with vertex at $(0, 2a)$, and revolve it about x-axis, the base, to generate the cycloidal body of revolution. In this case we have

$$\alpha = (\pi/2) - (t/2), R = 2a.\cos(t/2), \tag{5.2}$$

and so we get from (2.9) by replacing b by $2a$,

$$h = (9/32)a\pi. \tag{5.3}$$

The formula (2.9) provides, for the axial flow, the drag force,

$$Fc_x = (128/3)\mu Ua, \tag{5.4}$$

where the label c stands for cycloid. For the transverse flow, we have from (2.12)

$$h_y = (15/64)a\pi, \tag{5.5}$$

and then from (2.13) and (5.4), we have

$$Fc_y = (256/5)\mu Ua = (6/5)Fc_x = 1.20Fc_x. \tag{5.6}$$

The transverse drag force on the cycloid is 1.2 times the axial drag force. This may be used experimentally to confirm the conjectures proposed in this paper.

Case II: Next, we consider the body generated by the rotation about x-axis of the curve composed of arcs of two cycloidal parts represented parametrically by

$$x = a(1 + \cos t), \quad y = a(t + \sin t), \quad 0 \le t \le \pi; \tag{5.7a}$$

$$x = -a(1 + \cos t), \quad y = a(t + \sin t), \quad 0 \le t \le \pi. \tag{5.7b}$$

Thus, we have, for the first part

$$\alpha = t/2, \sin\alpha = \sin(t/2), R = a(t + \sin t)/\sin(t/2), \tag{5.8}$$

$$\alpha = \pi - t/2, \quad \sin\alpha = \sin t/2, \quad R = a(t + \sin t)/\sin t/2, \tag{5.9}$$

using the above values in (2.8), we have

$$\begin{aligned} h &= (3/8)\left[\int_{\alpha=0}^{\pi/2} R\sin^3\alpha\, d\alpha + \int_{\alpha=\pi/2}^{\pi} R\sin^3\alpha\, d\alpha\right] \\ &= (a/32)[3\pi^2 + 16], \end{aligned} \tag{5.10}$$

and hence from (2.9), we have the axial drag

$$Fc_x = (96\pi^3\mu Ua)/(3\pi^2 + 16). \tag{5.11}$$

Similarly, for the transverse flow, we have from (2.12) with R given by (5.9)

$$\begin{aligned} h_y &= (3/16)\left[\int_{\alpha=0}^{\pi/2} \{2R\sin\alpha - R\sin^3\alpha\}d\alpha + \int_{\alpha=\pi/2}^{\pi} \{2R\sin\alpha - R\sin^3\alpha\}d\alpha\right] \\ &= (a/64)[9\pi^2 + 32], \end{aligned} \tag{5.12}$$

and hence from (2.13), we have transverse drag

$$Fc_y = \lambda(a\pi)^2/2h_y = (92\pi^3\mu Ua)/(9\pi^2 + 32). \tag{5.13}$$

In Case II, we have

$$Fc_x/Fc_y = (9\pi^2 + 32)/(6\pi^2 + 32) \text{ or } Fc_x \approx 1.32\, Fc_y. \tag{5.14}$$

Keeping in view that in Case II, the axis have been interchanged, it is seen that result (5.14) compares with the result (5.6).

It will be interesting to compare that results for cycloid with those of a spheroid. For this purpose we consider a spheroid with major and minor axes of lengths $2\pi a$ and $4a$ respectively, corresponding to the maximum and minimum diameters of the cycloidal body. The eccentricity of such spheroid is $e = \sqrt{(1 - (4/\pi^2))} = 0.77$. Therefore, we have from (5.4) and (3.5) with a replaced by $a\pi$ and evaluated at $e = 0.77$ and the label p stands for the prolate spheroid.

$$Fc_x/Fp_x = (8/3\pi)[\{(1 + e^2)L - 2e\}/e^3] \approx 1.00. \tag{5.15}$$

From (5.6) and (3.7) with a replaced by $a\pi$,

$$Fc_y/Fp_y = (8/5\pi^2)[\{2e + (3e^2 - 1)L\}/e^3] \approx 1.1. \tag{5.16}$$

Next, from (5.11) and (3.10) with a replaced by $a\pi$, we have

$$Fc_x/Fo_x = (12\pi^2/(3\pi^2 + 16))[\{e(1 - e^2)^{1/2} - (1 - 2e^2)\sin^{-1} e\}/e^3] \approx 3.7, \tag{5.17}$$

and from (5.13) and (3.11) with a replaced by $a\pi$, we have

$$Fc_y/Fo_y=(12\pi^2/(9\pi^2+32))[\{-e(1-e^2)^{1/2}+(1+2e^2)\sin^{-1}e\}/e^3]\approx 3.06, \tag{5.18}$$

where the subcript o stands for oblate spheroid. It is interesting to note that while the ratios for the prolate case given by (5.15) and (5.16) also close to unity, the forces on the cycloidal body as given by (5.17) and (5.18) are much larger than those on the oblate spheroid.

6. Flow past an egg-shaped body

Continuing in the same manner, we have calculated the drag for an egg-shaped body in which the right portion is in the shape of a half prolate spheroid given parametrically by

$$x=a\cos t,\quad y=a\sin t,\quad 0\leq t\leq \pi/2, \tag{6.1}$$

and left portion is a hemisphere given by

$$x=b\cos t,\quad y=b\sin t,\quad \pi/2\leq t\leq \pi. \tag{6.2}$$

Now for the spheroidal portion, we have

$$\begin{aligned}\sin\alpha &= (b\sin t)/R, \cos\alpha=(b\cos t)/(b^2\cos^2 t+a^2\sin^2 t)^{1/2},\\ R &= (b/a)(a^2\sin^2 t+b^2\cos^2 t)^{1/2},\end{aligned} \tag{6.3}$$

and for the spherical part, we have

$$R=b. \tag{6.4}$$

Next, for (2.8), we can obtain

$$\begin{aligned}h &= (3/8)\int_{\alpha=0}^{\pi} R\sin^3\alpha\,\mathrm{d}\alpha=(3/8)\left[\int_{\alpha=0}^{\pi/2} R\sin^3\alpha\,\mathrm{d}\alpha+\int_{\alpha=\pi/2}^{\pi} R\sin^3\alpha\,\mathrm{d}\alpha\right]\\ &= (3/8)[(b^2/4ae^3)\{-2e+(1+e^2)\log(1+e/1-e)\}+(2/3)b],\end{aligned} \tag{6.5}$$

and then on substituting the above value of h in (2.9), we have axial drag

$$\begin{aligned}F_x &= 8\pi\mu Ua(1-e^2)^{1/2}[(2/3)+(1-e^2)^{1/2}/4e^3\\ &\quad\times\{-2+(1+e^2)\log(1+e/1-e)\}]^{-1}.\end{aligned} \tag{6.6}$$

When $e\to 0$, it may be confirmed that we get the classical result $F_x=6\pi\mu Ua$ for a sphere. Similarly, for transverse flow, from (2.12), we have

$$\begin{aligned}h_y &= (3/16)\int_0^{\pi} R(2\sin\alpha-\sin^3\alpha)\mathrm{d}\alpha\\ &= (3/16)\left[\int_{\alpha=0}^{\pi/2} R(2\sin\alpha-\sin^3\alpha)\mathrm{d}\alpha+\int_{\alpha=\pi/2}^{\pi} R(2\sin\alpha-\sin^3\alpha)\mathrm{d}\alpha\right]\\ &= (3/16)[(4/3)+(1-e^2)^{1/2}/4e^3\{2e+(3e^2-1)\log(1+e/1-e)\}].\end{aligned} \tag{6.7}$$

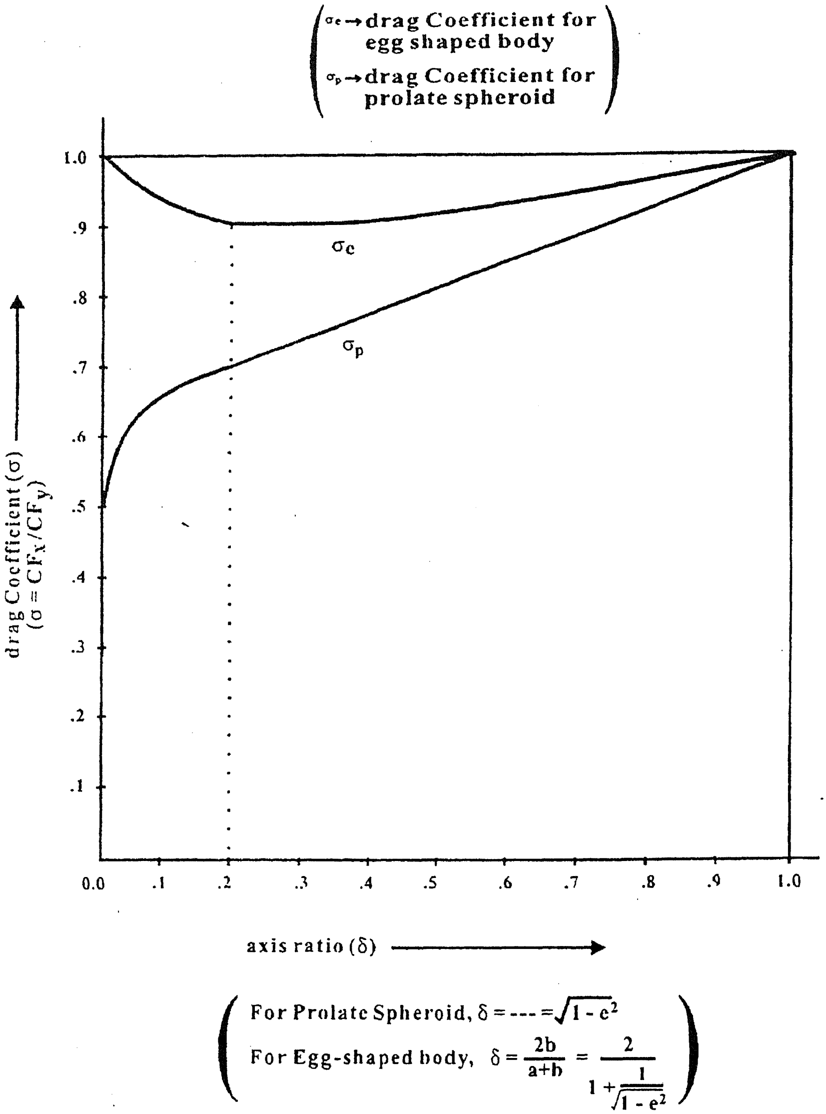

Figure 3. Effect of axis ratio δ on drag coefficient ratio σ.

On substituting the above h_y in (2.13), we get the transverse drag force

$$F_y = (16\pi\mu Ua)(1-e^2)^{1/2}[(4/3)+(1-e^2)^{1/2}/4e^3$$
$$\times\{2e+(3e^2-1)\log(1+e/1-e)\}]^{-1}. \quad (6.8)$$

When $e \to 0$, it may be confirmed that we get the classical result for a sphere. Since Stokes flow does not distinguish between fore and aft flow patterns, the same results will be obtained when the direction of flow is reversed.

It is instructive to analyse figure 3 depicting the ratio $\sigma = F_x/F_y$ for a prolate spheroid and an egg-shaped body. It is seen that while σ steadily increases with the axis ratio δ (for

prolate spheroid, $\delta = \sqrt{(1-e^2)}$ and for egg-shaped body $\delta = 2/\{1 + 1/\sqrt{(1-e^2)}\}$ for the former from the value 0.5 at $\delta = 0$, corresponding to a needle, to its maximum value 1 for a sphere now for the latter it begins with the maximum value 1, for a needle decreasing to its lowest value 0.90 at $\delta = 0.2$ and then rises again to the maximum value 1 for a sphere. This discrepancy indicates that the results for the egg-shaped body are not tenable for small values of δ (at least up to $\delta = 0.2$). The cause of this discrepancy may be due to the tremendous lack of fore and aft symmetry when δ is small. It is worth noting that not only the ratio b/a of right and left axial lengths is very large, the ratio of corresponding curvatures is also large.

7. Moment on a rotating body

The moment on a sphere of radius b with angular velocity Ω is given by the formula [4]

$$M = 6\pi\mu b^3 \int_{\alpha=0}^{\pi} \sin^3\alpha \, d\alpha \doteq \int_0^{\pi} dm(\text{say}), \tag{7.1}$$

where

$$dm = 6\pi\mu b^3 \sin^3\alpha \, d\alpha. \tag{7.2}$$

Comparing it with the elemental force df (with R replaced by b) as given by (2.7) and keeping in mind that two forces constitute a couple, we have

$$dm = (1/2)(3/4)(b^2\Omega/U)df = (2/3)a^2(1-e^2)(\Omega/U)df. \tag{7.3}$$

Prolate spheroid

Since the moment of a force is linearly related to the force, the relation (7.3) enables us to write down at once, on making use of the result (3.5), the moment on a prolate spheroid rotating about x-axis as

$$M_x = (32/3)\pi\mu a^3 e^3\Omega(1-e^2)[-2e + (1+e^2)L]^{-1}. \tag{7.4}$$

Oblate spheroid

In a similar fashion, the moment on an oblate spheriod rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (16/3)\pi\mu a^3 e^3\Omega[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1}e]^{-1}. \tag{7.5}$$

The above results (7.4), (7.5) are in agreement with classical results in the literature [4].

Cycloidal body

In a similar fashion, the moment on the cycloidal body (5.1) rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (1024/9)\mu\Omega a^3. \tag{7.6}$$

Again for the cycloidal body (5.7), the moment is found to be given by

$$M_x = (64\mu\Omega\pi^5 a^3)/(3\pi^2 + 16). \tag{7.7}$$

Egg-shaped body

Next, the moment on an egg-shaped body rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (16/3)\mu\Omega\pi a^3(1-e^2)^{3/2})[(2/3)+\sqrt{(1-e^2)}/4e^3(-2e+(1+e^2)L]^{-1}, \tag{7.8}$$

where L is given by (3.4).

Deformed sphere

In the end, the moment on a deformed sphere rotating with angular velocity Ω about x-axis is found by using (7.3) and (4.6) as

$$M_x = 4\pi\mu\Omega a^3\{1+\varepsilon(3d_0-(6/5)d_2)+o(\varepsilon^2)\}. \tag{7.9}$$

For $\varepsilon \to 0$, we get the classical result, $4\pi\mu\Omega a^3$, for a sphere with radius a and rotating with angular velocity Ω.

8. Conclusion

The simple drag formula (2.9) for axial free stream providing exact results for a spheroid may be fortuitous but the formula (2.13) derived by the help of (2.9), giving exact result for the non-axisymmetric situation cannot be dismissed as due to chance, Thus, it may be conjectured that the formulas do provide approximations to Stokes drag for axisymmetric bodies. These are, of course, subject to restrictions on the geometry of the meridional body profile $y(x)$ of continuously turning tangent implying that $y'(x)$ is continuous $y''(x) \neq 0$, thereby avoiding corners and straight line portions.

Since both axial and transverse flows have been considered in a free stream results of the force at an oblique angle of attack may be resolved into its components to get the required result. Flow with paraboloidal free stream has been represented through average velocity [2]. This may be exploited to generate a drag formula for axial symmetric bodies for more complex flows.

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Stability of an expanding bubble in the Rayleigh model

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Abstract. A bubble expands adiabatically in an incompressible, inviscid liquid. The variation of its radius R with time is given by the Rayleigh's equation. We find that the bubble is stable at the equilibrium point in this model.

Keywords. Bubble; Rayleigh's equation; stability; hamiltonian; Liapounov's function.

1. Introduction

When a bubble expands adiabatically in an inviscid liquid at rest at infinity, the variation of its radius R with time is given by Rayleigh's equation [4], which is a highly non-linear equation. This equation has been generalized including viscosity of surrounding liquid and surface tension by Noltingk and Neppiras [5] and Poritsky [6]. Progress in the study of this equation is generally expected numerically. However, when the liquid outside the bubble is inviscid, Rayleigh's modified equation can be transformed suitably and we can prove, analytically, that the expanding bubble in this model is stable at the equilibrium point.

2. Mathematical formulation and analysis

As the bubble expands adiabatically, its radius R is given by Rayleigh's equation, modified by surface tension σ. This equation can be written as (Chakraborty [1] and Chakraborty and Tuteja [2])

$$R\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\frac{1}{\rho}\left\{p_e-p_{g_0}\left(\frac{R_0}{R}\right)^{3\gamma}+\frac{2\sigma}{R}\right\}=0, \tag{1}$$

where ρ is the density of the outside inviscid liquid, p_{g_0} and R_0 are the gas pressure and radius of the bubble initially, p_e is the pressure, taken as constant, in the liquid at a large distance from the bubble and γ is the ratio of the two specific heats of the gas. We find that equation (1) can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(R^{5/2})+\frac{5}{2}\frac{R^{1/2}}{\rho}\left\{p_e-p_{g_0}\left(\frac{R_0}{R}\right)^{3\gamma}+\frac{2\sigma}{R}\right\}=0. \tag{2}$$

Defining

and taking $\gamma = 4/3$ for simplicity, we finally get from (2) the equation

$$\frac{d^2 r}{dt^2} + \frac{5}{2\rho} r^{1/5} \left\{ p_e - p_{g0} \left(\frac{r_0}{r} \right)^{8/5} + 2\sigma r^{-2/5} \right\} = 0, \tag{3}$$

where $r_0 = R_0^{5/2}$.

We use p_{g_0} and U_0 as characteristic pressure and speed respectively and r_0 as the characteristic value of r and T_0 as that of time, where $T_0 = R_0/U_0$. We define the dimensionless quantities r', t' and p'_e as

$$r' = r/r_0, \quad t' = t/T_0, \quad p'_e = p_e/p_{g0}.$$

We may now write eq. (3) as

$$\frac{d^2 r'}{dt'^2} = \frac{-5p_{g0}}{2\rho U_0^2} \left\{ p'_e r'^{1/5} + \frac{2\sigma}{p_{g_0} R_0} r'^{-1/5} - r'^{-7/5} \right\}. \tag{4}$$

Omitting the dashes from r', t' and p'_e and using from now on these undashed symbols for the corresponding quantities, we find that eq. (4) can be written in dimensionless form as

$$\frac{d^2 r}{dt^2} = \frac{-5p_{g0}}{2\rho U_0^2} \left\{ p_e r^{1/5} + \frac{2\sigma}{p_{g0} R_0} r^{-1/5} - r^{-7/5} \right\}. \tag{5}$$

Finally, defining x and y by the equations

$$x = r, y = \frac{dr}{dt}, \tag{6}$$

so that eq. (5) can be written as

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} y \\ \dfrac{-5p_{g0}}{2\rho U_0^2} \left(p_e x^{1/5} - x^{-7/5} + \dfrac{2\sigma}{p_{g0} R_0} x^{-1/5} \right) \end{bmatrix}. \tag{7}$$

Equation (7) defines a Hamiltonian system, with Hamiltonian $H(x, y)$ so that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \tag{8}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}. \tag{9}$$

In view of (7), equations (8) and (9) give

$$H = \frac{y^2}{2} + \frac{5p_{g0}}{2\rho U_0^2} \left(\frac{5}{6} p_e x^{6/5} + \frac{5}{2} x^{-2/5} + \frac{5\sigma}{2p_{g_0} R_0} x^{4/5} \right) + C, \tag{10}$$

where C is an arbitrary constant.

The equilibrium point (x, y) for the basic dynamical system, defined by (7), is given by

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0. \tag{11}$$

Equation (11), in view of (10), shows that an equilibrium point (x, y) satisfies the equations

$$y = 0, \quad p_e x^{1/5} - x^{-7/5} + \frac{2\sigma}{p_{g_0} R_0} x^{-1/5} = 0. \tag{12}$$

Writing (7) as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}), \tag{13}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} y \\ \dfrac{-5p_{g_0}}{2\rho U_0^2} \left(p_e x^{1/5} - x^{-7/5} + \dfrac{2\sigma}{p_{g_0} R_0} x^{-1/5} \right) \end{bmatrix}, \tag{14}$$

we find that $\mathbf{F}(\mathbf{x})$ vanishes at the equilibrium point. Also H and its partial derivatives are continuous at all points except when $x = 0$. If α is a positive real root of (12), then equation (14) shows that $x = \alpha$ and $y = 0$ is the equilibrium point of (13). Let us choose C in (10) in such a way that at this equilibrium point, the value of H vanishes. Hence

$$H(\alpha, 0) = \frac{5p_{g_0}}{2\rho U_0^2} \left(\frac{5}{6} p_e \alpha^{6/5} + \frac{5}{2} \alpha^{-2/5} + \frac{5\sigma}{2p_{g_0} R_0} \alpha^{4/5} \right) + C = 0. \tag{15}$$

H is minimum at the equilibrium point $x = \alpha, y = 0$ if

$$\frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y} \right)^2 > 0. \tag{16}$$

The condition (16), in view of (10), gives us the condition

$$p_e x^{1/5} + 7x^{-7/5} - \frac{2\sigma}{p_{g_0} R_0} x^{-1/5} > 0 \tag{17}$$

for the existence of a minimum at the equilibrium point $(\alpha, 0)$. Adding the left hand side of (12) to the left hand side of (17), we get

$$p_e x^{1/5} + 3x^{-7/5} > 0,$$

which is always satisfied as x is real and positive. Therefore, H is minimum at the equilibrium point $x = \alpha$, $y = 0$, but H vanishes at $(\alpha, 0)$. Thus H is always positive near $(\alpha, 0)$ and is therefore positive definite in the neighbourhood of the equilibrium point $(\alpha, 0)$. Also $-\mathbf{F}$. grad H vanishes in view of (8), (9) and (13). Thus $-\mathbf{F}$. grad H is positive semidefinite. We can therefore choose H as a Liapounov function and by Liapounov's theorem (Drazin [3]) we have the result that a bubble expanding through an inviscid liquid is stable at its equilibrium point.

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